INSTITUTO TECNOLÓGICO Y DE ESTUDIOS SUPERIORES DE MONTERREY

CAMPUS MONTERREY

GRADUATE PROGRAMS IN INFORMATION TECHNOLOGIES AND ELECTRONICS

DOCTOR OF PHILOSOPHY
INFORMATION TECHNOLOGIES AND COMMUNICATIONS
MAJOR IN TELECOMMUNICATIONS

# Mathematical Models of some Evolutionary Systems Under the Influence of Stochastic Factors

by

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December 2007

# Mathematical Models of Some Evolutionary Systems Under the Influence of Stochastic Factors

A Dissertation Presented by

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*Submitted in partial fulfillment of*
*the requirements for the degree of*

Doctor of Philosophy
in Information Technologies and Communications
Major: Telecommunications

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Campus Monterrey
December 2007

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# Declaration

I hereby declare that I composed this dissertation entirely myself and that it describes my own research.

Roberto David Rodriguez Said
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# Abstract

As it is known, the problem of availability of information is normally addressed using a buffer. Most of the times it is required that the effectiveness or reliability of the system is calculated to optimize the amount of stored information according to the customers random requests and to the amount of incoming information from the supply line. In this thesis, we consider the case of single buffer connected to any number of customers with bursty demands. We model the variation of the level of stored information in the buffer as an evolution in a random media. We assume that the customers can be modeled as semi-Markov stochastic processes and we use the *phase merging algorithm* to reduce the evolution process in a semi-Markov to an approximated evolutions in a Markov media. Then we obtain a general solution to the stationary probability density of the level of the buffer and general results for the stationary efficiency of the system.

# Acknowledgements

I would first like to express my gratitude to my advisors Dr. Rodríguez-Dagnino and Dr. Pogorui. I really appreciate all your support throughout this work. Also, I want to thank Dr. Campuzano, Dr. Mex and Dr. Rodríguez-Cruz for your valuable remarks and for making this possible.

# Dedication

This Thesis is dedicated to my wife Jessy and to mi kids Sofi and Said. Thank you for your patience and encouragement. Thank you for making this happen.

# Contents

# List of Figures

# List of Tables

# Chapter 1

# Introduction

## 1.1 Motivation

As it is known, the problem of availability of information is normally addressed using a buffer. Most of the times it is required that the effectiveness or reliability of the system is calculated to optimize the amount of stored information according to the customers random requests and to the amount of incoming information from the supply line.

Random evolutions are the mathematical model of the evolutionary systems under the influence of random factors. In a general form, these models are described based on stochastic operator integral equations in a separable Banach space [11, 10]. Random evolutions have proved to be a helpful mathematical formulation for different areas of knowledge such as biology, finance, information systems, etc. [22].

Many times, the random factors or processes that regard these schemes are considered to be Markov processes. Sometimes this consideration is considerably accurate. In some other cases it is not, and semi-Markov processes need to be modeled and solved.

Although, there exist techniques to addressed this kind of problems [11], quite often the calculations to solve exactly this problems turn cumbersome. In response to these situations some techniques to solve approximately semi-Markov schemes have been proposed. One of this techniques proposes the use of the *Phase Merging Algorithm* (PMA) to reduce evolutions in a semi-Markov media to evolutions in an approximated Markov media. We use this technique in this thesis.

As a matter of fact, the system presented here is a rather general scheme that may be used in different information schemes. This model and results may even be useful in some other engineering areas.

## 1.2 Problem Statement and Context

As it is known, the theory of the random evolutions was born after the application of some probabilistic methods to the solution of some partial differential equations such as the heat and the telegraph equations [5, 6] after a generalization of the work of Kac regarding the motion on the real line. Then, the term was introduced by Reuben Hersh and Richard Griego being suggested by Peter Lax. The theory was mostly developed by authors such as Papanicolau, Hersh, Pinsky, Kertz, Watkins and others no less important. There is a broad series of papers regarding several aspects of the evolutions such as limit theorems [8, 16] and diffusion processes and random motions [4, 3]. The semi-Markov case has been considered by authors such as Swishchuk, Turbin, and Korolyuk. Of course this list is far from being complete.

In this thesis we use the evolutionary formulation to study the stationary efficiency of a system consisting of a finite capacity buffer connected to different customers with bursty on-off demands.

The system functionality is as follows:

We assume that the buffer is filled up at a constant rate $F$.

The customers switch from the active or "ON" state to the inactive or "OFF" state, and we consider that the switching process of the customers can be modeled as a semi-Markov process.

When active, one customer demands information at a certain rate whereas, when $n$ customers are active, the sum of all demands is the rate at which information is required. If the buffer is empty ($v = 0$), an unproductive situation is considered. When no customer is active, then no product is required. The filling aggregate provides the buffer with product at a constant rate $F$. This aggregate is active as long as the volume of information is below the maximum capacity of the buffer ($V$), see Figure 1.1.

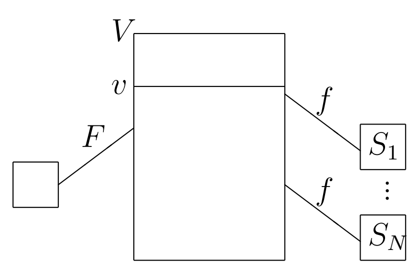


Figure 1.1: A system of $N$ independent customers and one buffer filled up at a constant rate $F$.

## 1.3 Research Questions

The dynamics of the overall system are modeled using a semi-Markov evolution environment and we derive design formulae involving the main parameters.

We show that it is possible to use the *Phase Merging Algorithm* to reduce the semi-Markov process to an approximated Markov process. Then, we use that result and we find the stationary distribution of the system for two different semi-Markov cases, namely, $m$-Erlang and hyper-exponential cases The Markov evolution environment is also included as a special case.

We study some limits of the stationary efficiency as the buffer capacity tends to infinity and we identify three different stationary behavior cases for the system.

## 1.4 Solution Overview

The dynamics of this linear system can be captured by a first order differential equation having a random component, or the so-called random evolution process [10]. We elaborate our semi-Markov mathematical modeling and after that deal with the special Markov case. We obtain the stationary compound probability distribution for the buffer content level and the mathematical expression of the efficiency parameter in terms of the system values. Then, with the help of some plots we analyze some numerical results for different Markov and semi-Markov cases. In particular, we include the $m$-Erlang, exponential, and hyper-exponential distributions for the active periods.

Let us denote $I(T)$ the amount of information delivered to customers $S_1$ and $S_2$, in a time interval $[0, T]$. Thus, we can define $K = \lim_{T\to\infty} \frac{I(T)}{T}$ as the steady state parameter for the system effectiveness, see Chapter 4 in [20]. Then, determine $K$ as a function of the system parameters, $\lambda_0$, $\lambda_1$, $\mu_0$, $\mu_1$, $f_0$, $f_1$ and $F$.

## 1.5 Main Contributions

The main contributions of this thesis are the following:

- Exact stationary probability density for the level of the buffer for the Markov case with two different customers and the exact expression of the stationary efficiency. Also, the approximated stationary probability density for the cases $m$-Erlang and hyper-exponential in addition to the stationary efficiency for these cases [18].

- Exact general solution of the stationary probability density of the system with any number $N$ of superposed Markov processes (customers) [19] in addition to generalized results for the stationary efficiency.

## 1.6 Thesis Organization

In Chapter 2, we introduce some basic notions of measure theory and probability for the reader to understand the framework of this research. Namely, we need to understand that we can see a random variable as a measurable function. It is necessary to show the way measurable functions are integrated using the Lebesgue integral. In addition, in Chapter 2 random evolutions are introduced with some simple examples. We are going to introduce the properties of the infinitesimal operator that is used in the random evolution formulation.

In Chapter 3, we address the problem of the superposition of two different processes (customers). This case may be useful for some applications. In Chapter 4, we consider the case where we have a superposition of two processes with the same parameters. It will be seen that the results of Chapter 3 regarding the stationary probability density cannot be reduced to the case of two processes with the same parameters. Also, it will be shown that this chapter is an important step toward the generalized results of the superposition of any number $N$ of processes with the same parameters. Such results are shown in Chapter 5. In Chapter 6 we summarize the most important results.

# Chapter 2

# Background and State of the Art

## 2.1 Chapter Summary

It is necessary to introduce some basic notions of set theory and measure theory to understand an important part of the framework of this research. Namely, we need to understand that we can see a random variable as a measurable function. Also, we need to understand the way these functions are integrated using the Lebesgue integral. We will try to explain these concepts in the first part of this chapter.

After that, we are going to define some random processes that will help us introduce the random evolutions. We will present the infinitesimal operator and we will talk about its properties. Then, we will show a simple example where random evolutions can be formulated. Namely, the Brownian motion.

## 2.2 Set theory

A set is a collection of elements from a defined space $U$. This is a basic concept that lines up with the notions learned on early school with the Venn Diagrams although the modern approach to the set theory is axiomatic. This means that sets or set memberships are defined by axioms that describe their properties.

We can mention some examples of sets: the set of positive integer numbers $\mathbf{N} = 1, 2, 3, ...$, the set of real numbers, i.e., $\mathbf{R} = (-\infty, \infty)$.

It is a common practice to use $U$ to denote a universal set as well as $\emptyset$ to denote an empty set. The universal set is the set that holds all elements of a certain space. Sets are commonly denoted with capital letters, such as $U$, $A$ or $B$. When one says that set $A$ and $B$ are equal, it means that these sets contain exactly the same members.

There are several ways to describe a set. One way is to give a description such as: $S$ is the set of all students in Monterrey. Another way to describe a set is by explicit enumeration, normally using braces. For example, $F = \{green, white, red\}$. Also some abbreviation can be used to describe sets such as in the case of the set of integer numbers $\mathbf{N} = \{1, 2, 3, ...\}$ or in the case of the set of real numbers $\mathbf{R} = (-\infty, \infty)$. The *set builder* notation can also be used such as in the case $\{k : n \in \mathbf{N} \wedge k = 2n\}$ which stands as the set of all even natural numbers. In this notation, the colon stands as "such that". Also, it is common to denote by $2^U$ all subsets of the universal set $U$ including $\emptyset$ and $U$.

In a set there cannot be two or more identical elements. Also, the order in which the elements of a set are listed is irrelevant.

To denote that an element is or it is not a member of a particular set, the symbols $\in$, $\notin$ are used respectively, for example, "*red*" $\in F$. There are several other symbols that describe relationships between sets or describe some operations. See the following table.

| | |
|---|---|
| $A \subseteq B$, | A is a subset of B or it is equal to B. |
| $A \subset B$, | A is a subset of B but it is not equal to B. |
| $A \cap B$, | The intersection of sets A and B. |
| $A \cup B$, | The union of sets A and B |
| $A \backslash B$, | The (relative) complement of B in A |
| $A \bigtriangleup B$, | The symmetrical difference of sets or disjunctive sum |

The relative complement of B in A refers to the set of all elements that belong to set A, but do not belong to set B. This complement is relative since it does not represent the complement with respect to the universal set $U$. Another way to represent the complement of B in A is by writing $A - B$.

The symmetrical difference of sets or disjunctive sum can be described by the expression $A \bigtriangleup B = (A \backslash B) \cup (B \backslash A)$. In words, the set of all elements that belong to A, but not to B and all elements that belong to B, but not to A.

Another important concept is the *monotonicity*. We say that a sequence of elements $\{A_1, A_2, \ldots A_n, \ldots\} \subset K$ is a monotonic set class if $A_1 \subset A_2 \subset \cdots A_n \subset \cdots$, i.e., $\lim_{n \to \infty} A_n \equiv \bigcup_{n=1}^{\infty} A_n \in K$. We also say that the set class $\{A_n : n \geq 1\} \subset K$, is a monotonic sequence of sets if $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$, i.e., $\lim_{n \to \infty} A_n \equiv \bigcap_{n=1}^{\infty} A_n \in K$

### 2.2.1 Rings, and algebras

A ring is a set with an algebraic structure in which addition and multiplication between its members are defined. Also, these operations fulfill the following axioms.

Let $R$ be a ring then, for all $a, b, c \in R$. We have two operations $\oplus$ and $\otimes$ such that

1. $a \oplus b = b \oplus a$,
2. $(a \oplus b) \oplus c = a \oplus b \oplus c$,

3. $\exists \theta \in R$, an identity element, such that $a \oplus \theta = a$,
4. $\exists -a \in R$, a symmetric element such that $a \oplus -a = \theta$,
5. $a \otimes (b \otimes c) = (a \otimes b) \otimes c$,
6. $a \otimes (b \oplus c) = (a \otimes (b \oplus a) \otimes) c$.

