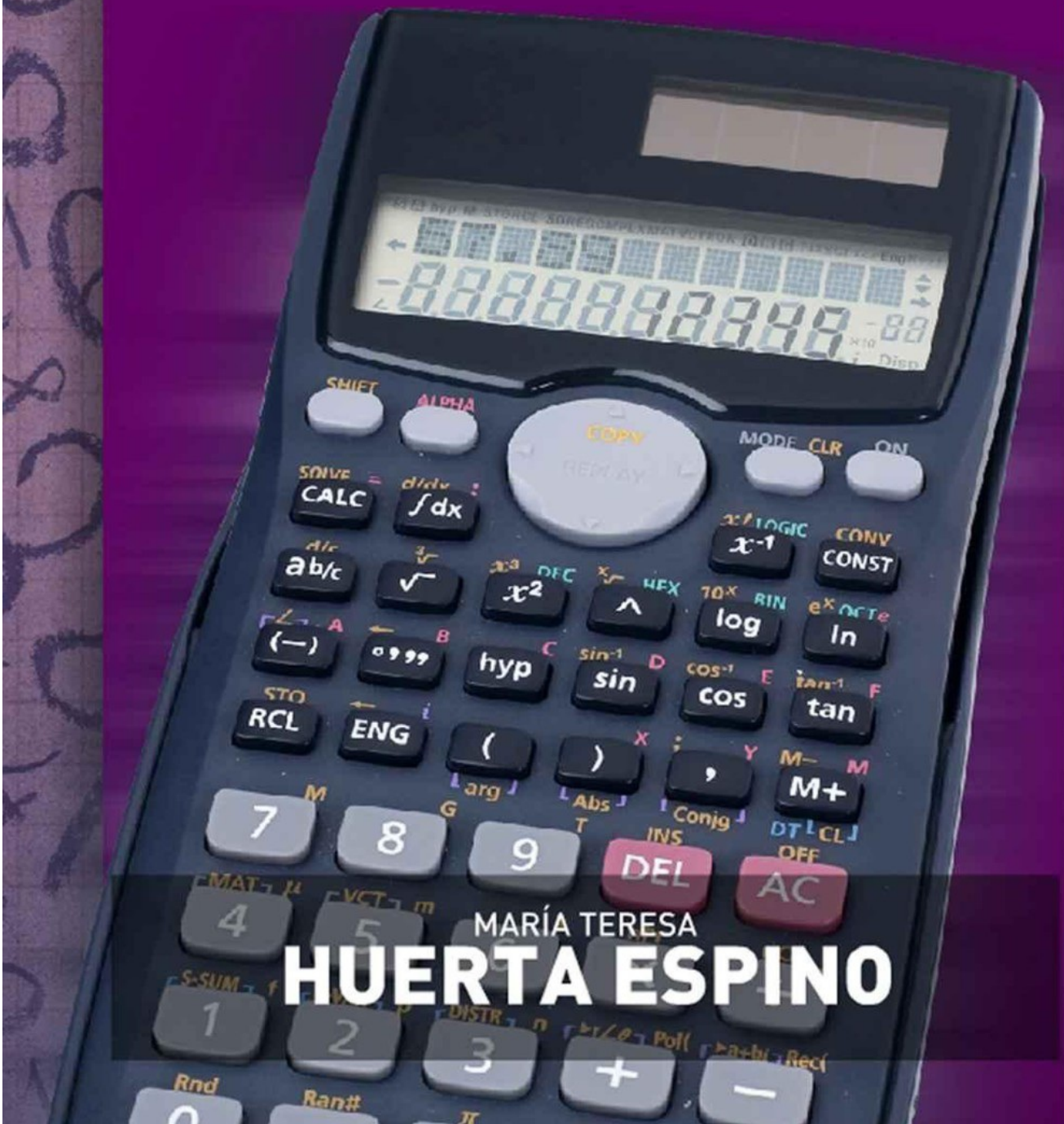
EDITORIAL
DIGITAL
TECNOLÓGICO DE MONTERREY

DIFFERENTIAL CALCULUS



MARÍA TERESA
HUERTA ESPINO

Table of Contents

Publishing House
About the author
Concept Map
About this eBook
Chapter 1. Functions and their graphs
1.1 Function concept
1.2 Linear functions
1.3 Quadratic functions
1.4 Rational functions
1.5 Trigonometric functions
1.6 Exponential functions
1.7 Logarithmic functions
1.8 Step functions
1.9 Absolute value functions
Chapter 1. Conclusion
Chapter 1. Activities
Chapter 1. Resources
Chapter 2. Algebra in functions
2.1 Basic operations
2.2 Composition of functions
Chapter 2. Conclusion
Chapter 2. Activities
Chapter 2. Resources
Chapter 3. Limits, continuity and their properties
3.1 Definition of limit
3.2 Properties of the limits of a function
3.3 Limits to infinity and infinite limits
3.4 Continuity
Chapter 3. Conclusion
Chapter 3. Activities
Chapter 3. Resources
Chapter 4. Differentiation
4.1 Rate of change and the tangent
4.2 Definition of the derivative of a function
4.3 General derivative rules
4.4 High-order derivatives
4.5 The product and quotient rule
4.6 The chain rule
4.7 Derivatives of trigonometric functions
4.8 Derivatives of exponential and logarithmic functions
4.9 Implicit differentiation
Chapter 4. Conclusion
Chapter 4. Activities
Chapter 4. Resources
Chapter 5. Properties of curves

5.1 Stationary points
5.2 Turning points by the 1st derivative test
5.3 Turning points by the 2nd derivative test
5.4 Curve sketching
Chapter 5. Conclusion
Chapter 5. Activities
Chapter 5. Resources
Chapter 6. Optimization
6.1 Solving method for optimization problems
6.2 Geometry
6.3 Economy
Chapter 6. Conclusion
Chapter 6. Activities
Chapter 6. Resources
Glossary

Publishing House

DIFFERENTIAL CALCULUS

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MARÍA TERESA HUERTA ESPINO

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About the author

Professor at Tecnológico de Monterrey, Preparatoria Esmeralda del Campus Estado de México since 2006. She obtained an electronics and communications engineering degree at this same institution. During two years she was part of the TI staff at Daimler-Chrysler, where she worked planning, maintaining and supporting structural networks in the whole continent.

She obtained a Master in Mechatronics by the FH Ravensburg Weingarten University at Germany, where she worked as an investigation assistant in the Steinbeis TransferZentrum – Weingarten. She collaborated by designing and building corrosion- proof equipment for the Mercedes Benz.

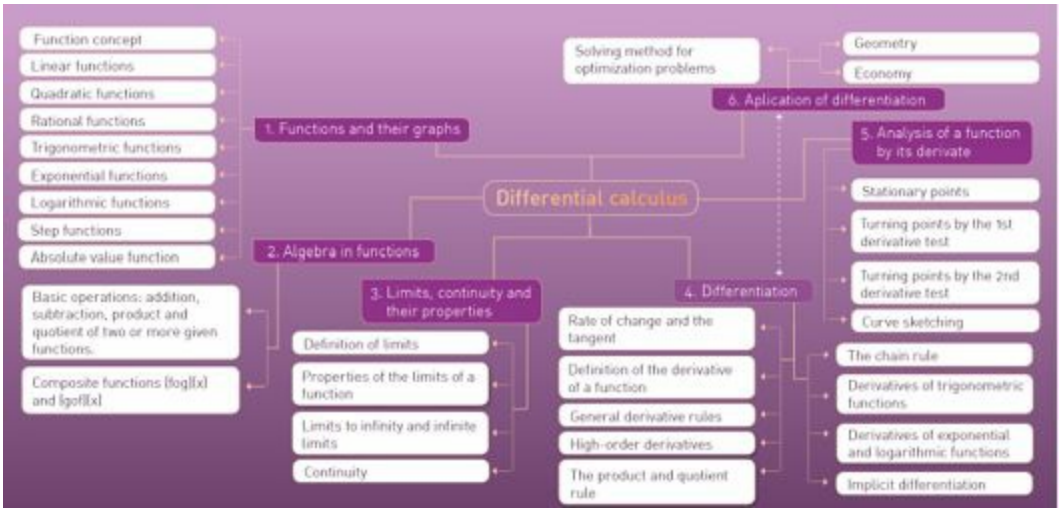
She has participated at British Columbia's summer program for professors in Canada, Colegio de St. George´s math workshop in Argentina and at UDEM's Physics workshop in Monterrey, México.

She also redesigned the Math and TI courses; coordinate the quality improvement process of the multimedia virtual modality. Developed workbooks for Math I, II and III focused in Tecnológico de Monterrey's high school study programs.

In 2011 she performed as head coach for students' group "Tec Balam", who participated at "FIRST Robotics Competition" in Toronto, winning the "Rookie All-Star Award" that celebrates the rookie team exemplifying a young but strong partnership effort. Later, this team won the "Rookie Inspiration Award", which celebrates a rookie team's outstanding success in advancing respect and appreciation for engineering, at Saint Louis World Championship.

Nowadays she is part of a project for mobile learning; she develops resources and activities that support the teaching-learning experience with mobile devices.

Concept Map


Differential calculus
1. Functions and their graphs
Function concept
Linear functions
Quadratic functions
Rational functions
Trigonometric functions
Exponential functions
Logarithmic functions
Step functions
Absolute value function
2. Algebra in functions
Basic operations: addition, subtraction, product and quotient of two or more given functions.
Composite functions (fog)(x) and (gof)(x)
3. Limits, continuity and their properties
Definition of limits
Properties of the limits of a function
Limits to infinity and infinite limits
Continuity
4. Differentiation
Rate of change and the tangent
Definition of the derivative of a function
General derivative rules
High-order derivatives
The product and quotient rule
6. Aplication of differentiation
Solving method for optimization problems
Geometry
Economy
5. Analysis of a function by its derivate
Stationary points
Turning points by the 1st derivative test
Turning points by the 2nd derivative test
Curve sketching
The chain rule
Derivatives of trigonometric functions
Derivatives of exponential and logarithmic functions
Implicit differentiation

About this eBook

The main intention of this eBook is to assist student in learning process, throughout the use of additional resources and better comprehension of real life applications of Calculus.

The proposed activities aim at promoting and consolidating the use of technical communication skills and proper mathematical notations.

An auto evaluation section is provided at the end of each chapter, to check whether topic was properly understood; that is after going through the explanations and detailed procedures of various problems. It is also important to recall that each chapter contains the sections shown on the Figure 0.1.

The use of Problem-Based Learning methodology helps the student to learn the fundamental concepts and rules of differential calculus using an easy mathematical terminology. Some theorems may be explained by the use of technological resources that shows how mathematics is a tool for exploring our world.

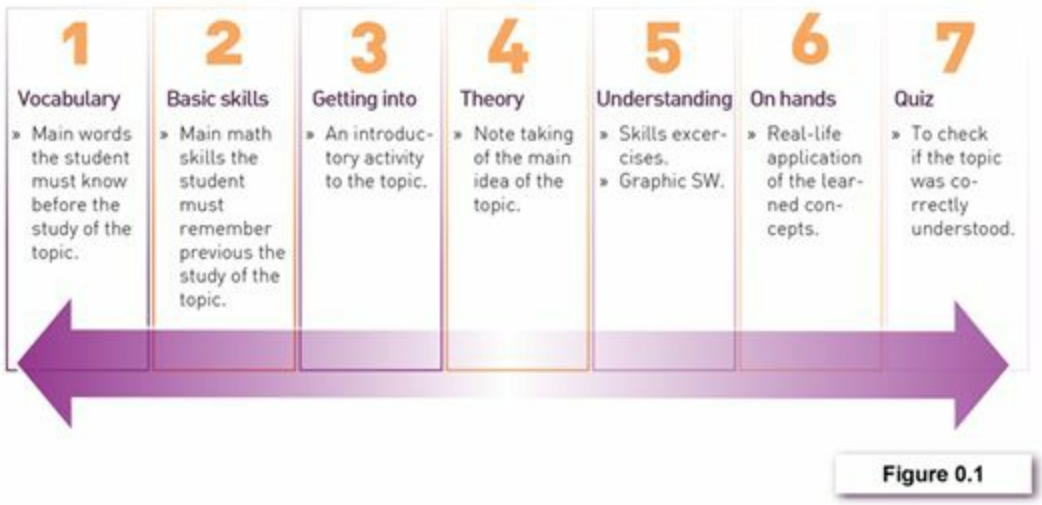
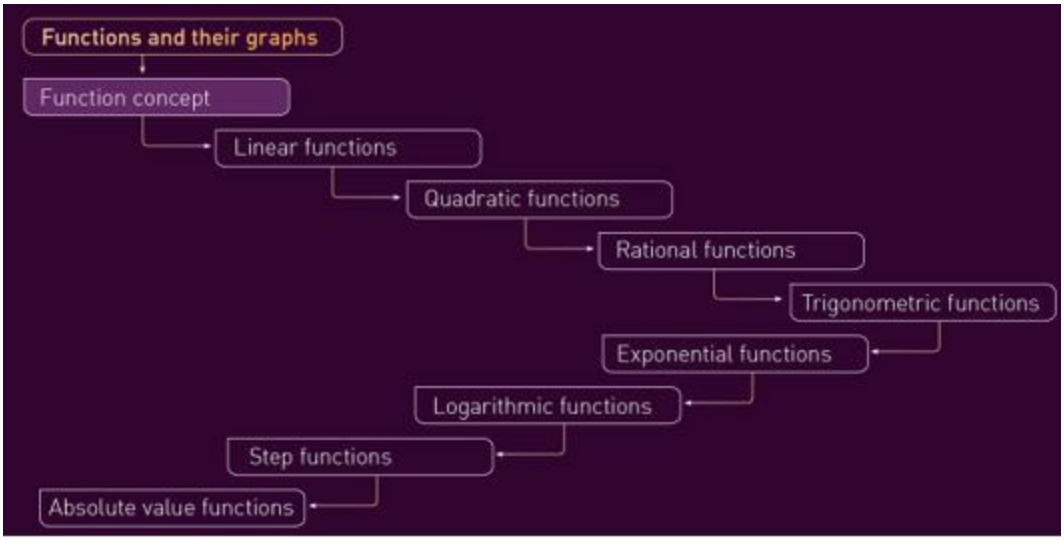


Figure 0.1

Chapter 1. Functions and their graphs



In order to explain a specific mathematical behavior, one quantity is described in terms of another. To represent this, we can use ordered pairs. These related quantities could be represented by a function.

In this chapter, we recall the proper use of the function's notation, domain and range, sketching, transformations and function combinations.

BASIC SKILLS

Practice some of your math skills through the next activity. You can evaluate your skills with the answer sheet.

Basic skills 1

Basic skills 1 - Answer sheet

GETTING INTO

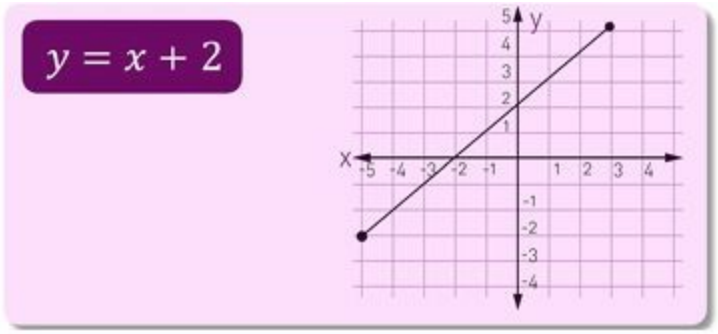
Introductory activity to the topics on this chapter:

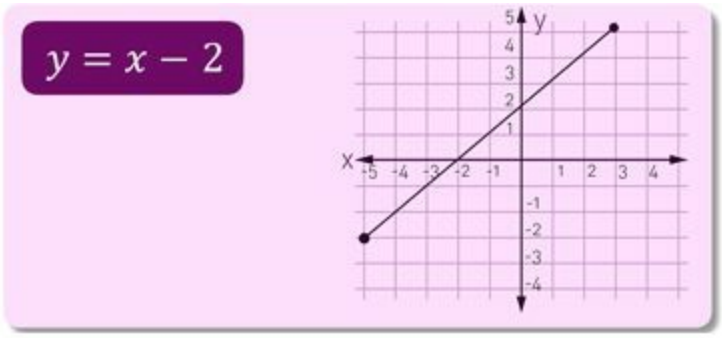
Getting into 1

1.1 Function concept

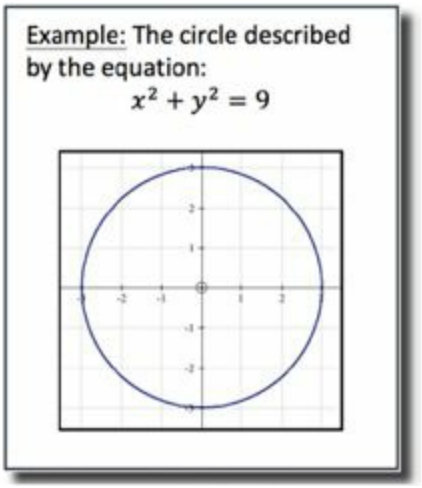
A relationship between two sets that matches each member of the first set with a unique member of the second set.