Assume a subset class $K \subset U$ is an algebraic ring, in the case that it holds the commutative property $a \otimes b = b \otimes a$ and if $a \oplus a = 0$, $\forall a \in R$ this ring is called a *Boolean ring.*

Assume $K$ is an algebraic ring with the addition defined as the symmetric difference and the multiplication as the intersection. This is, for any $a, b \in K$, $a \oplus b = (A \backslash B) \cup (B \backslash A)$, also $a \otimes b = a \cap b$. It can easily be proved that $K$ is also a Boolean ring.

A boolean ring can also be defined in the following way. Let $R \subset 2^U$ then $\forall A, B \in R$, $a \cup b \in K$ and $A \backslash B \in R$. It is easy to prove that the ring $K$ holds this properties. It is said that a Boolean ring is closed under finite union and relative complement.

A ring is called an algebra if $U \in K$. This is, if the universal set is an element of the ring.

In a Boolean Algebra there is a correspondence between the set operations and some basic logical operations, i.e., AND, OR and NOT. Furthermore, it is important to remember the properties given on sets by the DeMorgan theorems in Boolean Algebras. Basically, these theorems state that $\overline{A \cap B} = \overline{A} \cup \overline{B}$, also, $\overline{A \cup B} = \overline{A} \cap \overline{B}$, given $\overline{A} = U \backslash A$.

It is said that if a non empty set class $S \subset 2^U$ has all the properties of a ring, except that it has no symmetric (or additive inverse) elements for each of its member elements, then $S$ is called a semi-ring.

Formally speaking, if $\forall A, B \in S$, $A \cap B \in S$, and $\exists A_1, A_2, ...A_3 \in S$ such that $A \backslash B = \bigcup_{k=1}^{n} A_k$, then $S$ is a semi-ring.

A ring $R$ is called a $\sigma$-ring if, given $A_n \in R\ \forall n \in N$, $\bigcup_{n=1}^{\infty} A_n \in R$. In this case, it is said that the ring is closed under countable union.

In the case the universal set $U$ is an element of the $\sigma$-ring, this is called a $\sigma$-algebra.

Consider the set $k(H) = \bigcap_{R \text{ is ring and } R \supset H} R$. It is said that $k(H)$ is a ring generated by the set class $H$. In fact, the ring $k(H)$ is the intersection of all rings containing $H$. Then, $k(H)$ is the minimum ring containing $H$ [2].

We can also define [7]:

- $\sigma k(H)$, the minimal $\sigma$-ring containing $H$.

- $m(H)$, the minimal monotonic class containing $H$.
- $a(H)$, the minimal algebra containing $H$
- $\sigma a(H)$, the minimal $\sigma$-algebra containing $H$.

### 2.2.2 Functions over sets

Assume $H \subset 2^U$ is a class of sets. Also, assume $\lambda$ is a real-valued function on $H$.

If $\forall A \in H$, $\lambda(A) \geq 0$, then it is said that $\lambda$ is a non-negative function.

In the case that $\forall A_i \in H$, $i = 1, \ldots, n$ and $\bigcup_{i=1}^{n} A_i \in H$ where $A_i \cap A_j = \emptyset$ for $i \neq j$, if $\lambda\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \lambda(A_i)$ then, it is said that $\lambda$ is a finitely additive function. If, instead, we have $\lambda\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \lambda(A_i)$, it is said that $\lambda$ is a semi-additive function.

Also, one can recall the infinite case, where we have $\forall A_i \in H, i = \{1, 2...\}$ and $\bigcup_{i=1}^{\infty} A_i \in H$, where $A_i \cap A_j = \emptyset$ for $i \neq j$. In this case that, $\lambda\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda(A_i)$, it is said that $\lambda$ is a infinite additive function, i.e., a $\sigma$-additive function. Instead, if we have that $\lambda\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \lambda(A_i)$, it is said that $\lambda$ is infinite semi-additive function, i.e., a $\sigma$-semi-additive function.

We say that $\lambda$ is a monotone function if for $A, B \in H$ and given $A \subset B$, we have $\lambda(A) \leq \lambda(B)$.

Also, we say that $\lambda$ is a finite function if $\forall A \in H$ we have $\lambda(A) < +\infty$.

Furthermore, in the case that $\exists\{A_i : i \geq 1\} \in H$ such that $\bigcup_{i=1}^{\infty} A_i = U$ we say that $\lambda$ is a $\sigma$-finite function if $\lambda(A_i) < \infty$, $\forall i \geq 1$.

## 2.3 Measure

It is said that a measure $\mu$ is a real-valued function defined on a $\sigma$-algebra $S \subset 2^U$ if it satisfies the following conditions.

1. $\mu$ is nonnegative.

2. $\mu(\emptyset) = 0$.
3. $\mu$ is $\sigma$-additive.

If $\mu$ is a measure it has the following properties:

1. $\mu$ is a monotone function. Furthermore, if $\{A, B_1, B_2, B_3, \ldots, B_n\} \in S$ given $A \subset \bigcup_{i=1}^{n}(B_i)$ then $\mu(A) \leq \sum_{i=1}^{n} \mu(B_i)$.
2. For a sequence $\{A_i : i \geq 1\} \subset S$ such that $\bigcup_{i=1}^{\infty} A_i \in S$ we have that
$$\mu\left(\bigcup_{n=1}^{\infty} A_i\right) \leq \sum_{n=1}^{\infty} \mu(A_i),$$
if sets $A_i$ and $A_j$ may not be disjoint for any $i \neq j$, and $i, j \geq 1$. In words $\mu$ is semi-additive for a sequence of sets which may not be disjoint.
3. For any $A \in R$, $B \in R$ such that $\mu(A) < +\infty$ and $\mu(B) < +\infty$, the equality $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ holds true.
4. $\mu$ is a continuous function from upper and from down. This is,
$$\mu\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n),$$
for any given monotonic sequence of sets $\{A_n | n \geq 1\}$.

A set $U$ with a $\sigma$-algebra $S$ of subsets of $U$ is called a measurable space and is denoted $(U,\ S)$. The triple $(U, S, \mu)$ is called a space with measure or a measure space. If $\mu(U) = 1$, $\mu$ is called a probability measure.

Consider the space with measure $(U, S, \mu)$. We say that the measure is discrete if it is concentrated on no more than a enumerable infinite set of points. Any singleton (one point subset) $x \in S$ which has measure $\mu(x) > 0$ is called an atom of $\mu$.

On the other hand, we say that a measure is continuous if $\mu$ is defined but equals zero on any singleton of $U$.

**Theorem 2.1 (Lebesgue)** *Any $\sigma$-additive measure $\mu$ can be uniquely represented as*
$$\mu(x) = \mu_d(x) + \mu_c(x),\ \forall x \in S,$$
*where $\mu_d$ is a discrete measure and $\mu_c$ is a continuous measure.*

See [7], [23].

### 2.3.1 Completeness and extension of a measure

Let us introduce the concept of completeness.

Consider the space $(U, S, \mu)$. A set $N \subset U$ is called a $\mu$-null set if $\exists A \in S$ such that $\mu(A) = 0$ given $A \supset N$. $(U, S, \mu)$ is said to be a *complete* space if every $\mu$-null set $N \in S$. The measure $\mu$ is said to be also complete.

**Theorem 2.2** *Consider the space $(U, S, \mu)$, any null set $N$ in the given space and the set class $\overline{S} = \{A \cup N : A \in S\}$. Then, the measure $\overline{\mu}$ over the set class $\overline{S}$ is defined as*

$$\overline{\mu}(A \cup N) \ := \mu(A), \ \forall A \in S.$$

*Then, $\overline{S}$ is a $\sigma$-algebra.*

See [7], [23].

It is said that a space can be extended to a complete one if the symmetric difference of both measurable sets is a null set.

It was said that a measure is a function defined on a $\sigma$-algebra, and this is the common notion. In fact, a measure can also be defined on a semi-ring and there exists a couple of important theorems that state that the measure over a semi-ring may be uniquely extended to a measure over a $\sigma$-ring. Given this, the only difference of a measure over a $\sigma$-ring and a measure over a $\sigma$-algebra is that in the latter case the measure over the universal set $U$ is defined.

**Theorem 2.3** *Every $\sigma$-finite measure on a semi-ring $H$ can be uniquely extended to a measure over $\sigma$-finite measure on $R(H)$, i.e., over the minimum ring $R$ containing the semi-ring $H$.*

**Theorem 2.4 (Caratheodory)** *Every $\sigma$-finite measure over a ring $R$ can be uniquely extended to a $\sigma$-finite measure over $\sigma K(R)$, i.e., over the minimum $\sigma$-ring $\sigma K$ that contains the ring $R$. Also, every $\sigma$-finite measure over an algebra $A$ can be uniquely extended to a $\sigma$-finite measure over $\sigma T(A)$, i.e., over the minimum $\sigma$-algebra $\sigma T$ containing the algebra $A$.*

Proofs are omitted. See Chapter 1 in [23].

### 2.3.2 Lebesgue Measure

Consider de following semi-ring. $P = \{(a, b] | a < b \in \ \mathbf{R}\} \cup \{\emptyset\}$. Then the following theorem is fulfilled.

**Theorem 2.5** *If $\mu$ is a measure on $P$ then there exists a nondecreasing and continuous from the right function $F$ on $\mathbf{R}$ such that*

$$\mu((a, b]) = F(b) - F(a). \tag{2.1}$$

*Viceversa, if there exists a nondecreasing and continuous from the right function $F$ on $\mathbf{R}$ then there exists a measure $\mu$ such that 2.1 is fulfilled.*

Proof is omitted. See [23].

Now, consider the function $F(x)$. This function defines the measure $\mu((a, b]) = b - a$ on the semi-ring $P$. By using Theorem 2.3, this measure can be extended to a measure on $R(P)$, i.e., over the minimum ring containing the semi-ring $P$. Then, using the Theorem 2.4 the measure $\mu$ can be extended to a measure on $\sigma K(P)$, indeed, over the minimum $\sigma$-ring containing $P$. Using Theorem 2.2 we can obtain the complete space $(U, \overline{\sigma a(P)}, \overline{\mu})$.

The measure $\overline{\mu}$ is called the *Lebesgue measure* on $\mathbf{R}$ and $\overline{a\sigma(P)}$, the minimum $\sigma$-algebra containing $P$, is called the algebra of Lebesgue sets on $\mathbf{R}$. Then, the Lebesgue measure is a complete measure [7].

The $\sigma$-algebra $\sigma a(P)$, which is the set class before completion, is called the *algebra of Borel* sets on $\mathbf{R}$. The measure over this $\sigma$-algebra is called the *Borel measure* and, of course, it is not complete.

### 2.3.3 Measure on $\mathbf{R}^m$

The Lebesgue measure can also be defined on the space $\mathbf{R}^m$. Consider the semi-ring $P_m = \left\{ \prod_{i=1}^{m} (a_i, b_i] \middle| \, a_i < b_i, \;\; a_i, b_i \in \mathbf{R} \right\} \cup \{\emptyset\}$. Take the measure over $P_m$ as

$$\mu_m \left( \prod_{i=1}^{m} (a_i, b_i] \right) = \prod_{i=1}^{m} (b_i - a_i), \qquad \mu_m(\emptyset) = 0.$$

For $m = 2$ we have $\mu_2 \left( \prod_{i=1}^{2} (a_i, b_i] \right) = (a_1, b_1] \times (a_2, b_2]$ which is the area of a rectangle.

For $m = 3$ we have $\mu_3 \left( \prod_{i=1}^{3} (a_i, b_i] \right) = (a_1, b_1] \times (a_2, b_2] \times (a_3, b_3]$ which is the volume of a parallelepiped.

The measure on $P_m$ can be extended to the measure over $\overline{\sigma a(P_m)}$ and it is called the Lebesgue measure on $\mathbf{R}^m$. Thereafter, $\overline{\sigma a(P_m)}$ is called the Lebesgue $\sigma$-algebra and sometimes is referred to as $L(\mathbf{R}^m)$.

It is said that the Lebesgue measure is a standard way of assigning a length, area or volume to the subsets of an $m$-dimensional Euclidean space such as $\mathbf{R}^m$.