For example $y=x+2$ and $y=x^2$ are the equations of two relations. These equations can be modeled by sets of ordered pairs or graphs as follows:





But not all the relations can be described by an equation. Figure 1.1 shows an example.



A **function** is a relation in which each element of the domain is paired with exactly one element in range. There are two ways to determine whether a relation is a function, algebraic and geometric test.

BASIC SKILLS

Function Machine

A funny game for you to practice discovering the function that is used to get a result given a certain input.

The Wolfram Functions Site

This site will let you know a wide variety of functions from the elementary ones to the polynomials, and even more advanced. In this website you will also find a section to plot functions.

1.1.1 Algebraic Test

If a **relation** is given as an equation, verify that for each x-value, there is one and only one corresponding y-value. If this is true, the equation is a **function**, as illustrated in the following figures:

(1) $y = x + 2$

x	y
-3	-1
-2	0
-1	1
0	2
1	3
2	4

(2) Is a function, as for any value of x there is only one value of y.

Figure 1.1.1.1 Algebraic function

(2) $x = y^2$

x	y
9	-3
4	-2
1	-1
0	0
1	1
4	2

(1) Is not a function, since if $x = 4$, then $y = \pm 2$

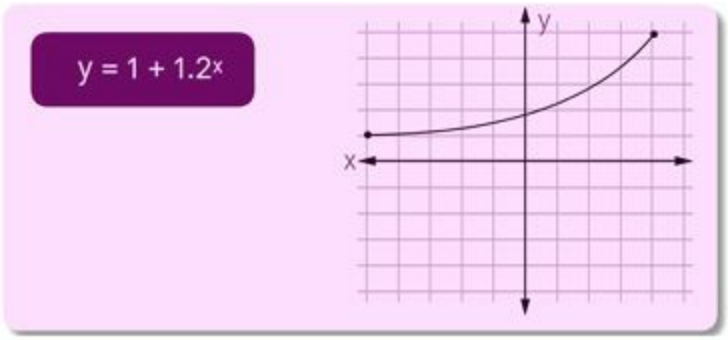
Figure 1.1.1.2 Algebraic function

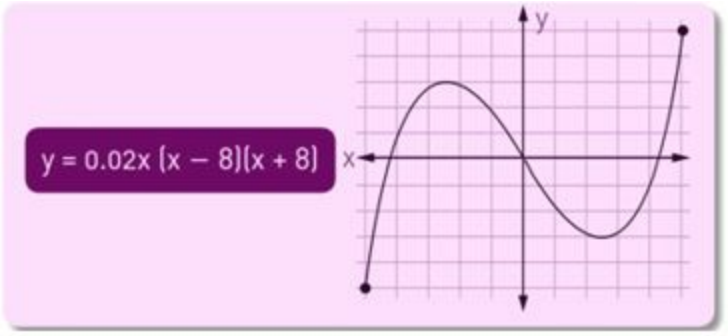
1.1.2 Geometric Test

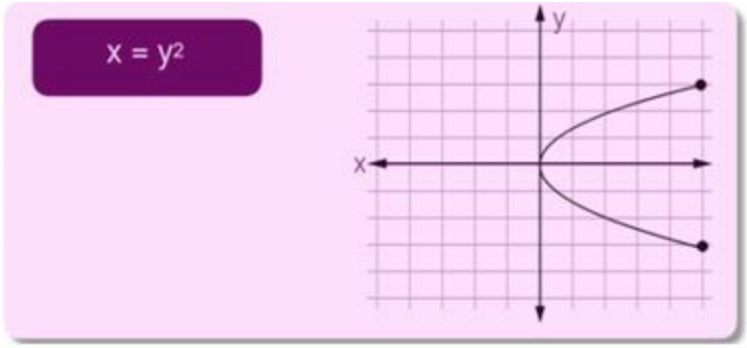
This is also known as “Vertical Line Test”. If a vertical line is drawn on the graph of the relation:

» Is a function if it intersects the graph no more than once.

» Is not a function if one line intersects the graph more than once.







Remember:

» The domain is the set of all x-coordinates of the ordered pairs of a relation.

» The range is the set of all y-coordinates of a relation.

Math functions

Short video with an example on determining if a relation represents a function.

Relations and Functions Problems

A set of 63 problems of relations and functions.

Domain and Range of a Function

Four examples on how to obtain the domain and range of functions.

Domain, Range and Codomain

An article that explains the concept of domain, range and codomain.

Math Functions, Relations, Domain & Range

An article that explains the concepts of function, relation, domain and range. It includes some exercises and interactive examples.

Domain and Range

This video shows an example of stating the domain and range of a relation.

Function Grapher and Calculator

Here you will find a Grapher and Calculator that will help you to plot functions.

What is a function

You will find what is a function.

1.2 Linear functions

An equation is called linear equation if it has no operations other than addition, subtraction, and multiplication of a **variable** by a constant. The variable must not be multiplied together or appear in a denominator. Also, a linear equation does not have variables with exponents different than one. As its name indicates, the graph of a linear equation is a line.

Examples:

Linear equation	No linear equation
$5x-2y=2$	$7a-3b^2=-6$
$x=2$	$y=\sqrt{5-x}$
$3s+3t=-15$	$xy-x=3$
$y=\frac{1}{2}x-1$	$y=\frac{1}{x+1}$

A linear function is a function whose ordered pairs satisfy a linear equation. This function can be written in the form: $f(x)=mx+b$, where m states for slope and b for y-intercept; and both are real numbers. Therefore, this form is called the slope-intercept form.

Any linear equation can be written in standard form, $Ax+By+C=0$, where A,B and C area integers with 1 as a greatest common factor.

The intercept form of a linear equation, $x/a+y/b=1$, states a as the **x-intercept** and b as the **y-intercept**.

LINKS

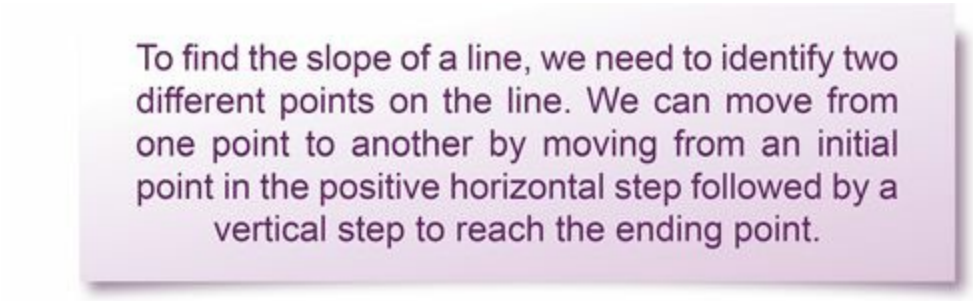
An interactive tutorial on Linear Functions

Linear function Machine

1.2.1 Slope

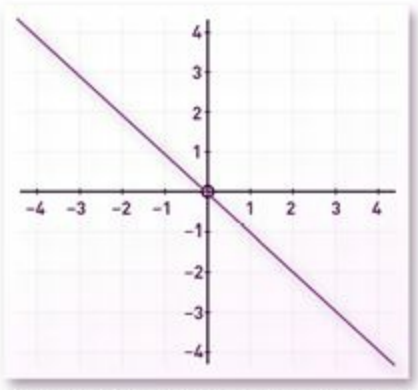
The slope of a line is a measure of its steepness. Every straight line that is not vertical cuts the y-axis at a single point. The y-coordinate of this point is called the y- intercept of the line.

These interceptions depend on the slope.

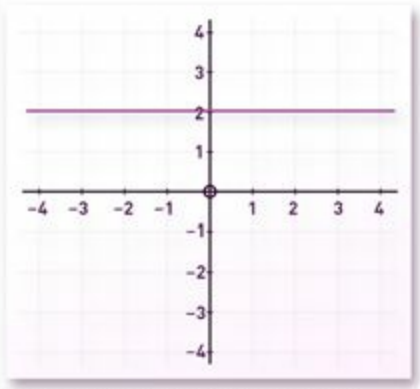


Given two points passing through (x_1, y_1) and (x_2, y_2), we can calculate the slope by:

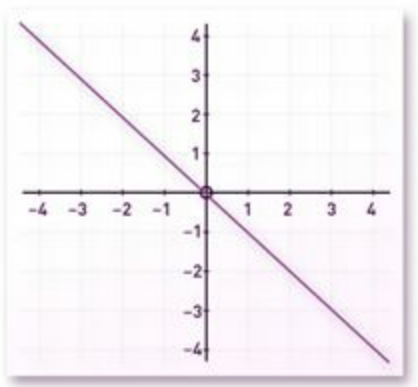
$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$



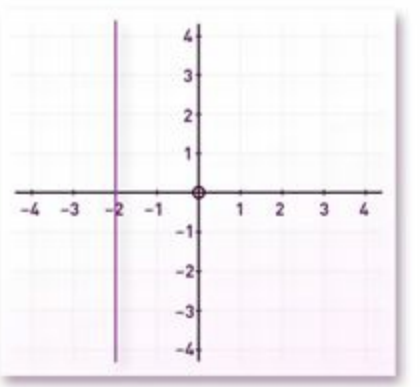
If the line rises to the right, then the slope is positive.



If the line is horizontal, then the slope is zero.

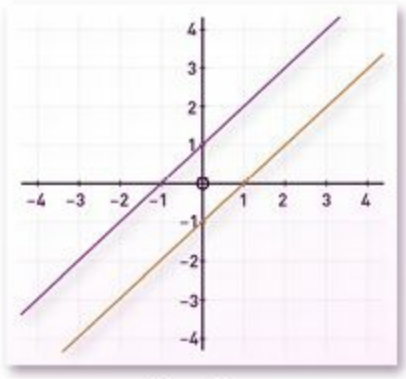


If the line falls to the right, then the slope is negative.



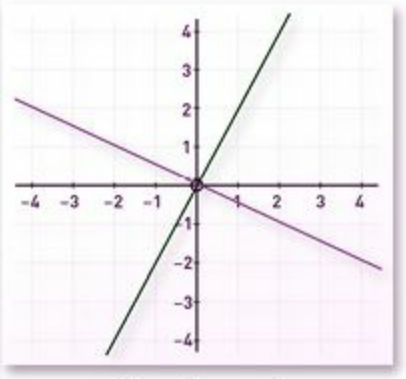
If the line is vertical, then the slope is undefined.

Additional to this, remember that:



$m_1 = m_2$

Lines with the same slope are **parallel.**



$m_1,\ m_2 = -1$

If the result of product of the slopes of two lines is -1, then the lines are **perpendicular.**

LINKS

Linear function

A tool for you to graph linear functions of the form $f(x)=ax+b$.

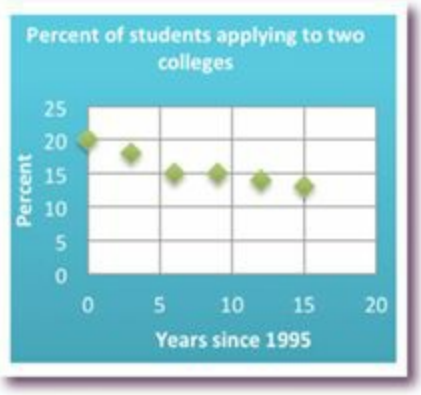
1.2.2 Real-World solved example

The table 1.2.2 shows the approximate percent of students who sent applications to two colleges in various years since 1995.

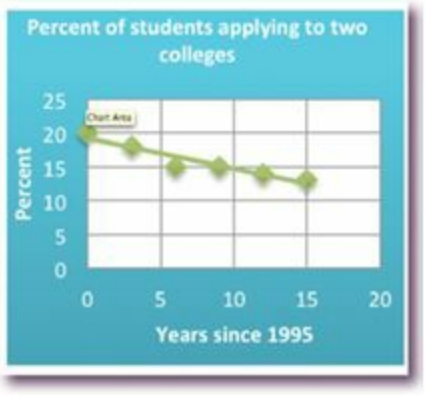
Years since 1995	Percent
0	20
3	18
6	15
9	15
12	13
15	14

Step 1. Make a **scatter plot** and a line of fit, and describe the **correlation**.

Scatter plot by graphing sw:



Line of fit:



The data show a strong negative correlation due the negative slope of the line of fit.

Step 2. Use two ordered pairs to write a predictive equation.

Using (3,18) and (15,13):

Slope:

$$m=\frac{13-18}{15-3}=-\frac{5}{12}$$

Using the **point-slope form**, $y-y_1=m(x-x_1)$; we get the following equation:

$$y=-\frac{5}{12}x+\frac{77}{4}$$

The slope indicates that the percent is falling by about 42% each year. The y-intercept indicates the percent in 1995 would have been about 19%.

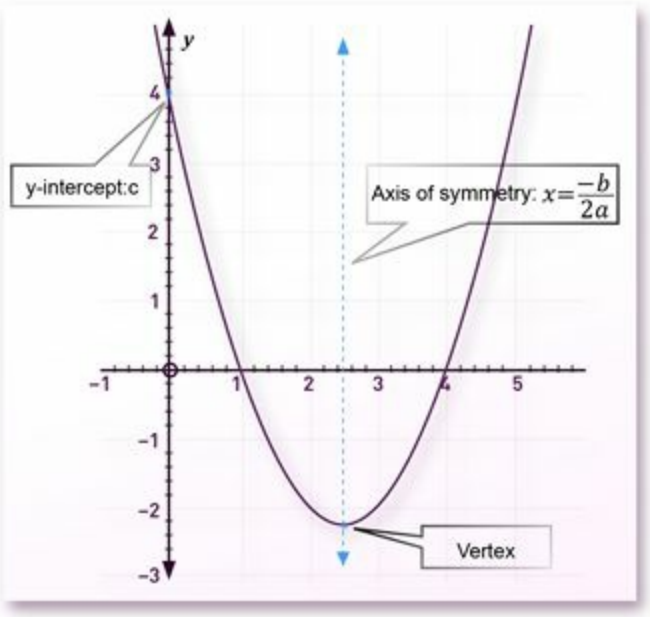
Step 3. Predict the percent of students who will send applications to two colleges in 2020.

Given x as the number of years since 1995 and y the percent of students, we find that:

$$y=-\frac{5}{12}(2020-1995)+\frac{77}{4}=88.3\%$$

1.3 Quadratic functions

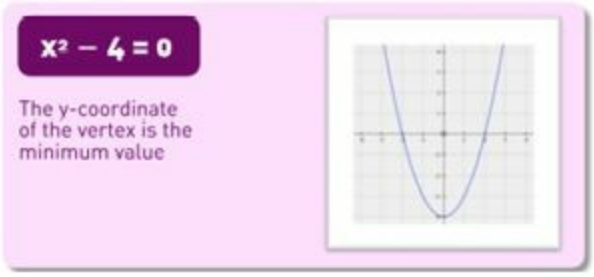
The graph of a **quadratic function** is called a **parabola**. All parabolas have an axis of symmetry, a vertex, and a y-intercept. Some parabolas open up, and others open down.

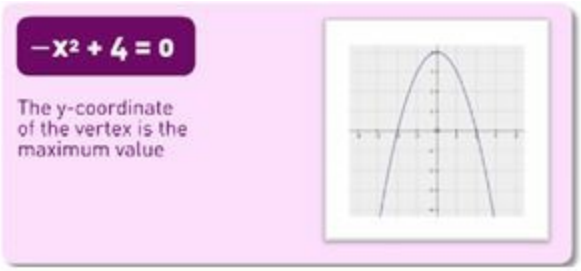


For vertical parabolas, the y-coordinate of the vertex is the **maximum value** or the **minimum value** of the function. These values represent the greatest or lowest possible value the function can reach.

The leading coefficient indicates if the function has a maximum or minimum value:

- If the leading coefficient is positive, then the parabola opens up. Therefore, it has a minimum value.
- If the leading coefficient is negative, then the parabola opens down. Therefore, it has a maximum value.





The domain of a quadratic function will always be all real numbers. The range will either be all real numbers less than or equal to the maximum value or all real numbers greater than or equal to the minimum value.

LINKS

An interesting intro video to Quadratic Functions.

Quadratic Function Intro Video

A tutorial on quadratic function.