## 2.4 Measurable Functions

Consider that $S$ is a $\sigma$-algebra over the set $U$. Also consider that $B$ is a $\sigma$-algebra over the set $\Omega$. Then a function between these measurable spaces, i.e., $f : U \to \Omega$ is said to be $S/B$-measurable if $\{u|f(u) \in A\} \in S,\ \forall A \in B$. In words, we say that $f$ is $S/B$-measurable if

the *preimage* of every set in $B$ is in $S$. Sometimes this is also written as $f^{-1}(A) \in S,\ \forall A \in B$.

In the case that $\Omega = \mathbf{R}$, $B$ is considered the Borel $\sigma$-algebra of sets over $\mathbf{R}$, i.e., $B(\mathbf{R})$ and we may just say that $f : U \to \mathbf{R}$ is $S$-measurable.

$\Omega = \bar{\mathbf{R}}$, where $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty\} \cup \{\infty\}$, may also be used to define the Borel $\sigma$-algebra over this set, namely $B(\bar{\mathbf{R}})$. Then a function $f : U \to \bar{\mathbf{R}}$ would be also $S$-measurable [1].

**Theorem 2.6** *Consider, $H \subset 2^{\mathbf{R}}$ such that $\sigma a(H) = B(\mathbf{R})$, i.e., the minimum $\sigma$-algebra containing the set $H$ is the Borel $\sigma$-algebra of sets over $\mathbf{R}$. Then a real function $f$ is $S$-measurable if and only if $\{u|f(u) \in A\}$ for all $A \in H$.*

See [7], [23].

Of course there is much more to say about measures.

Random variables are by definition measurable functions defined on sample spaces. An important example, especially in the theory of probability, is the Borel algebra on the set of real numbers. It is the algebra on which the Borel measure is defined. Given a real random variable defined on a probability space, its probability distribution is by definition also a measure on the Borel algebra.

### 2.4.1 Simple Functions

Consider the measurable space (U,S). A simple function $f(x)$ is a function that can be expressed in the following form

$$f(x) = \sum_{i=1}^{n} x_i I_{A_i}(x),$$

where $\bigcup_{i=1}^{n} A_i = U$ and $A_i \in S$ for $x_i \in R$, $i = 1, 2, \ldots, n$. $I_{A_i}$ stands as the indicator function of the set $A_i$, i.e.,

$$I_{A_i}(x) = \begin{cases} 1, & x \in A_i, \\ 0, & x \notin A_i. \end{cases}$$

It can be proved that the simple function $f(x)$ is $S$-measurable.

Indeed, it is said that the Borel $\sigma$-algebra is the minimum $\sigma$-algebra in $R^n$ that contains the open intervals in $R^n$. These open intervals may obtained by a numerable collection of disjoint intervals of the form $(a, b]$ [15].

Then, there are some remarkable sets that belong to the $\sigma$-algebra of Borel in $\mathbf{R}$, say $B(\mathbf{R})$. As some examples we may mention:

1. $(a, b) \in B(\mathbf{R})$ given $a, b \in R$, since this is an open interval.
2. $(a, b] \in B(\mathbf{R})$ since $(a, b] = \bigcap_{i=1}^{\infty} \left(a, b + \frac{1}{i}\right)$
3. $[a, b) \in B(\mathbf{R})$ since $[a, b) = \bigcap_{i=1}^{\infty} \left(a - \frac{1}{i}, b\right)$
4. $[a, b] \in B(\mathbf{R})$ since $[a, b] = \bigcap_{i=1}^{\infty} (a - \frac{1}{i}, b + \frac{1}{i})$.
5. $(-\infty, b) \in B(\mathbf{R})$ since this is a numerable union of sets such as those mentioned above.
6. Any point in $\mathbf{R}$ belongs to $B(\mathbf{R})$, say $b = [b, b]$. Furthermore, any collection of points in $\mathbf{R}$ also belongs to $B(\mathbf{R})$. For example, the collection of all the rational numbers, say $\mathbf{Q}$.

Given this, one can see that a simple function $f(x)$ is $S/B(\mathbf{R})$-measurable.

There are some important theorems for measurable and simple functions which may be useful ahead.

**Theorem 2.7** *Suppose $f$ and $g$ are simple functions. Then, $cf$ (given $c \in \mathbf{R}$), $f + g$, $fg$, $\frac{1}{f}$ (given $f(u) \neq 0, \forall u \in U$), are simple functions.*

**Theorem 2.8** *Given $f : U \to \mathbf{R}^+$ is a $S$-measurable function, then, there exists a sequence of simple functions $f_1, f_2, \ldots, f_n$ such that $f_1 \leq f_2 \leq \ldots \leq f_n$ and $\lim_{n\to\infty} f_n(u) = f(u)$ for all $u \in U$.*

**Theorem 2.9** *If $f : U \to \mathbf{R}$ is $S$-measurable function, there exists a sequence of simple functions $f_1, f_2, \ldots, f_n$ such that $|f_n| \leq |f|$ and $\lim_{n\to\infty} f_n(u) = f(u)$, $\forall u \in U$.*

See [7] for proofs.

## 2.5 Lebesgue Integral

Assume the measure space $(U, S, \mu)$ and the $S$-measurable function $f : U \to \mathbf{R}$. Sometimes it is preferred to present the integral of Lebesgue of the function $f$ with respect to the measure $\mu$, i.e.,

$$\int_A f \mathrm{d}\mu, \ A \subset S,$$

in stages as follows.

### 2.5.1 Simple Functions

Consider the indicator function $I_{A_i}$, $A_i \in S$ as defined above. We have

$$\int_U I_{A_i} \mathrm{d}\mu = \mu(A_i).$$

Now, consider the simple function $f(x) = \sum_{i=1}^{n} x_i I_{A_i}(x)$. The Lebesgue integral of this function over a set $A \subset S$ with respect to the measure $\mu$ is

$$\int_A f \mathrm{d}\mu = \sum_{i=1}^{n} x_i \mu(A_i \cap A).$$

There are some important properties of the Lebesgue integral of simple functions that are important to mention at this point.

1. Given $A \in S$ and $\mu(A) = 0$, then $\int_A f \mathrm{d}\mu = 0$ for any simple function $f$.
2. Given $A \in S$ and $f(x) \geq 0$, $\forall x \in A$, then $\int_A f \mathrm{d}\mu \geq 0$.
3. $\int_A (af + bg) \mathrm{d}\mu = a \int_A f \mathrm{d}\mu + b \int_A g \mathrm{d}\mu$, $\forall a, b \in \mathbf{R}$.
4. Given $f > g$, then $\int_A f \mathrm{d}\mu \geq \int_A g \mathrm{d}\mu$.
5. $\left| \int_A f \mathrm{d}\mu \right| \geq \int_A |f| \, \mathrm{d}\mu$.

### 2.5.2 Nonnegative functions

Thereafter, we continue with the construction of the Lebesgue integral for other kinds of functions.

Consider $f : U \rightarrow \mathbf{R}^+$ is a $S$-measurable function. Then, according to Theorem 2.8 there exists a sequence of non-negative simple functions such that

$$\lim_{n\to\infty} f_n(u) = f(u),\ \forall u \in U.$$

Furthermore, we have, for the Lebesgue integral of $f$ over a set $A \in S$ with respect to the measure $\mu$, that

$$\int_A f \mathrm{d}\mu = \lim_{n\to\infty} \int_A f_n \mathrm{d}\mu.$$

### 2.5.3 Signed Functions

Consider $f : U \rightarrow \mathbf{R}$ is a S-measurable function and $f^+ = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$. We may also say that

$$f^+(x) = \begin{cases} f(x), & \text{if} f(x) \geq 0 \\ 0, & \text{if} f(x) < 0. \end{cases},$$

$$f^-(x) = \begin{cases} f(x), & \text{if} f(x) \leq 0 \\ 0, & \text{if} f(x) > 0. \end{cases}$$

Notice that $f^+$ and $f^-$ are non negative functions. The Lebesgue integral of $f$ over a set $A \in S$ with respect to the measure $\mu$ is defined if any of the integrals, $\int_A f^+ \mathrm{d}\mu$, or $\int_A f^- \mathrm{d}\mu$ is finite.

From now on let us denote as $L(A, \mu)$ the set of different functions for which the integral of Lebesgue, over a set $A$ and with respect to the measure $\mu$, is defined.

### 2.5.4 Properties of the Lebesgue Integral of $L(A, \mu)$ functions

Next we will show some of the basic properties of the Lebesgue integral of $L(A, \mu)$ functions. Let us consider, as usual, the measure space $(U, S, \mu)$.

1. Given $A \in S$ and $\mu(A) = 0$, then $\int_A f \mathrm{d}\mu = 0$ for any $S$-measurable function $f$.
2. Given $A \in S$ and $f(x) \geq 0$, $\forall \in A$. Then $\int_A f \mathrm{d}\mu$.
3. $\int_A c \cdot f \mathrm{d}\mu = c \int_A f \mathrm{d}\mu$, $\forall c \in \mathbf{R}$.
4. Given $A \in S$ $\int_A \mathrm{d}\mu = c\mu(A)$, $\forall c \in \mathbf{R}$.
5. $\int_A (f + g) \mathrm{d}\mu = \int_A f \mathrm{d}\mu + \int_A g \mathrm{d}\mu$

6. Given $f \geq g$, then $\int_A f \mathrm{d}\mu \geq \mathrm{d}\mu$.

7. Given $f : U \to \mathbf{R}^+$ is a $S$-measurable function, then

$$\int_{A \cup B} f \mathrm{d}\mu = \int_A f \mathrm{d}\mu + \int_B f \mathrm{d}\mu$$

   if $A, B \in S$ and $A \cap B = \emptyset$.

8. Given $A, B \in S$ and $A \subset B$, then $\int_A f \mathrm{d}\mu \geq \int_B f \mathrm{d}\mu$, for any $S$-measurable function $f : U \to \mathbf{R}^+$.

The proofs of some of these properties are evident. For most of the properties whose proof is not evident, the procedure is to take the sequence of simple functions $f_n \uparrow f$ as $n \to \infty$. Then one can recall the properties of the integral for simple functions. In the case of signed functions, one must consider the definition of the integral for signed functions to complete the proof.

### 2.5.5 Properties almost everywhere

Consider a property $P$ that is fulfilled by some elements of $U$. We say that this property is fulfilled *almost everywhere* with respect to a measure $\mu$ if the set of elements $N$, which do not fulfill the property, is a null set. This is, $\mu(N) = 0$.

**Theorem 2.10** *Given $f : U \to \mathbf{R}^+$ and $f \in L(A, \mu)$, $A \in S$. If $\int_A f \mathrm{d}\mu = 0$, then* $f(x) = 0$ almost everywhere.

## 2.6 Introduction to Probability Theory

### 2.6.1 Probability Measure

Consider the measurable space $(\Omega, \mathcal{F})$. A measure $P$ on $\mathcal{F}$ is called a *probability measure* if $\mu(\Omega) = 1$. Also, any set $A \in \mathcal{F}$ is called a random event.

The triplet $(\Omega, \mathcal{F}, P)$ is called a *probability space*.

A real function or map $\xi : \Omega \to \mathbf{R}$ is called a *random variable* if it is $\mathcal{F}$-measurable. This is, if $\{\omega : \xi(\omega) \in A\} \in \mathcal{F}$, $\forall A \in B(\mathbf{R})$. If $\xi$ is a simple function, then it is said that the random variable is simple.

If we take the Lebesgue integral of the random variable $\xi$ with respect to the measure $P$, we obtain the expectation $\mathbf{E}\xi$ of the random variable. This is,

$$\int_\Omega \xi \mathrm{d}P = \mathbf{E}\xi.$$

### 2.6.2 Stochastic Processes

Consider the measurable space $(X, \Sigma)$ and the probability space $(\Omega, \mathcal{F}, P)$. A *stochastic process* is a two component function $\xi(t, \omega)$ defined on $[0, +\infty) \times \Omega$ which satisfies the following conditions.

1. For every fixed $t_0 \in [0, +\infty)$, $\xi(t_0, \omega) : \Omega \to X$ is a $\Sigma$-measurable function.
2. For every fixed $\omega_0 \in \Omega$, $\xi(t, \omega_0)$, is a function with values in $X$ for all $t \in [0, \infty)$.

One common case for the space of a stochastic processes would be $(X, \Sigma) = (\mathbf{R}^n, B(\mathbf{R}^n))$, $n \geq 1$.

Consider the probabilistic space $(\Omega, \mathcal{F}, P)$, a stochastic process $\zeta(t)$ (or $\zeta(t, \omega)$) with values in a measurable space $(X, \Sigma)$, is called a *Markovian processes* if [12]

$$P\{\zeta(t) \in A, t > t_0 / \zeta(t_0) = x,\ \zeta(s) \in B, s < t_0\} = P\{\zeta(t) \in A, t > t_0 / \zeta(t_0) = x\},$$

for every $x, A, B \in \Sigma$.

It is said that the evolution of a Markov process after any point of time $t_0$ is independent of the evolution of this process before $t_0$. It is normally assumed that the value of the process at $t_0$ is known.

Now it is important to continue defining a few more concepts regarding stochastic processes in order to be able to mention important characteristics of these functions.

A *stochastic kernel* is a two-component function, say $P(x, B)$, for $x \in X$ and $B \in \Sigma$ where

1. $P(x_0, B)$ is a measure on $\Sigma$ for every fixed $x_0 \in X$. Also, $P(x_0, X) = P(x, X) = 1$, $\forall x \in X$.
2. $P(x, B_0) : X \to R$ is a measurable function for every fixed $B_0 \in \Sigma$.

### 2.6.3 Markov processes

A family of stochastic kernels $P(t, x, B)$, $t \in T$, where $T$ is a finite or infinite interval of nonnegative real numbers or integers is said to be Markovian if $P(t, x, B)$ satisfy the Chapman-Kolmogorov equations. This means that

$$P(x + t, x, B) = \int_X P(s, x, dy) P(t, y, B),\ \forall s, t, s + t \in T,$$

given $B \in \Sigma$.

$P(t, x, B)$ are transition probabilities and $(X, \Sigma)$ is the space of all possible states of the process, i.e., a *phase space*. A family of stochastic kernels $P(t, x, B) \in T$ on a measurable space $(X, \Sigma)$ is called a *Homogeneous Markov Process*

Consider as $M = M(\Sigma)$ the collection of all finite measures on the $\sigma$-algebra $\Sigma$. Then, let us define the operator $T^*$ on $M$ as

$$T^* m = \int_X P(t, x, A) m(\mathrm{d}x),$$

given $t \in T$, $A \in \Sigma$.

If we take the initial distribution of a homogeneous Markov process as $m(B) = P\{\xi(0) \in B\}$, $B \in \Sigma$ then we have that the distribution of the process at time $t$ is

$$m_t(A) = \int_X P(t, x, A) m(\mathrm{d}x),$$

for $m \in M$. Then, the operator $T^* : M \to M$ determines the distribution of the process at time $t$ given the initial distribution.

Using the properties given by the Chapman-Kolmogorov equation we have that, given $s, t, t + s \in T$,

$$\begin{aligned}
m_{s+t}(A) &= \int_X P(t + s, x, A) m(\mathrm{d}x) && (2.2)\\
&= \int_X \int_X P(t, x, \mathrm{d}y) P(s, y, A) m(\mathrm{d}x) && (2.3)\\
&= \int_X P(s, y, A) \left( \int_X P(t, x, \mathrm{d}y) m(\mathrm{d}x) \right) && (2.4)\\
&= \int_X P(s, y, A) m_t(\mathrm{d}y). && (2.5)
\end{aligned}$$

It is not difficult to see that $T^*_{s+t} = T^*_s T^*_t$. When a family of operators owns this property it is called a *semi-group* of operators.

Now, let us consider the collection of all bounded non-negative $\Sigma$-measurable functions on X as $B(\Sigma)$. Let us also define another family of operators in the same manner as above for $f \in B(\Sigma)$. We have

$$f_t(x) = T_t f(x) = \int_X f(y) P(t, x, \mathrm{d}y).$$

$f_t(x)$ is the conditional expectation of the random process $f(\xi(t))$ with respect to the random event $\xi(0) = x$. It can be proved that $f_t$ is bounded and nonnegative since $f$ is bounded and nonnegative. Then $T_t : B(\Sigma) \to (\Sigma)$.

Using the Chapman-Kolmogorov equation, we can obtain

$$\begin{aligned}
f_{s+t}(x) &= \int_X f(y) P(s + t, x, \mathrm{d}y) && (2.6)\\
&= \int_X f(y) \int_X P(s, x, \mathrm{d}u) P(t, u, \mathrm{d}y) && (2.7)\\
&= \int_X f(y) P(t, u, \mathrm{d}y) \int_X P(s, x, \mathrm{d}u) && (2.8)\\
&= \int_X f_t(u) P(s, x, \mathrm{d}u). && (2.9)
\end{aligned}$$

It is easy to see that $T_{s+t} = T_s T_t$. Then this is also a semi-group of operators.

Consider $D \subset B(\Sigma)$ where $D = \{\varphi : \exists A\varphi\}$ given

$$A\varphi = \lim_{\Delta t \to 0^+} \frac{T_{\Delta t}\varphi - \varphi}{\Delta t}, \quad \lim_{\Delta t \to 0^+} T_{\Delta t}\varphi = \varphi. \tag{2.10}$$

The operator $A$ is called the infinitesimal operator of the semi-group $T_t$. It is said, according to the Hille-Yosida theorem, that under general conditions, the infinitesimal operator $A$ uniquely determines the semi-group of operators $T_t$ and in consequence, the Markov process [12].

Given $I_B(x) \in D$, where $I_B(x)$ is the indicator function of the event $B$, we have that

$$T_t I_B(x) = \int_X I_B(y) P(t, x, \mathrm{d}y) = P(t, x, B).$$

Then,

$$\begin{aligned}
\frac{\partial P(t,x,B)}{\partial t} &= \lim_{\Delta t \to 0} \frac{P(t+\Delta t, x, B) - P(t,x,B)}{\Delta t}, \\
&= \lim_{\Delta t \to 0} \frac{T_{\Delta t} P(t,x,B) - P(t,x,B)}{\Delta t}, && (2.11) \\
\frac{\partial P(t,x,B)}{\partial t} &= AP(t,x,B). && (2.12)
\end{aligned}$$

$P(t, x, B)$ stands as the probability distribution that the state of the process after time $t$ belongs to the set or event $B$ given any initial state $x$. Equation 2.12 is called the first (or backward) Kolmogorov equation. In a similar way we can obtain the second (or forward) Kolmogorov equation if we consider the semigroup $T_t^*$, $t \in T$.

Let $D^* \subset M$, where $D^* = \{\phi : \exists A\phi\}$ given

$$A\phi = \lim_{\Delta t \to 0^+} \frac{T_{\Delta t}^* \phi - \phi}{\Delta t}, \qquad \lim_{\Delta t \to 0^+} T_{\Delta t}^* \phi = \phi. \tag{2.13}$$

Take, $\phi(t, B) = T_t^* \phi = \int_X P(t, x, B)\phi(\mathrm{d}x)$. Then, for $\phi$ such that $\phi(t, B) \in D^*$, then

$$\begin{aligned}
\frac{\partial \phi(t,B)}{\partial t} &= \lim_{\Delta t \to 0} \frac{\phi(t+\Delta t, B) - \phi(t,B)}{\Delta t}, \\
&= \lim_{\Delta t \to 0} \frac{T_{t+\Delta t}^* \phi(t,B) - \phi(t,B)}{\Delta t}, && (2.14) \\
\frac{\partial \phi(t,B)}{\partial t} &= A^* \phi(t,B). && (2.15)
\end{aligned}$$

Furthermore, $\phi(t, B) = \int_X P(t, y, B)\phi_x(\mathrm{d}y) = P(t, x, B)$ for $\phi_x = \delta(x)$ is the indicator function of the initial state of the process.

We have

$$\frac{\partial P(t,x,B)}{\partial t} = A^* P(t,x,B), \tag{2.16}$$

which is known as the second (or forward) Kolmogorov equation. It is said that the forward equation is used when one needs to know the probability distribution of the state of the process at some time given an initial distribution, usually represented by a delta function.

As a conclusion, we can say that the infinitesimal operator characterizes the Markov process. As an example we can mention the Brownian motion on **R**.

### 2.6.4 Brownian Motion on R

Given $\xi(t)$ is the state of the process at time t, this process has the following properties.

1. $\mathbf{E}\Delta\xi = 0$.
2. $\mathbf{E}(\Delta\xi)^2 = b\Delta t$, for $b \in \mathbf{R}$.
3. $\mathbf{E}(|\Delta|^2)^{2+\delta} = o(\Delta)$, $\forall \delta > 0$.

If $\varphi$ is any real function such that $u(t,x) = E(\varphi(\xi(t))/\xi(0) = x)$ is twice differentiable, then it can be proved that

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2} b \frac{\partial^2 u(t,x)}{\partial x^2},$$

with the boundary condition $u(0,x) = \varphi(x)$.

Then, for this case, the infinitesimal operator has the form of

$$A\phi = \frac{1}{2} b \frac{\partial^2}{\partial x^2} \phi.$$

This Brownian motion belongs to a more general class of processes called diffusion processes for which the infinitesimal operator has the general form

$$A\phi = a(x) \frac{\partial}{\partial x} \phi + b(x) \frac{\partial^2}{\partial x^2}.$$

For this processes, $a(x)$ is called the drift coefficient and $b(x)$ is called the diffusion coefficient.

### 2.6.5 Substochastic Kernel

In the measurable space $(X, \Sigma)$, a *substochastic kernel* $P(x, B)$ is a real-valued function which accomplishes the following properties.

1. $P(x_1, B)$, is a measure on $\Sigma$ for all fixed $x_1$ such that $P(x, X) \leq 1$.
2. $P(x, B_1)$ is a measurable function on $x$ for every fixed $B_1 \in \Sigma$.

### 2.6.6 Semi-Markov Kernel

A positive-valued function $Q(x, B, t)$, $x \in X$, $B \in \Sigma$, $t \geq 0$ is a *semi-Markov kernel* if the following conditions are satisfied.

1. $Q(x, B, t_1)$ is a substochastic kernel on $(X, \Sigma)$ for every fixed $t_1 > 0$.
2. $Q(x_1, B_1, t)$ is a non-decreasing right-continuous function on $t \geq 0$ for a every fixed $x_1$ and $B_1$. Also, $Q(x, B, 0) = 0$.
3. $Q(x, B, +\infty) = P(x, B)$ is a stochastic kernel.
4. $Q(x, X, t) = G_x(t)$ is a distribution function on $t \geq 0$ for every fixed $x \in X$.

## 2.7 Markov Renewal Processes

Consider the measurable space $(X, \Sigma)$. In probability theory, $X$ represents the phase space of the process, i.e., the space of all the states of the process. $\Sigma$ stands as the $\sigma$-algebra of all the events from $X$. It is assumed that $\Sigma$ contains all singletons from $X$.

Consider the Markov chain $\{\xi_n, n \geq 0\}$ is a Markov chain with the phase space $X, \Sigma$. Also, consider the sequence of positive independent random variables $\theta_i$, for $i \geq 0$ on the probability space $(\Omega, \mathcal{F}, P)$. Let us construct a homogeneous two-component Markov process $\{\xi_n, \theta_n; n \geq 0\}$, $\xi_n \in X$, $\Theta_n \in [0, +\infty)$ given $P\{\xi_0 \in B\} = p_0(B)$, $B \in \Sigma$. It is not very difficult to prove that

$$P\{\xi_{n+1} \in B, \theta_{n+1} \leq 1/\xi_0 \in B_0, \theta_0 \leq t_0, \xi_1 \in B_1, \theta_1 \leq t_1, \ldots, \xi_n \in B_n, \theta_n \leq t_n\} = P\{\xi_{n+1} \in B, \theta_{n+1} \leq t/\xi_n \in B_n\},$$

where $B_i \in \Sigma$ and $i = 0, 1, \ldots, n$. In addition, $P\{\xi_{n+1} \in X, \theta_{n+1} \leq \infty/\xi_n = x\} = 1$.

Since, $Q(x, B, t) = P\{\xi_{n+1} \in B, \theta_{n+1} \leq t/\xi_n = x\}$ is a semi-Markov kernel that determines the transition probabilities of the process then, the two component homogeneous Markov process $\{\xi_n, \theta_n; n \geq 0\}$ is called a *Markov Renewal Process.*

The component $\xi_n$, $n \geq 0$ is called the *embedded Markov Chain.* The non-negative random variables $\theta_n$, $n \geq 0$ are called the *renewal times* and they define the intervals between Markov renewal moments $\tau_n = \sum_{i=0}^{n} \theta_i$.