Quadratic Function (General form)

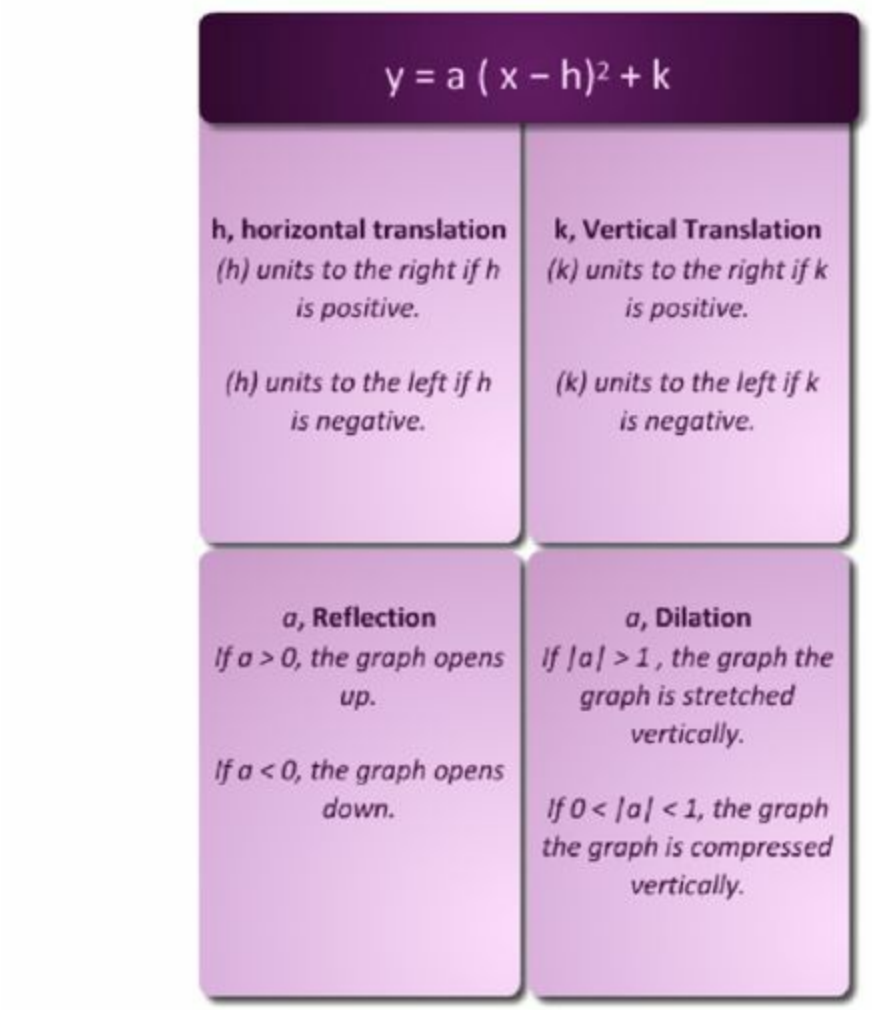
1.3.1 Transformations with quadratic functions

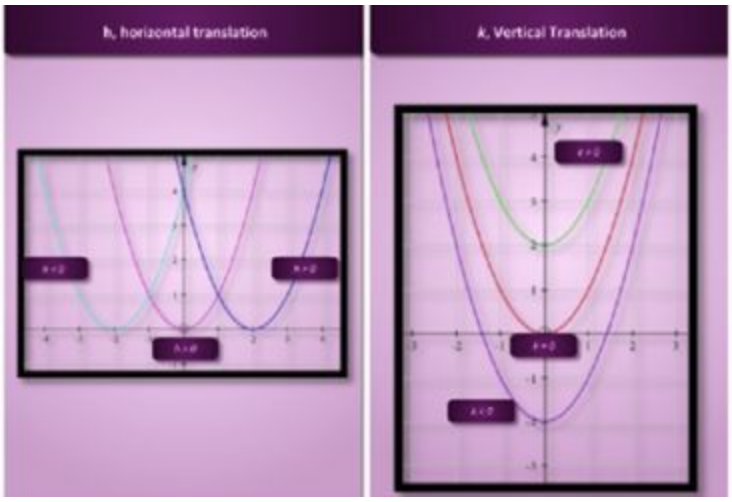
Quadratic equations can be written in *vertex* form:

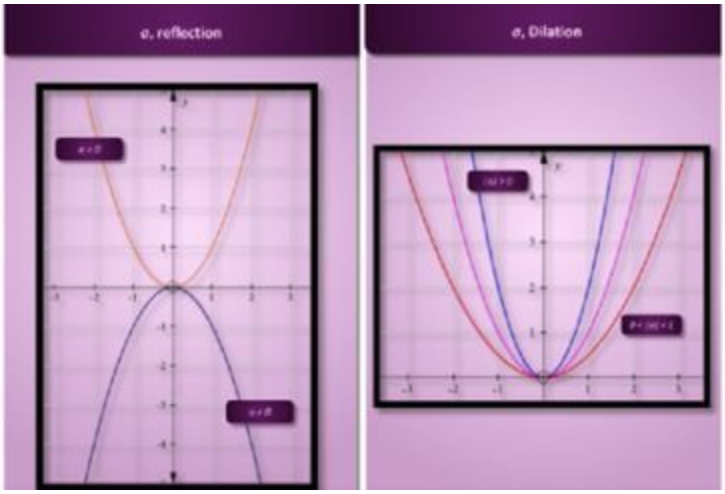
$$y = a(x-h)^2 + k$$

Where the vertex of the graph of the equation is at (h,k) and the axis of symmetry is the line $x=h$.

- The value of k determines the graph's vertical **translation.**
- The value of h determines the graph's horizontal translation.
- The value of a determines the direction of opening of the graph and the shape of the parabola.







LINKS

[The Product of two Linear Functions Gives a Quadratic Function]

An interactive tutorial on quadratic functions from linear functions

[Quadratic Function Explorer]

A tool that will help you explore the graph of a quadratic function of the form

$y = ax^2 + bx + c$

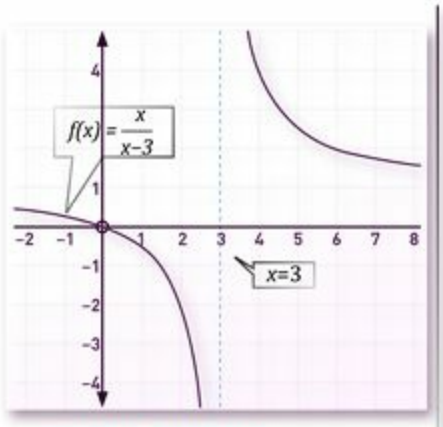
1.4 Rational functions

A **rational function** has an equation of the form

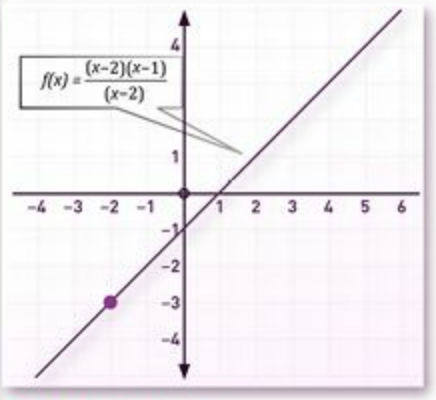
$$f(x) = \frac{a(x)}{b(x)}$$

where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$

Some graphs of rational functions have breaks in continuity. A break in continuity may appear as:



A vertical asymptote whenever $b(x) = 0$.



Point discontinuity, a hole in a graph. This type of discontinuity appears at $x=c$ whenever $x-c$ is a factor of both $a(x)$ and $b(x)$.

A rational function has at most one **horizontal asymptote:**

Case 1: If the degree of $a(x)$ is greater than the degree of $b(x)$, there is no horizontal asymptote.

Case 2: If the degree of $a(x)$ is less than the degree of b(x), the horizontal asymptote is the line $y=0$.