To obtain the transition probabilities of the embedded Markov chain, one can evaluate $t = +\infty$ in the semi-Markov kernel in the following form.

$$P(x, B) = P\{\xi_{n+1} \in B/\xi_n = x\} = Q(x, B, +\infty).$$

The distribution of the renewal time is a conditional distribution depending on the states of the embedded Markov. This is,

$$G_x(t) = Q(x, X, t) = P\{\theta_{n+1} \leq t/\xi_n = x\}.$$

Sometimes the notation $P\{\theta_x \leq t\} = P\{\theta_{n+1} \leq t/\xi_n = x\}$ is also used.

## 2.8 Semi-Markov Processes

Consider the probability space $(\Omega, \mathcal{F}, P)$ and the measurable space $(X, \Sigma)$. Let us denote as $\{\xi_n,\ n \geq 0\}$ the Markov chain in the phase space $(X, \Sigma)$. Also, let us denote as $\theta_i$, $i \geq 1$ a sequence of positive independent random variables. Then, let us construct the Markov renewal process $\{\xi_n, \theta_n, n \geq 0\}$ with the renewal moments given by the process $\tau_n = \sum_{i=0}^{n} \theta_i$ on $(\Omega, \mathcal{F}, P)$ and with the embedded Markov chain $\{\xi_n\}$. The transitional probabilities are given by a semi-Markov kernel

$$Q(x, B, t) = P\{\xi_{n+1} \in B, \theta_{n+1} \leq t/\xi_n = x\}.$$

Then, the transition probabilities of the embedded Markov chain $\{\xi_n, n \geq 0\}$ are given by

$$P(x, B) = P\{\xi_{n+1} \in B/\xi_n = x\} = Q(x, B, +\infty).$$

The distribution function of the sojourn time $\theta_x$ in every state $x \in X$ is given by

$$G_x(t) = P\{\theta_{n+1} \leq 1/\xi_n = x\}.$$

The Radon-Nikodym theorem states that since $Q(x, B, t) \leq P(x, B)$ for every fixed $x \in X$ and $t \geq 0$, there exists a measurable function $G_{xy}(t)$ such that

$$Q(x, B, t) = \int_B G_{xy}(t) P(x, \mathrm{d}y).$$

It can proved that the function $G_{xy}(t)$ is of the following form

$$G_{xy}(t) = P\{\theta_{n+1} \leq t/\xi_n = x, \xi_{n+1} = y\}.$$

This is the distribution function of the sojourn time of the MRP in a state $x$ and in transition to state $y$. If the distribution is independent of $y$, i.e., $G_{xy}(t) = G_x(t)$, we have that

$$G_{xy}(t) = P\{\theta_{n+1} \leq t/\xi_n = x\}.$$

Then,

$$Q(x, B, t) = P(x, B) G_x(t).$$

As an example, consider the case of a discrete phase space $X = \{1, 2, \ldots, n\}$. The semi-Markov kernel is then given by a semi-Markov matrix of the form

$$Q(t) = \{Q_{ij}(t); i, j \in X\},$$

where $Q_{ij}$ can be represented as $Q_{ij}(t) = p_{ij}G_i(t)$, given that the sojourn time distribution depends only on the current state. $G_i(t)$ stand as the distribution function of the sojourn time in state $i \in X$ and $p_{ij}$ stand as the transition probabilities of the embedded Markov chain.

Given $v(t) = \max\{n \geq 0; \tau_n \leq 1\}$, $t \geq 0$, the process $\xi(t) : \xi_{v(t)}$ is called a *semi-Markov process*. This process changes state at renewal times $t_n$ and the sojourn time in every state $\xi_n = x$ is given by $\theta_{n+1} = \theta_x$. It is easy to see that this process remains constant in the intervals $[\tau_n, \tau_{n+1})$ and that it is right continuous.

Consider the process $\{\xi_n, \theta_n, n \geq 0\}$ with a discrete phase space $X$ for the embedded Markov chain $\{\xi_n, n \geq 0\}$. In the case the semi-Markov kernel is of the form

$$Q_{ij}(t) = p_{ij}(1 - e^{-\lambda_i t}),$$

Then, the semi-Markov process is in fact a Markov process with the known property, called by A. Khinchin, the absence of aftereffect.

In a more general form, the semi-Markov kernel of this sort of Markov process is given by

$$Q(x, B, t) = P(x, B)\left(1 - e^{-q(x)t}\right).$$

Here, $q(x) \geq 0$, $x \in X$ stand as the sojourn time intensities of the renewal times. Given this, we can construct a generating kernel for the process in the following way,

$$Q(x, B) = q(x)(P(x, B) - 1),$$

for $x \in X$ and $B \in \Sigma$. It is known that the generating kernel uniquely determines the Markov *jump* process.

In the case $X$ is a discrete phase space, we have a generating matrix of the form

$$Q = q(P - I),$$

where $q$ is diagonal matrix of sojourn time intensities and $I$ is the identity matrix.

## 2.9 Transfer Processes

Consider a space $(E, \Sigma)$ and a jump Markov process $\eta(t)$ on this phase space. Let the semi-Markov kernel of the process be determined by

$$Q(x, B, t) = P(x, B)(1 - e^{-\lambda_x t}).$$

Then, the infinitesimal operator of the process $\eta(t)$ is

$$Q(x, B) = \lambda_x[P(x, B) - 1]$$

It is not difficult to see that the infinitesimal operator uniquely determines the semi-Markov kernel.

Denote by $C(u, x)$, a function in $u \in R$, $x \in E$ which is differentiable with respect to $x$ and it is one value solvable for

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = C(u(t)),$$

given $t \geq 0$ and $u(0) = u_0$.

A stochastic transfer process $u^s(t)$ is given by

$$\frac{\mathrm{d}u^s}{\mathrm{d}t} = C\left(u^s(t), \eta(t)\right), \tag{2.17}$$

for $u^s(0) = u_0^s$.

It is well known that the two-component process $(u^s(s), \eta(t))$ is a Markov process and its infinitesimal operator is of the following form.

$$A\phi(u, x) = Q\varphi(u, x) + C(u, x)\frac{\partial}{\partial u}\varphi(u, x),$$

where

$$Q\varphi(u, x) = \int_E Q(x, \mathrm{d}y)\varphi(u, y) - \lambda_x\varphi(u, x),$$

and $Q(x, \mathrm{d}y)$ is the infinitesimal operator of $\eta(t)$.

In the case that $\eta(t)$ is a semi-Markov process, then equation 2.17 determines the stochastic transfer process in a semi-Markov media. If a solution of a equation of this kind is to be intended, one needs to consider the three component Markov process $\varsigma(t) = (u^s(t), \eta(t), \gamma(t))$. $\gamma(t) = t - \tau(t)$ is called a past holding time process.

The infinitesimal operator of $\varsigma(t)$ is of the following form

$$A\varphi(u, x, \tau) = Q\varphi(u, x, \tau) + C(u, x, \tau)\frac{\partial}{\partial u}\varphi(u, x, \tau) + \frac{\partial}{\partial \tau}\varphi(u, x, \tau),$$

where,

$$Q\varphi(u, x, \tau) = \int_E \frac{G'_x(t)}{1 - G_x(t)}P(x, \mathrm{d}y)\varphi(u, y, \tau) - \frac{G'_x(t)}{1 - G_x(t)}\varphi(u, x, 0).$$

To show the benefits of this formulation just presented known as the *evolutionary formulation* let us consider as an example the mathematical model of the Brownian motion in $\mathbf{R}^2$.

## 2.10 Brownian Motion in $\mathbf{R}^2$

Consider that a particle can move in one of three directions in $\mathbf{R}^2$. The angle between any two directions is equal to $\frac{2\pi}{3}$. Starting at $x_0 \in \mathbf{R}^2$ assume the particles can move at a velocity $v > 0$ in any of the three directions during a random time interval that is exponentially distributed with parameter $\lambda_0$. After that random interval, the particle switches to another direction with a probability of 1/2 on each one and so on and so forth.

To proceed, let us denote by $x(t)$ and $y(t)$ the abscissa and the ordinate of the particle at time $t$. We have

$$f_n(x, y, t)\mathrm{d}x\mathrm{d}y = P\{x \leq x(t) \leq x + \mathrm{d}x, y \leq y(t) \leq y + \mathrm{d}y/E_i\},$$

for $n = 1, 2, 3$. Also,

$$\frac{\partial}{\partial t} f_n = A f_n(t, x, y).$$

We have the evolution equation

$$\frac{\partial \vec{s}}{\partial t} = \vec{c}(\vec{s}(t), \xi_\lambda(t)), \tag{2.18}$$

where $\xi_\lambda$ stands as the embedded Markov chain.

The infinitesimal operator for this process is

$$A\varphi(s, n) = c(s, n)\frac{\partial}{\partial s}\varphi(s, n) + Q\varphi(s, n),$$

given $Q = q(P - I)$, where $I$ stands as the identity matrix.

Here,

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix},$$

also,

$$q = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

Then,

$$Q \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\lambda f_0 & \lambda/2 f_1 & \lambda/2 f_2 \\ \lambda/2 f_0 & -\lambda f_1 & \lambda/2 f_2 \\ \lambda/2 f_0 & \lambda/2 f_1 & -\lambda/2 f_2 \end{pmatrix}$$

Then, since $\cos\pi/3 = \sin\pi/6 = 1/2$ and $\sin\pi/3 = \cos\pi/6 = \sqrt{3}/2$ we have for Eq. (2.18) that

$$\frac{\partial f_0(t,x,y)}{\partial t} = v\frac{\partial}{\partial x}f_0 - \lambda f_0 + \frac{\lambda}{2}f_1 + \frac{\lambda}{2}f_2, \tag{2.19}$$

$$\frac{\partial f_1(t,x,y)}{\partial t} = -\frac{1}{2}v\frac{\partial}{\partial x}f_1 + \frac{\sqrt{3}}{2}v\frac{\partial}{\partial y}f_1 + \frac{\lambda}{2}f_0 - \lambda f_1 + \frac{\lambda}{2}f_2, \tag{2.20}$$

$$\frac{\partial f_2(t,x,y)}{\partial t} = -\frac{v}{2}\frac{\partial}{\partial x}f_2 - \frac{\sqrt{3}}{2}v\frac{\partial}{\partial y}f_2 + \frac{\lambda}{2}f_0 + \frac{\lambda}{2}f_1 - \lambda f_2 \tag{2.21}$$

Then, we need to solve the system $D\vec{f} = \vec{0}$. Namely,

$$D = \begin{pmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & D_{12} \\ D_{20} & D_{21} & D_{22} \end{pmatrix}.$$

Here, $D_{01} = D_{02} = D_{10} = D_{12} = D_{20} = D_{21} = -\frac{1}{2}$ and

$$D_{00} = \frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + \lambda$$
$$D_{11} = \frac{\partial}{\partial t} + \frac{v}{2}\frac{\partial}{\partial x} - \frac{\sqrt{3}}{2}v\frac{\partial}{\partial y} + \lambda$$
$$D_{22} = \frac{\partial}{\partial t} + \frac{v}{2}\frac{\partial}{\partial x} + \frac{\sqrt{3}}{2}v\frac{\partial}{\partial y} + \lambda.$$

Assuming $D_{ij}$ are commutative we have

$$\det(D)\cdot(f_0 + f_1 + f_2) = \det(D)\cdot(f) = 0.$$

Then, we obtain

$$\frac{\partial^3}{\partial t^3}f + 3\lambda\frac{\partial^2}{\partial t^2}f + \frac{9}{4}\lambda^2\frac{\partial}{\partial t}f - \frac{3}{4}v^2\frac{\partial}{\partial t}\triangle f - \frac{3}{4}v^2\lambda\triangle f - \frac{v^3}{4}\left[\frac{\partial^3}{\partial x^3}f - 3v^3\frac{\partial^3}{\partial x\partial y^2}\right] = 0,$$

where, $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Although evolution of systems in semi-Markov media have been formulated, most of the times it turns too difficult to solve exactly. Many times the exact solution of these systems using the evolutionary formulation involves the solution of several integral-differential partial coupled equations. For example, a superposition of two-simple semi-Markov processes may involve as much as eighteen or more integral-differential partial coupled equations to be solved simultaneously to satisfied also initial conditions. These calculations may be cumbersome and this is why many times some other methods are used to approximately solved this systems. One of these techniques involves the *phase merging algorithm* to reduce the evolution of the system in a semi-Markov media to an approximated evolution in a Markov media. This technique is explained in more detail ahead.

# Chapter 3

# Two Different Customers

## 3.1 Chapter Summary

In this paper we study the stationary efficiency of a system consisting of a finite capacity buffer connected to two different customers with bursty on-off demands. We assume that the buffer is filled up at a constant rate $F$. The dynamics of the overall system are modeled using a semi-Markov evolution environment and we derive design formulae involving the main parameters.

It has been shown that it is possible to use the *Phase Merging Algorithm* to reduce the semi-Markov process to an approximated Markov process [9], [11], [13]..

Here, we use that result and we find the stationary distribution of the system. As examples, we consider the $m$-Erlang and hyper-exponential probability distributions for the sojourn times. The Markov evolution environment is also included as a special case. We study some limits of the stationary efficiency as the buffer capacity tends to infinity and we identify three different stationary behavior system cases.

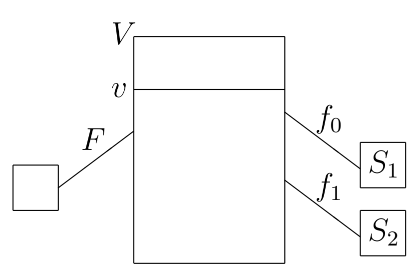


Figure 3.1: A system of two independent random state switching customers and one buffer filled up at a constant rate.

The system functionality is as follows:

The customers switch from the active or “ON” state to the inactive or “OFF” state, and we consider that the switching process of the customers can be modeled as a semi-Markov process.

When active, one customer demands information at a rate $f_0$ whereas the other customer demands information at a rate $f_1$. When both customers are active, information is required at a rate $f_1 + f_0$. In each of these cases, if the buffer is empty ($v = 0$), an unproductive situation is considered. When no customer is active, then no product is required. The filling aggregate provides the buffer with product at a constant rate $F$. This aggregate is active as long as the volume of information is below the maximum capacity of the buffer ($V$), see Figure 3.1.

Let us denote $I(T)$ the amount of information delivered to customers $S_1$ and $S_2$, in a time interval $[0, T]$. Thus, we can define $K = \lim_{T\to\infty} \dfrac{I(T)}{T}$ as the steady state parameter for the system effectiveness, see Chapter 4 in [20]. Our main purpose in this work is to determine $K$ as a function of the system parameters, $\lambda_0$, $\lambda_1$, $\mu_0$, $\mu_1$, $f_0$, $f_1$ and $F$.

The dynamics of this linear system can be captured by a first order differential equation having a random component, or the so-called random evolution process [10]. In Section 2, we elaborate our semi-Markov mathematical modeling. In Sections 3 and 4 we deal with the special Markov case. We obtain the stationary compound probability distribution for the buffer content level and the mathematical expression of the efficiency parameter in terms of the system values. Then, in Section 5, with the help of some plots we analyze some numerical results for different Markov and semi-Markov cases. In particular, we include the $m$-Erlang, exponential, and hyper-exponential distributions for the active periods.

## 3.2 Semi-Markov Mathematical Model

We begin studying case $N = 2$. Consider the semi-Markov process $\{\chi(t)\}$ which is the superposition of two independent alternating semi-Markov processes with phase space $\mathbf{Z} = \left\{(h, x^i) : h \in \mathbf{H}, x^i \in \mathbf{R}^2_+\right\}$, where $\mathbf{H} = \{h : h = (h_1, h_2), \;\; h_i = 0, 1;\, i = 1, 2\}$, and $\mathbf{R}^2_+ = \{\vec{x} : \vec{x} = (x, 0), x \geq 0\} \bigcup \{\vec{x} : \vec{x} = (0, x), x \geq 0\}$. We have defined $h_i$ as

$$h_i = \begin{cases} 1, & \text{if } S_i \text{ is active;} \\ 0, & \text{if } S_i \text{ is not active,} \end{cases}$$

where $S_i$ stands for subsystem $i$. The component $x$ of the vector $(x, 0)$ (respectively $(0, x)$) is the residual life from the last state change of $S_1$ (respectively $S_2$). The initial distribution of $\chi(t)$ is $P\{\chi(0) = (1, 1; 0, 0)\} = 1$.

Let us write this in more detail:

$(1, 1; 0, x)$ - subsystem $S_1$ starts to be active and subsystem $S_2$ has been active for the time $x$,

$(1,1;x,0)$ - subsystem $S_2$ starts to be active and subsystem $S_1$ has been active for the time $x$,

$(1,0;0,x)$ - subsystem $S_1$ starts to be active and subsystem $S_2$ has been inactive for the time $x$,

$(1,0;x,0)$ - subsystem $S_2$ starts to be inactive and subsystem $S_1$ has been active for the time $x$,

$(0,1;0,x)$ - subsystem $S_1$ starts to be inactive and subsystem $S_2$ has been active for the time $x$,

$(0,1;x,0)$ - subsystem $S_2$ starts to be active and subsystem $S_1$ has been inactive for the time $x$,

$(0,0;0,x)$ - subsystem $S_1$ starts to be inactive and subsystem $S_2$ has been inactive for the time $x$,

$(0,0;x,0)$ - subsystem $S_2$ starts to be inactive and subsystem $S_1$ has been inactive for the time $x$.

The embedded Markov chain of this semi-Markov process has the following transition probabilities [9]:

$$\begin{aligned}
P\left[(h_1,h_2;0,x),\left\{(\bar{h}_1,h_2;0,u),\ u\le y\right\}\right] &= \frac{1}{\bar{F}^{(2)}_{h_2}(x)}\int_0^{y-x}\bar{F}^{(2)}_{h_2}(x+u)\mathrm{d}F^{(1)}_{h_1}(u),\\
P\left[(h_1,h_2;0,x),\left\{(h_1,\bar{h}_2;u,0),\ u\le y\right\}\right] &= \frac{1}{\bar{F}^{(2)}_{h_2}(x)}\int_x^{y+x}\bar{F}^{(1)}_{h_1}(u-x)\mathrm{d}F^{(2)}_{h_2}(u),\\
P\left[(h_1,h_2;x,0),\left\{(\bar{h}_1,h_2;0,u),\ u\le y\right\}\right] &= \frac{1}{\bar{F}^{(1)}_{h_1}(x)}\int_x^{y+x}\bar{F}^{(2)}_{h_2}(u-x)\mathrm{d}F^{(1)}_{h_1}(u),\\
P\left[(h_1,h_2;x,0),\left\{(h_1,\bar{h}_2;u,0),u\le y\right\}\right] &= \frac{1}{\bar{F}^{(1)}_{h_1}(x)}\int_0^{y-x}\bar{F}^{(1)}_{h_1}(u+x)\mathrm{d}F^{(2)}_{h_2}(u),
\end{aligned}\tag{3.1}$$

where $\bar{h}_i = 1-h_i$, $\bar{F}(x) = 1-F(x)$, and $F(x)$ is the cumulative distribution function.

The sojourn times corresponding to the stochastic process $\chi(t)$ with phase space $\mathbf{Z}$, have the following expected values

$$m(h_1,h_2;x,0) = \frac{1}{\bar{F}^{(1)}_{h_1}(x)}\int_0^\infty \bar{F}^{(1)}_{h_1}(x+y)\bar{F}^{(2)}_{h_2}(y)\,\mathrm{d}y,$$

$$m(h_1,h_2;0,x) = \frac{1}{\bar{F}^{(2)}_{h_2}(x)}\int_0^\infty \bar{F}^{(1)}_{h_1}(y)\bar{F}^{(2)}_{h_2}(x+y)\,\mathrm{d}y.$$

The density of the stationary distribution for the Markov chain embedded in $\chi(t)$ is of the following form [9]:

$$\rho(h_1, h_2; 0, x) = c_{s0}\,\bar{F}^{(2)}_{h_2}(x) \qquad \text{and} \qquad \rho(h_1, h_2; x, 0) = c_{s0}\,\bar{F}^{(1)}_{h_1}(x), \tag{3.2}$$

where

$$c_{s0}^{-1}(h_1, h_2) = \int_0^\infty \left(\bar{F}^{(1)}_{h_1}(x) + \bar{F}^{(2)}_{h_2}(x)\right) \mathrm{d}x.$$

Consider a function $C(w)$ on the space $\mathbf{W} = \mathbf{Z} \times [0, V]$ that expresses the rate of change of the amount of information in the buffer and is defined as

$$C(w) = \begin{cases} F, & w = \{(0,0;\vec{x}), v\},\ \{\vec{x} = (x,0)\} \text{ or } \{\vec{x} = (0,x)\},\ 0 < v < V; \\ F - f, & w = \{(0,1;\vec{x}), v\},\ \{\vec{x} = (x,0)\} \text{ or } \{\vec{x} = (0,x)\},\ 0 < v < V; \\ F - f, & w = \{(1,0;\vec{x}), v\},\ \{\vec{x} = (x,0)\} \text{ or } \{\vec{x} = (0,x)\},\ 0 < v < V; \\ F - 2f, & w = \{(1,1;\vec{x}), v\},\ \{\vec{x} = (x,0)\} \text{ or } \{\vec{x} = (0,x)\},\ 0 < v < V; \\ 0, & \text{in other cases.} \end{cases} \tag{3.3}$$

Let $v(t)$ be the amount of information in the buffer at time $t$. Hence, it is easily verified that $v(t)$ obeys to the differential equation:

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = C\left(\chi(t), v(t)\right), \tag{3.4}$$

with the initial condition $v(0) = v_0 \in [0, V]$. $C(w) = C(\chi(t), v(t))$ for $\chi(t) \in \mathbf{Z}$ and $v(t) \in [0, V]$. It can be said that Eq. (3.4) determines the random evolution of the system. Meaning that the process $v(t)$ is the stochastic transfer process in the semi-Markov medium $\chi(t)$ [9], [11]. By using the *phase merging algorithm* with the merging function $k(h_1, h_2; \vec{x}) = (h_1, h_2)$, we can obtain a Markov averaged evolution $\bar{v}(t)$ that is a close approximation to the original semi-Markov case, see Chapter 5 in [11]. Hence the averaged evolution $\bar{v}(t)$ obeys the following differential equation

$$\frac{\mathrm{d}\bar{v}(t)}{\mathrm{d}t} = \bar{C}\left(\bar{\chi}(t), \bar{v}(t)\right),\ \bar{v}(0) = \bar{v}_0 \in [0, V],$$

where

$$\bar{C}((h_1, h_2), v) = \int_0^\infty C\{(h_1, h_2; \vec{x}), v\}[\rho(h_1, h_2; 0, x) + \rho(h_1, h_2; x, 0)]\,\mathrm{d}x$$

is a function on $\mathbf{X} \times [0, V]$, and $\mathbf{X} = \{00, 01, 10, 11\}$. The stochastic process $\{\bar{\chi}(t)\}$ is a Markov process with the phase space $\mathbf{X}$. Let us write $\bar{C}((h_1, h_2), v)$ in more detail,

$$\bar{C}\left((h_1, h_2),\ v\right) =$$

$$\begin{cases} Fc_{s0}\int_0^\infty \left(\bar{F}^{(1)}_{h_1}(x)+\bar{F}^{(2)}_{h_2}(x)\right)\mathrm{d}x, & (h_1,h_2)=(0,0),\ 0<v<V; \\ (F-f)c_{s0}\int_0^\infty \left(\bar{F}^{(1)}_{h_1}(x)+\bar{F}^{(2)}_{h_2}(x)\right)\mathrm{d}x, & (h_1,h_2)=(1,0),\ 0<v<V; \\ (F-f)c_{s0}\int_0^\infty \left(\bar{F}^{(1)}_{h_1}(x)+\bar{F}^{(2)}_{h_2}(x)\right)\mathrm{d}x, & (h_1,h_2)=(0,1),\ 0<v<V; \\ (F-2f)c_{s0}\int_0^\infty \left(\bar{F}^{(1)}_{h_1}(x)+\bar{F}^{(2)}_{h_2}(x)\right)\mathrm{d}x, & (h_1,h_2)=(1,1), 0<v<V; \\ 0, & \text{in other cases.} \end{cases} \tag{3.5}$$