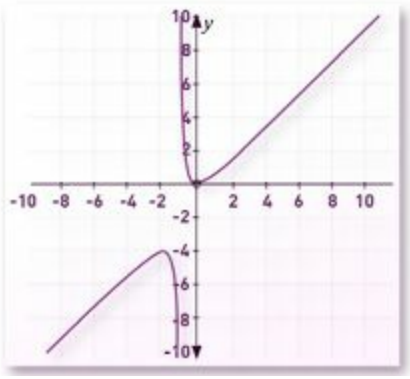
Case 3: If the degree of $a(x)$ equals the degree of $b(x)$, the horizontal asymptote is the line

$$y = \frac{leading\ coefficient\ of\ a(x)}{leading\ coefficient\ of\ b(x)}$$

Examples

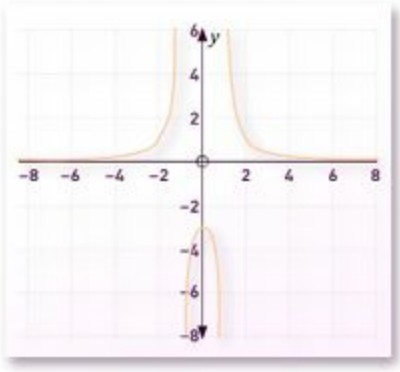
Case 1:

$$y = \frac{x^2}{x+1}$$



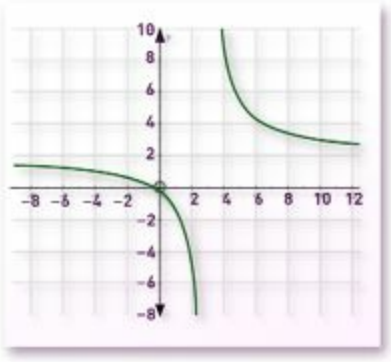
Case 2:

$$y = \frac{3}{x^2-1}$$



Case 3:

$$y = \frac{2x+1}{x-3}$$



Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of

$$f(x) = \frac{x^2 - 1}{x^2 - 6x + 5}$$

First, factor the numerator and denominator of the rational expression.

This shows us that the function is undefined for X = 1 and X = 5

The function is undefined for $x = 1$ and $x = 5$

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of

$$f(x) = \frac{x^2 - 1}{x^2 - 6x + 5}$$

$$= \frac{(x - 1)(x + 1)}{(x - 1)(x - 5)}$$

$$= \frac{(x + 1)}{(x - 5)}$$

Symplifying the expression yields...(x + 1) / (x – 5)

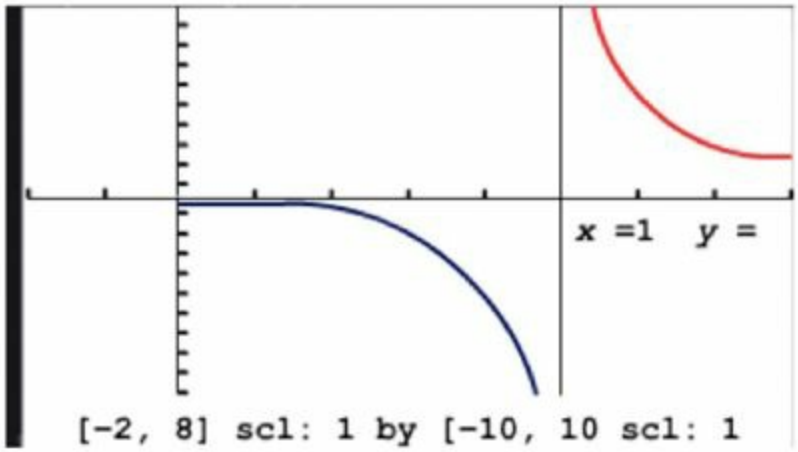
Since X = 1 is undefined in the original expression, but is defined in the simplified expression, X = 1 is a hole in the graph.

Since X = 5 is undefined in both the original and the simplified expression, X = 5 is a vertical asymptote of the graph. You can use a graphic calculator to check this finding.

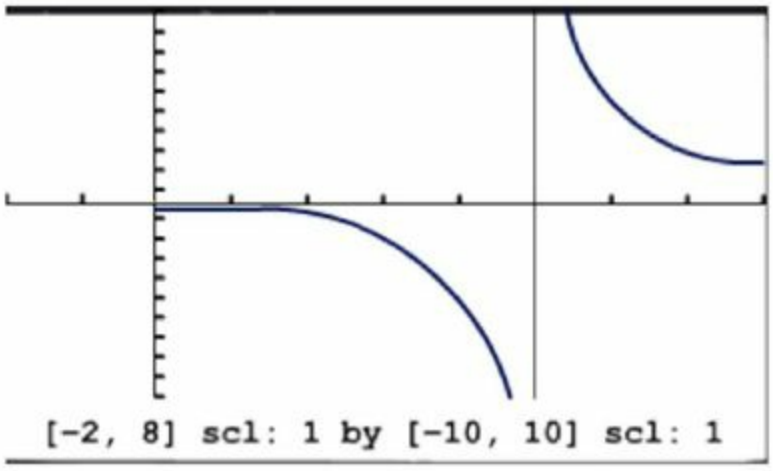
$x = 1$ is a whole in the graph.

$x = 5$ is a vertical asymptote

It is not clear from the graph that the function is not defined at 1. However if you use the value function of the CALC menu, and enter 1 at the x prompt, you will see that no value is returned for y= This shows that f(x) is not defined at x = 1



This is the graph of f(x).



The graph shows the vertical asymptote at x = 5.

LINKS

Rational Functions with Slant Asymptotes

An applet you can use to explore rational functions with slant asymptotes.

Rational Functions with two Vertical Asymptotes

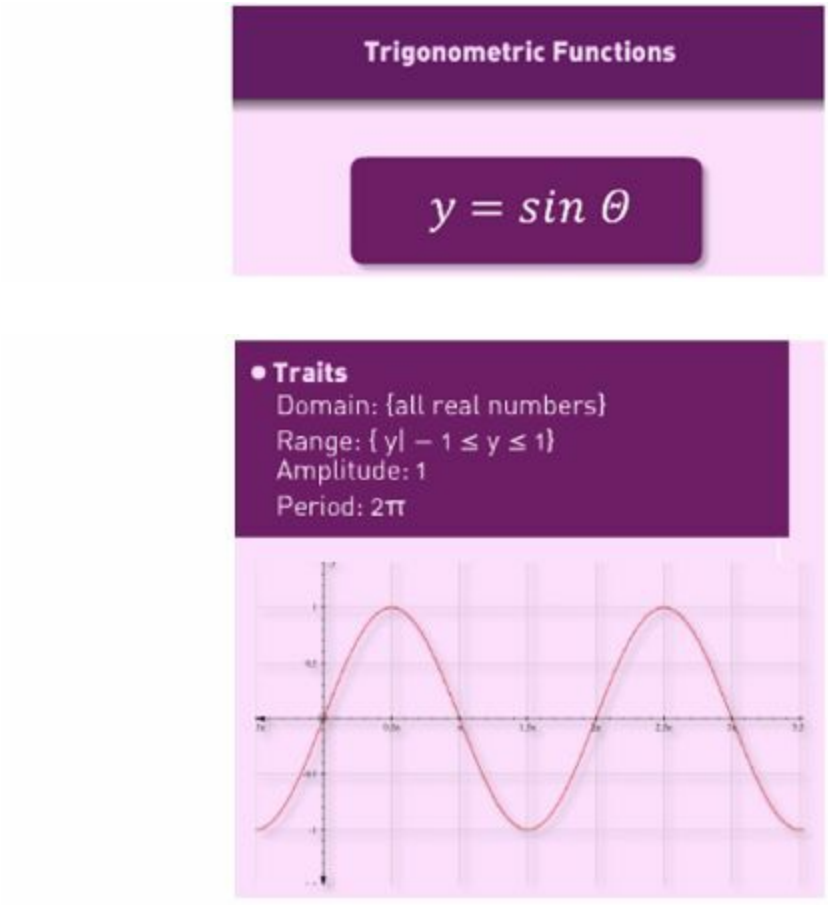
An applet you can use to explore rational functions with two vertical asymptotes.

Rational Functions - Dynamic Visualization

A tool that will let you explore the visualization of a rational function.

1.5 Trigonometric functions

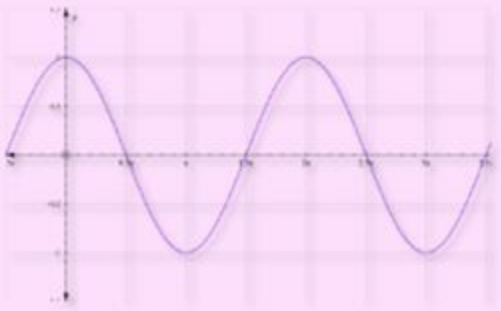
The graphs of **sine**, **cosine** and **tangent** functions have repeating patterns, called cycles. The horizontal length of each cycle is its **period**. For the sine and cosine functions, the period is 2π radians or 360°. For the tangent function, the period is π radians or 180°. For sine and cosine functions half the difference between the maximum and minimum value is called the **amplitude** of the graph.