From Eqs. (3.1) and (3.2), we can obtain the following equations:

$$P\left\{(h_1,h_2)(\bar{h}_1,h_2)\right\}= \frac{1}{c_{s1}}\left[\int_0^\infty \rho(h_1,h_2;0,x)P\left[(h_1,h_2;0,x),\{(\bar{h}_1,h_2;0,u)\},\ u\le\infty\right]\mathrm{d}x + \int_0^\infty \rho(h_1,h_2;x,0)P\left[(h_1,h_2;x,0),\left\{(\bar{h}_1,h_2;0,u),\ u\le\infty\right\}\right]\mathrm{d}x\right], \tag{3.6}$$

$$P\left\{(h_1,h_2)(h_1,\bar{h}_2)\right\}= \frac{1}{c_{s1}}\left[\int_0^\infty \rho(h_1,h_2;0,x)P\left[(h_1,h_2;0,x),\{(h_1,\bar{h}_2;0,u)\},\ u\le\infty\right]\mathrm{d}x + \int_0^\infty \rho(h_1,h_2;x,0)P\left[(h_1,h_2;x,0),\left\{(h_1,\bar{h}_2;0,u),\ u\le\infty\right\}\right]\mathrm{d}x\right], \tag{3.7}$$

where

$$c_{s1}=\int_0^\infty (\rho(h_1,h_2;0,x))+\rho(h_1,h_2;x,0))\,\mathrm{d}x.$$

The transition probabilities of the corresponding embedded Markov chain can be obtained from Eqs. (3.6) and (3.7) and they are as follows:

$$P\left\{(h_1,h_2)\left(\bar{h}_1,h_2\right)\right\}= \frac{\int_0^\infty\int_0^\infty \bar{F}^{(2)}_{h_2}(x+u)\mathrm{d}F^{(1)}_{\bar{h}_1}(u)\mathrm{d}x+\int_0^\infty\int_0^\infty \bar{F}^{(2)}_{h_2}(u)\mathrm{d}uF^{(1)}_{\bar{h}_1}(x+u)\mathrm{d}x}{\int_0^\infty \bar{F}^{(1)}_{h_1}(x)\mathrm{d}x+\int_0^\infty \bar{F}^{(1)}_{h_2}(x)\mathrm{d}x}, \tag{3.8}$$

$$P\left\{(h_1,h_2)\left(h_1,\bar{h}_2\right)\right\}= \frac{\int_0^\infty\int_0^\infty \bar{F}^{(1)}_{h_1}(x+u)\mathrm{d}F^{(2)}_{\bar{h}_2}(u)\mathrm{d}x+\int_0^\infty\int_0^\infty \bar{F}^{(1)}_{h_1}(u)\mathrm{d}uF^{(2)}_{\bar{h}_2}(x+u)\mathrm{d}x}{\int_0^\infty \bar{F}^{(1)}_{h_1}(x)\mathrm{d}x+\int_0^\infty \bar{F}^{(1)}_{h_2}(x)\mathrm{d}x}. \tag{3.9}$$

The mean sojourn times of the process $\bar{\chi}(t)$ in states from $\mathbf{X}$ are given by

$$
\begin{aligned}
m(h_1,h_2) = & \\
& \int_0^\infty \rho(h_1,h_2;x,0)m(h_1,h_2;x,0)\mathrm{d}x + \int_0^\infty \rho(h_1,h_2;0,x)m(h_1,h_2;0,x)\mathrm{d}x, \\
& = c_{s0}\left(\int_0^\infty\int_0^\infty \bar{F}_{h_1}^{(1)}(y)\bar{F}_{h_2}^{(2)}(x+y)\mathrm{d}y\mathrm{d}x + \int_0^\infty\int_0^\infty \bar{F}_{h_1}^{(1)}(x+y)\bar{F}_{h_2}^{(2)}(y)\mathrm{d}y\mathrm{d}x\right).
\end{aligned}
\tag{3.10}
$$

Let us define the function $f(w)$, where $w \in \mathbf{W} = \Theta \times [0, V]$ as follows,

$$
f(w) := \begin{cases} f_0, & \text{if } \ w = \{1, v\},\ 0 < v \le V; \\ f_1, & \text{if } \ w = \{2, v\},\ 0 < v \le V; \\ f_0 + f_1, & \text{if } \ w = \{3, v\},\ 0 < v \le V; \\ 0 & \qquad\quad \text{in other cases.} \end{cases} \tag{3.11}
$$

This is the productivity of the system.

Let us denote $I(T)$ the amount of information delivered to customers $S_1$ and $S_2$, in a time interval $[0, T]$. Let us consider the joint stochastic process with a two-dimensional phase space $\xi(t) = (\bar{\chi}(t), \bar{v}(t))$.

Then, we can state the following equality

$$
K = \lim_{T\to\infty} \frac{I(T)}{T} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f\left(\xi(t)\right)\,\mathrm{d}t. \tag{3.12}
$$

It follows from ergodic theory [21] that if the process $\xi(t)$ has a stationary distribution $\rho(\cdot)$, then

$$
\lim_{T\to\infty} \frac{1}{T}\int_0^T f\left(\xi(t)\right)\,\mathrm{d}t = \int_{\mathbf{W}} f(w)\mathrm{d}\rho(w). \tag{3.13}
$$

Hence, by using Eq. (3.12) we obtain

$$
K = \int_{\mathbf{W}} f(w)\mathrm{d}\rho(w) = \int_{\mathbf{W}} f(w)\rho(\mathrm{d}w). \tag{3.14}
$$

In summary, by using the merging algorithm, the random evolution $v(t)$ in the semi-Markov medium $\chi(t)$ can be reduced to the Markov evolution $\bar{v}(t)$ in the Markov medium $\bar{\chi}(t)$. So, as an example we consider an evolution in a Markov medium.

## 3.3 Markov Mathematical Model

Let us introduce the following stochastic process $\{\bar{\chi}(t)\}$ where

$$\bar{\chi}(t) = \begin{cases} 0, & \text{if no customer is active;} \\ 1, & \text{if customer } S_1 \text{ is active;} \\ 2, & \text{if customer } S_2 \text{ is active;} \\ 3, & \text{if customers } S_1 \text{ and } S_2 \text{ are active.} \end{cases}$$

The stochastic process $\bar{\chi}(t)$ is a Markov process on the phase space (or states) $\Theta = \{0, 1, 2, 3\}$. Hence, the generating operator (or matrix) of $\bar{\chi}(t)$ can be written as [21]

$$Q = q[P - I] = \begin{bmatrix} -(\lambda_0 + \lambda_1) & \lambda_0 & \lambda_1 & 0 \\ \mu_0 & -(\mu_0 + \lambda_1) & 0 & \lambda_1 \\ \mu_1 & 0 & -(\mu_1 + \lambda_0) & \lambda_0 \\ 0 & \mu_1 & \mu_0 & -(\mu_0 + \mu_1) \end{bmatrix},$$

where $q = [q_i \delta_{ij}; i, j \in \{0, 1, 2, 3\}]$ is a diagonal matrix of sojourn times intensities of different states and $q_0 = \lambda_1 + \lambda_0$, $q_1 = \lambda_1 + \mu_0$, $q_2 = \lambda_0 + \mu_1$, and $q_3 = \mu_1 + \mu_0$. Here, as usual, the Kronecker's delta is defined as

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

We should notice that $q_\theta = (m(h_1, h_2))^{-1}$, with the equivalence $(h_1 h_2) = \{00, 01, 10, 11\} \Leftrightarrow \{0, 1, 2, 3\} = \Theta \ni \theta$.

The elements of the $P$ matrix are the transition probabilities of the Markov chain embedded in the Markov process $\bar{\chi}(t)$, i.e.,

$$P = \begin{bmatrix} 0 & \dfrac{\lambda_0}{\lambda_1 + \lambda_0} & \dfrac{\lambda_1}{\lambda_1 + \lambda_0} & 0 \\ \dfrac{\mu_0}{\lambda_1 + \mu_0} & 0 & 0 & \dfrac{\lambda_1}{\lambda_1 + \mu_0} \\ \dfrac{\mu_1}{\mu_1 + \lambda_0} & 0 & 0 & \dfrac{\lambda_0}{\mu_1 + \lambda_0} \\ 0 & \dfrac{\mu_1}{\mu_0 + \mu_1} & \dfrac{\mu_0}{\mu_0 + \mu_1} & \mu_0 \end{bmatrix}.$$

Consider a function $\bar{C}(w)$ on the space $\mathbf{W} = \{0, 1, 2, 3\} \times [0, V]$ defined as

$$\bar{C}(w) = \begin{cases} F & w = \{0, v\}, \quad 0 < v < V; \\ F - f_0 & w = \{1, v\}, \quad 0 < v < V; \\ F - f_1 & w = \{2, v\}, \quad 0 < v < V; \\ F - (f_0 + f_1) & w = \{3, v\}, \quad 0 < v < V; \\ 0 & \text{in other cases.} \end{cases} \tag{3.15}$$

Denote by $\bar{v}(t)$ the amount of information in the buffer at time $t$. It is easily verified that $\bar{v}(t)$ satisfies the following equation

$$\frac{\mathrm{d}\bar{v}(t)}{\mathrm{d}t} = \bar{C}\left(\bar{\chi}(t), \bar{v}(t)\right), \tag{3.16}$$

with the initial condition $\bar{v}(0) = \bar{v}_0 \in [0, V]$. Eq. (3.16) determines the random evolution of the system in the Markov medium $\bar{\chi}(t)$ [17].

Assume now the joint stochastic process with a two-dimensional phase space $\xi = (\bar{\chi}(t), \bar{v}(t))$. Then, the parameter $K$ can be calculated from the stationary distribution $\rho$ of the process $\xi(t)$, as it is shown in Eqs. (3.12), (3.13) and (3.14).

## 3.4 Stationary Distribution and Stationary Efficiency

The sojourn time distribution functions, say $F_\theta(t)$, have the following form for the different states: $F_0(t) = 1 - e^{-(\lambda_1+\lambda_0)t}$, $F_1(t) = 1 - e^{-(\lambda_1+\mu_0)t}$, $F_2(t) = 1 - e^{-(\mu_1+\lambda_0)t}$, and $F_3(t) = 1 - e^{-(\mu_1+\mu_0)t}$. Now, denote as $f_\theta(t) = \frac{\mathrm{d}F_\theta(t)}{\mathrm{d}t}$ and $r_\theta = \frac{f_\theta(t)}{1-F_\theta(t)}$ for all $\theta \in \Theta$, i.e., $r_0 = \lambda_1 + \lambda_0$, $r_1 = \lambda_1 + \mu_0$, $r_2 = \mu_1 + \lambda_0$, and $r_3 = \mu_1 + \mu_0$. Then, the two component process $\xi(t) = (\bar{\chi}(t), \bar{v}(t))$ is a Markov process with the generator [17, 10]

$$A\varphi(\theta, \bar{v}) = \bar{C}(\theta, \bar{v})\frac{\partial}{\partial \bar{v}}\varphi(\theta, \bar{v}) + r_\theta[P\varphi(\theta, \bar{v}) - \varphi(\theta, \bar{v})],$$

where $P\varphi(\theta, \bar{v}) = \sum_{y\in\Theta} p_{\theta y}\varphi(y, \bar{v})$, or equivalently,

$$A\varphi(\theta, \bar{v}) = \bar{C}(\theta, \bar{v})\frac{\partial}{\partial \bar{v}}\varphi(\theta, \bar{v}) + Q\varphi(\theta, \bar{v}),$$

where $Q = r[P - I]$.

Denote as $\rho$ the stationary distribution of the process $\xi(t)$. Then, for every function $\varphi(\cdot)$ belonging to the domain of the operator $A$ we have

$$\int_{\mathbf{W}} A\varphi(w)\rho(\mathrm{d}w) = 0. \tag{3.17}$$

The analysis of the process $\xi(t)$ properties leads up to the conclusion that, for the case $\max(f_0, f_1) < F < f_1 + f_0$, the stationary distribution $\rho$ has atoms at points $(3, 0)$, $(0, V)$, $(1, V)$, and $(2, V)$. We denote them as $\rho[3, 0]$, $\rho[0, V]$, $\rho[1, V]$ and $\rho[2, V]$. We denote the continuous part of $\rho$ as $\rho(\theta, v)$.