Trigonometric Functions

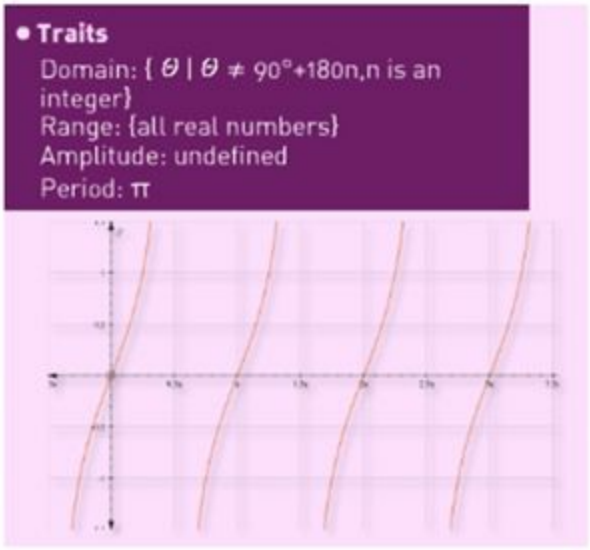
$$y = cos\ \Theta$$

- **Traits**
  Domain: {all real numbers}
  Range: $\{y \mid -1 \leq y \leq 1\}$
  Amplitude: 1
  Period: 2π



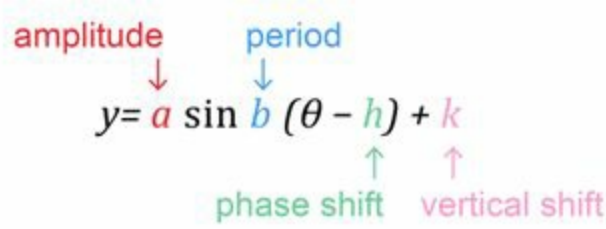
Trigonometric Functions

$$y = tan\ \Theta$$



1.5.1 Translations of trigonometric graphs

The graphs of the functions $y=\alpha \sin b\,(\theta-h)+k$, $y=\alpha \cos b\,(\theta-h)+k$, and $y=\alpha \tan b\,(\theta-h)+k$ are affected by changing the values of a,b,h, and k.



To graph a **trigonometric function**, follow the next steps:

- Determine the vertical shift, and graph the midline.
- Determine the amplitude, if it exists. Use dashed lines to indicate the maximum and minimum values of the function.
- Determine the period of the function, and graph the appropriate function.
- Determine the phase shift, and translate the graph accordingly.

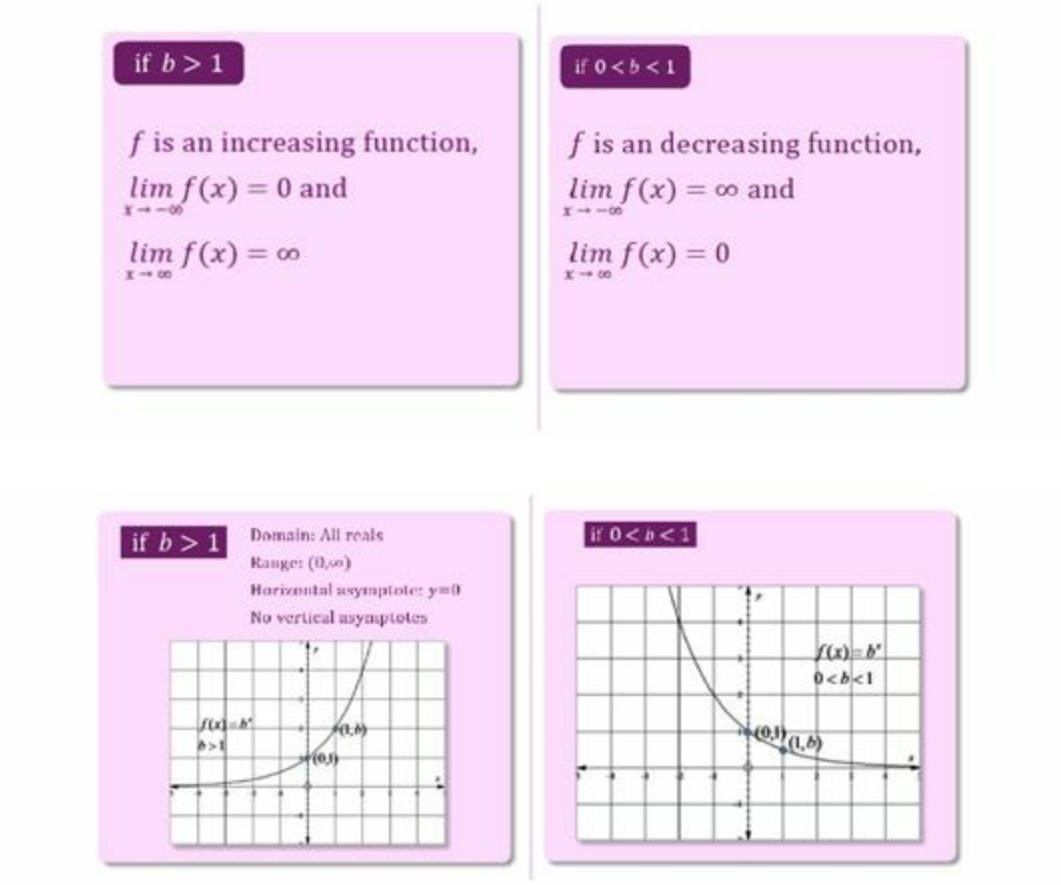
1.6 Exponential functions

An equation of the form $y = a \cdot b^x$, where $b > 0$, and $b \neq 1$ is called an exponential function. The domain of an **exponential function** is the set of all real numbers. There are two types of exponential functions:

Exponential growth, where $b > 1$, and

Exponential decay, where $0 < b < 1$.

The same transformations that have been performed on previously parent functions can be applied to the graphs of exponential functions.



LINKS

Exponential function

A tutorial on exponential functions

Exponential function

A tutorial on exponential functions with graphics.

A complete lesson from the MIT Open

A complete lesson from the MIT Open Course Ware that explains the concept of the exponential function.

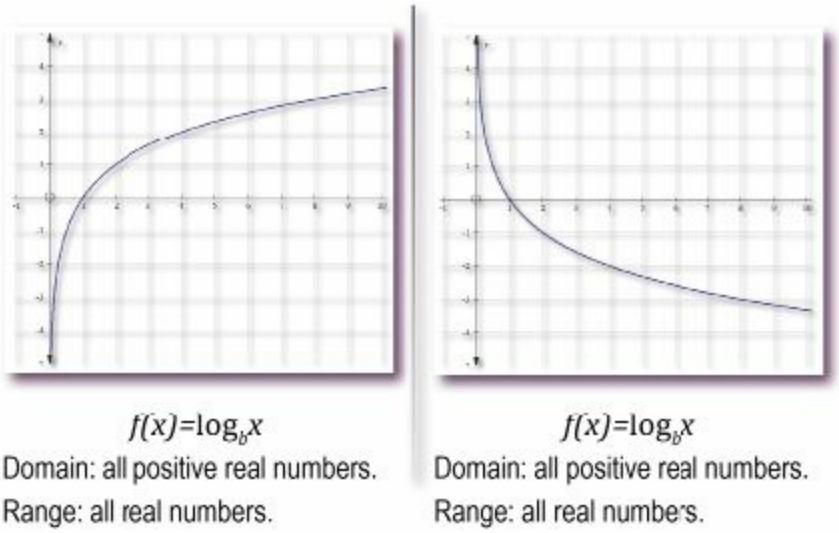
1.7 Logarithmic functions

In general, the inverse of $y = b^x$ is $x = b^y$, where b is the base and y is referred to as the **logarithm** of x. This relationship is usually written as $y = \log_b x$, where $b > 0$ and $b \neq 1$.

The graph of $f(x) = \log_b x$ is continuous, **one-to-one**, and has a domain of all positive real numbers. The y-axis is a vertical asymptote of the graph.

» Logarithms with a base of 10 are called *common logarithms.*

» Logarithms with a base of e are called **natural logarithms.**



$f(x)=\log_b x$
Domain: all positive real numbers.
Range: all real numbers.

$f(x)=\log_b x$
Domain: all positive real numbers.
Range: all real numbers.

LINKS

Logarithm function

An interactive tutorial on logarithmic functions.

Logarithm function

A tutorial on logarithmic functions, it includes graphing, many examples and problems for you to practice.

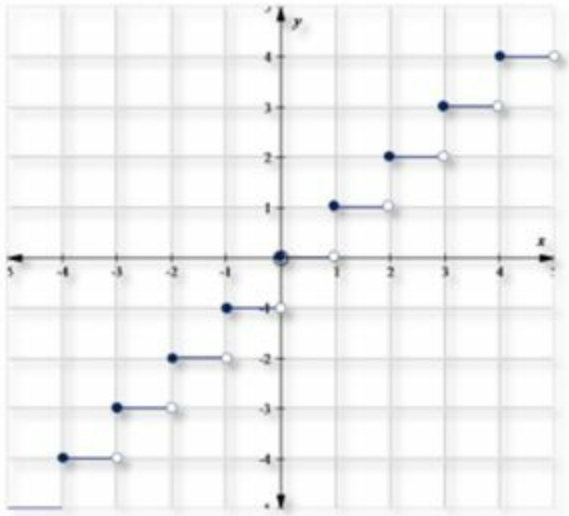
Domain and Range of Logarithmic Function

A set of examples on finding the domain and range of logarithmic functions.

1.8 Step functions

A **step function** consists of a series of line segments that look like steps. An open circle is used to indicate that the point is not included in that part of the graph.

Example: $f(x)=[x]$



LINKS

Step function

A tutorial on step functions and how to read and interpret them.

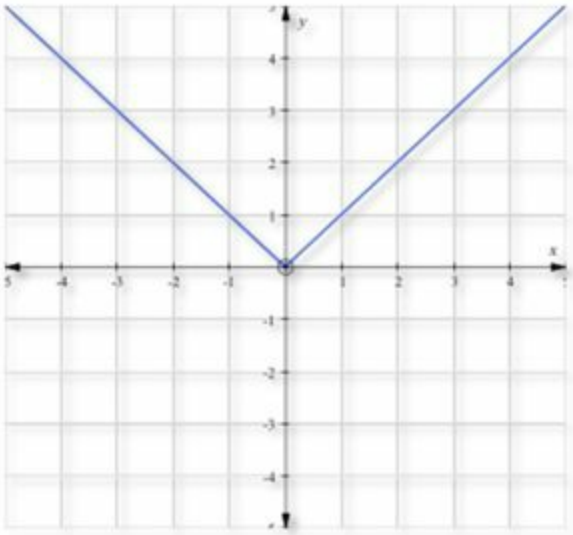
Step function

A set of videos on step functions

1.9 Absolute value functions

An **absolute value function** is shaped like a V and is made up of portions of two lines.

Example: $f(x)=|x|$



LINKS

Absolute value function

A tutorial on Absolute Value Functions, it includes an applet to show the graphic of a function of the form $f(x) = ax^2 + bx + c$

Graphing Absolute Value Functions

A tutorial on how to graph absolutevalue functions.

Review the activities at the end of this chapter

Chapter 1. Conclusion

There are different functions to represent to model situations in real life. Each function has a domain, range and ending behavior which help to predict its changes.

Chapter 1. Activities

Activities

- Activity 1
- Vocabulary 1
- Basic Skills 1
- Getting into 1
- Activity 1. Functions
- Activity 2. Domain and range
- Activity 4. Linear equations
- Activity 5. Writing and graphing linear equations
- Activity 6. Real World problems
- Activity 7. Graphing quadratic functions
- Activity 8. Graphs of quadratic functions
- Activity 9. Graph with no horizontal asymptote
- Activity 10. Oblique asymptote
- Activity 11. Readiness Skills
- Activity 12. Trigonometric functions
- Activity 13. Exponential functions
- Activity 14. Logarithmic functions
- Activity 15. Step and absolute value functions
- On hands 1

Answer Sheets (upon availability)

- Basic Skills 1
- Activity 1. Functions
- Activity 2. Domain and range
- Activity 4. Linear equations
- Activity 5. Writing and graphing linear equations
- Activity 6. Real World problems

- Activity 7. Graphing quadratic functions
- Activity 8. Graphs of quadratic functions
- Activity 9. Graph with no horizontal asymptote
- Activity 10. Oblique asymptote
- Activity 11. Readiness Skills
- Activity 12. Trigonometric functions
- Activity 13. Exponential functions
- Activity 14. Logarithmic functions
- Activity 15. Step and absolute value functions
- On hands 1

Chapter 1. Resources

» The Exponential Function

A complete lesson from the MIT Open Course Ware that explains the concept of the exponential function.

» Graphical Function Explorer

A function graphing tool to plot up to three functions on the same set of axes.

» Functions

A funny game for you to practice discovering the function that is used to get a result given a certain input.

» Finding the Domain and Range of a Function

An applet that will let you understand how to find domain and range of functions with graphics.

» Rational Functions with Slant Asymptotes

An applet you can use to explore rational functions with slant asymptotes.

» Rational Functions with two Vertical Asymptotes

An applet you can use to explore rational functions with two vertical asymptotes.

» Domain, Range and Codomain

An article that explains the concept of domain, range and codomain.

» Domain and Range

An article that explains the concepts of function, relation, domain and range. It includes some exercises and interactive examples.

» What is a Function

An article that explains what a function is, it includes many examples that will let you understand all the concepts and characteristics of functions.

» Linear Function Explorer

An easy-to-use applet that will let you explore the graph of a linear function of the form y=ax+b.

» Logarithmic Function

An interactive tutorial on logarithmic functions.

» Quadratic Function Intro Video

An interesting intro video to Quadratic Functions.

» Linear function Machine

An interactive tutorial on Linear Functions.

» The Product of two Linear Functions Gives a Quadratic

An interactive tutorial on quadratic functions from linear functions.

» A Real Life Application of Rational Functions

An opportunity for you to apply your knowledge on Rational Functions into a real life problem.

» Rational Functions in the Real World - Lens Sculptors

Another real life problem to be solved using rational functions.

» Applications of Exponential Functions

Applications and examples of exponential functions in real life.

» Domain and Range of Logarithmic Function

A set of examples on finding the domain and range of logarithmic functions.

» Step Function

A set of videos on step functions.

» Linear Functions Problems

A set of 25 problems on Linear Functions.

» Rational Functions Problems

A set of 35 problems on Rational Functions.

» Exponential Functions Problems

A set of 37 problems on Exponential Functions.

» Quadratic Functions Problems

A set of 39 problems on Quadratic Functions.

» Relations and Functions Problems

A set of 63 problems of relations and functions.

» Linear Function

A tool for you to graph linear functions of the form f(x)=ax +b.

» Quadratic Function Explorer

A tool that will help you explore the graph of a quadratic function of the form y = ax2 + bx + c

» Rational Fusnctions - Dynamic Visualization

A tool that will let you explore the visualization of a rational function.

» Absolute Value Functions

A tutorial on Absolute Value Functions, it includes an applet to show the graphic of a function of the form f(x) = ax2 + bx + c

» Exponential Function

A tutorial on exponential functions.

» Exponential Function

A tutorial on exponential functions with graphics.

» Graphing Absolute Value Functions

A tutorial on how to graph absolute value functions.

» Graphing Quadratic Functions

A tutorial on how to graph quadratic functions.

» Logarithmic Function

A tutorial on logarithmic functions, it includes graphing, many examples and problems for you to practice.

» Rational Function

A tutorial on Rational Functions. It includes some exercises.

» Step Function

A tutorial on step functions and how to read and interpret them.

» Quadratic Function (General Form)

A tutorial on quadratic function.

» Applications of Exponential Functions

Five real life problems to be solved using exponential functions.

» Domain and Range of a Function

Four examples on how to obtain the domain and range of functions.

» Calculator

Here you will find a Grapher and Calculator that will help you to plot functions.

» Parent Functions

Short presentation that shows the graphics of different kinds of Parent Functions.

» Math Functions

Short video with an example on determining if a relation represents a function.

» Functions vs. Relations

This article explains how to determine if a relation is a function.

» Domain & Range

This video shows an example of stating the domain and range of a relation.

» The Wolfram Functions Site

This site will let you know a wide variety of functions from the elementary ones to the polynomials, and even more advanced. In this website you will also find a section to plot functions.