Let us write Eq. (3.17) in more detail for the case, $\max(f_0, f_1) < F < f_1 + f_0$ as follows

$$\int_{\mathbf{W}} A\varphi(w)\rho(\mathrm{d}w)$$

$$
\begin{aligned}
= & \int_{0+}^{V-} \left\{ \left[ F\frac{\partial}{\partial v}\varphi(0,v) - (\lambda_0+\lambda_1)\varphi(0,v) + \lambda_0\varphi(1,v) + \lambda_1\varphi(2,v) \right] \rho(0,v) \right. \\
& + \left[ (F-f_0)\frac{\partial}{\partial v}\varphi(1,v) + \mu_0\varphi(0,v) - (\mu_0+\lambda_1)\varphi(1,v) + \lambda_1\varphi(3,v) \right] \rho(1,v) \\
& + \left[ (F-f_1)\frac{\partial}{\partial v}\varphi(2,v) + \mu_1\varphi(0,v) - (\mu_1+\lambda_0)\varphi(2,v) + \lambda_0\varphi(3,v) \right] \rho(2,v) \\
& \left. + \left[ (F-f_0-f_1)\frac{\partial}{\partial v}\varphi(3,v) + \mu_1\varphi(1,v) + \mu_0\varphi(2,v) - (\mu_0+\mu_1)\varphi(3,v) \right] \rho(3,v) \right\} \mathrm{d}v \\
& + \left[ -(\lambda_0+\lambda_1)\varphi(0,V) + \lambda_0\varphi(1,V) + \lambda_1\varphi(2,V) \right] \rho[0,V] \\
& + \left[ \mu_0\varphi(0,V) - (\mu_0+\lambda_1)\varphi(1,V) + \lambda_1\varphi(3,V) \right] \rho[1,V] \\
& + \left[ \mu_1\varphi(0,V) - (\mu_1+\lambda_0)\varphi(2,V) + \lambda_0\varphi(3,V) \right] \rho[2,V] \\
& + \left[ \mu_1\varphi(1,0) + \mu_0\varphi(2,0) - (\mu_0+\mu_1)\varphi(3,0) \right] \rho[3,0] = 0. \qquad (3.18)
\end{aligned}
$$

Let $A^*$ be the conjugate or adjoint operator of $A$. Then, by changing the order of integration in Eq. (3.18) we can obtain the following expressions for the continuous part of $A^*\rho$

$$
\left\{
\begin{aligned}
-F\frac{\partial}{\partial v}\rho(0,v) - (\lambda_0+\lambda_1)\rho(0,v) + \mu_0\rho(1,v) + \mu_1\rho(2,v) &= 0 \\
-(F-f_0)\frac{\partial}{\partial v}\rho(1,v) + \lambda_0\rho(0,v) - (\mu_0+\lambda_1)\rho(1,v) + \mu_1\rho(3,v) &= 0 \\
-(F-f_1)\frac{\partial}{\partial v}\rho(2,v) + \lambda_1\rho(0,v) - (\mu_1+\lambda_0)\rho(2,v) + \mu_0\rho(3,v) &= 0 \\
-(F-f_0-f_1)\frac{\partial}{\partial v}\rho(3,v) + \lambda_1\rho(1,v) + \lambda_0\rho(2,v) - (\mu_0+\mu_1)\rho(3,v) &= 0
\end{aligned}
\right. . \qquad (3.19)
$$

The expressions for the atoms for the case $\max(f_1, f_0) < F < f_1 + f_0$ are given by

$$
\left\{
\begin{aligned}
-F\rho(0,0+) &= 0 \\
-(F-f_0)\rho(1,0+) + \mu_1\rho[3,0] &= 0 \\
-(F-f_1)\rho(2,0+) + \mu_0\rho[3,0] &= 0 \\
-(F-f_0-f_1)\rho(3,0+) - (\mu_0+\mu_1)\rho[3,0] &= 0
\end{aligned}
\right. , \qquad (3.20)
$$

and

$$
\left\{
\begin{aligned}
F\rho(0,V-) - (\lambda_0+\lambda_1)\rho[0,V] + \mu_0\rho[1,V] + \mu_1\rho[2,V] &= 0 \\
(F-f_0)\rho(1,V-) + \lambda_0\rho[0,V] - (\mu_0+\lambda_1)\rho[1,V] &= 0 \\
(F-f_1)\rho(2,V-) + \lambda_1\rho[0,V] - (\mu_1+\lambda_0)\rho[2,V] &= 0 \\
(F-f_0-f_1)\rho(3,V-) + \lambda_1\rho[1,V] + \lambda_0\rho[2,V] &= 0
\end{aligned}
\right. . \qquad (3.21)
$$

In these equations we have defined the notation

$$\rho(\theta, 0+) := \lim_{v \downarrow 0} \rho(\theta, v),$$

and

$$\rho(\theta, V-) := \lim_{v \uparrow V} \rho(\theta, v).$$

It follows from Eqs. (3.19) that

$$F\rho(0,v) + (F-f_0)\rho(1,v) + (F-f_1)\rho(2,v) + (F-f_0-f_1)\rho(3,v) = c = \text{constant} \quad (3.22)$$

The constant $c$ can be proved to be equal to 0 from Eqs. (3.20) and (3.21).

Using Eqs. (3.19) we obtain

$$\begin{aligned}
\frac{\partial^3}{\partial v^3}\rho(0,v) &= \frac{A_0}{D_0}\frac{\partial^2}{\partial v^2}\rho(0,v) + \frac{B_0}{D_0}\frac{\partial}{\partial v}\rho(0,v) + \frac{G_0}{D_0}\rho(0,v), && (3.23)\\
\rho(1,v) &= \frac{A_1}{D_1}\frac{\partial^2}{\partial v^2}\rho(0,v) + \frac{B_1}{D_1}\frac{\partial}{\partial v}\rho(0,v) + \frac{G_1}{D_1}\rho(0,v), && (3.24)\\
\rho(2,v) &= \frac{A_2}{D_2}\frac{\partial}{\partial v}\rho(0,v) + \frac{B_2}{D_2}\rho(0,v) + \frac{G_2}{D_2}\rho(1,v), && (3.25)\\
\rho(3,v) &= \frac{A_3}{D_3}\rho(0,v) + \frac{B_3}{D_3}\rho(1,v) + \frac{G_3}{D_3}\rho(2,v), && (3.26)
\end{aligned}$$

where

$$\begin{aligned}
A_0 &= F^3\lambda_1 + (F-f_0)\Big(F^2\{2(\lambda_0+\mu_0+\mu_1)+\lambda_1\} - F\{f_0(2\lambda_0+\lambda_1+\mu_1)\\
&\quad + f_1[2(\lambda_1+\mu_0+\mu_1)+3\lambda_0]\} + \lambda_0 f_1^2 + \lambda_0 f_0 f_1\Big) - (F-f_1)\Big(F[\lambda_1 f_0\\
&\quad + f_1(2\lambda_1+\mu_0)] - \lambda_1 f_0 f_1\Big) - \lambda_1 f_1 f_0^2,\\
B_0 &= -F\Big(\lambda_0\mu_0 f_0 + f_1[\lambda_1(\lambda_0+\mu_1)+\mu_0\mu_1]\Big) + (F-f_0)\Big(F\{\mu_1[2(\lambda_1+\mu_0)\\
&\quad +3\lambda_0+\mu_1] + 3\lambda_0\lambda_1\} - \lambda_0 f_0(\lambda_0+\lambda_1+\mu_1) - f_1(2\lambda_0(\lambda_1+\mu_1)+\lambda_1\mu_1)\Big)\\
&\quad +(F-f_1)\Big(F[\lambda_1^2+(\lambda_0+\mu_0)^2+\mu_0(3\lambda_1+\mu_1)] - f_0[\lambda_0(\lambda_0+\mu_0)\\
&\quad +\lambda_1(\lambda_1+2\mu_0)] - \lambda_1 f_1(\lambda_0+\lambda_1+\mu_0)\Big),\\
G_0 &= (\lambda_0+\lambda_1+\mu_0+\mu_1)(\lambda_1+\mu_1)(\lambda_0+\mu_0)\left(F - \frac{\lambda_1 f_1}{\mu_1+\lambda_1} - \frac{\lambda_0 f_0}{\lambda_0+\mu_0}\right),\\
D_0 &= -F(F-f_1)(F-f_0)(F-f_1-f_0),
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= F(F-f_1)(F-f_0)(F-f_0-f_1),\\
B_1 &= (F-f_0)\{F^2(\lambda_1+\mu_1) - F[f_0(2\lambda_0+\lambda_1+\mu_1)+f_1(\lambda_1+\mu_1)] + f_0 f_1(\lambda_0+\lambda_1)\}\\
&\quad +(F-f_1)\{F^2(2\lambda_0+2\mu_0) - F[f_0(2\lambda_0+\mu_0)+f_1(\lambda_0+\lambda_1+\mu_0)] + f_0 f_1(\lambda_0+\lambda_1)\},\\
G_1 &= (F-f_0)\{F[\lambda_0^2+\mu_0(\mu_1+\lambda_1)] - \lambda_0 f_0(\lambda_0+\lambda_1+\mu_1) - f_1(\lambda_0^2+\lambda_1\mu_0)\}\\
&\quad +(F-f_1)\{F[\lambda_0(\mu_1+\lambda_1)+\mu_0(\lambda_0+\lambda_1+\mu_1)] - \lambda_0 f_0(\lambda_1+\mu_1) - \lambda_1\mu_0 f_1\},\\
D_1 &= \mu_0\{-(F-f_0)(\mu_1 F+\lambda_0 f_0) + (F-f_1)[F(\lambda_0-\lambda_1+\mu_0) - f_0(\lambda_0-\lambda_1) + f_1\lambda_1]\},
\end{aligned}$$

as well as, $A_2 = F$, $B_2 = \lambda_0 + \lambda_1$, $G_2 = -\mu_0$, $D_2 = \mu_1$, and, $A_3 = -F$, $B_3 = -(F - f_0)$, $G_3 = -(F - f_1)$, and $D_3 = F - f_0 - f_1$.

Solving Eq. (3.23) we obtain

$$\rho(0, v) = c_1 e^{\delta_1 v} + c_2 e^{\delta_2 v} + c_3 e^{\delta_3 v}, \tag{3.27}$$

where $\delta_1$, $\delta_2$ and $\delta_3$ are the roots of the polynomial

$$x^3 + \frac{A_0}{D_0} x^2 + \frac{B_0}{D_0} x + \frac{G_0}{D_0} = 0.$$

Using expression (3.27) into (3.24) we obtain

$$\rho(1, v) = k_{11} c_1 e^{\delta_1 v} + k_{12} c_2 e^{\delta_2 v} + k_{13} c_3 e^{\delta_3 v}, \tag{3.28}$$

where

$$\begin{aligned}
k_{11} &= \frac{G_1 + B_1\delta_1 + A_1\delta_1^2}{D_1}, \\
k_{12} &= \frac{G_1 + B_1\delta_2 + A_1\delta_2^2}{D_1}, \\
k_{13} &= \frac{G_1 + B_1\delta_3 + A_1\delta_3^2}{D_1}.
\end{aligned}$$

Also, using expression (3.28) into (3.25) we obtain

$$\rho(2, v) = k_{21} c_1 e^{\delta_1 v} + k_{22} c_2 e^{\delta_2 v} + k_{23} c_3 e^{\delta_3 v}, \tag{3.29}$$

where

$$\begin{aligned}
k_{21} &= \frac{B_2D_1 + G_2G_1 + (G_2B_1 + A_2D_1)\delta_1 + G_2A_1\delta_1^2}{D_1D_2}, \\
k_{22} &= \frac{B_2D_1 + G_2G_1 + (G_2B_1 + A_2D_1)\delta_2 + G_2A_1\delta_2^2}{D_1D_2}, \\
k_{23} &= \frac{B_2D_1 + G_2G_1 + (G_2B_1 + A_2D_1)\delta_3 + G_2A_1\delta_3^2}{D_1D_2}.
\end{aligned}$$

Using expression (3.29) into (3.26) we obtain

$$\rho(3, v) = k_{31} c_1 e^{\delta_1 v} + k_{32} c_2 e^{\delta_2 v} + k_{33} c_3 e^{\delta_3 v}. \tag{3.30}$$

where

$$\begin{aligned}
k_{31} &= \frac{A_3}{D_3} + B_3\frac{G_1 + B_1\delta_1 + A_1\delta_1^2}{D_1D_3} + G_3\frac{B_2D_1 + G_2G_1 + (G_2B_1 + A_2D_1)\delta_1 + G_2A_1\delta_1^2}{D_1D_2D_3}, \\
k_{32} &= \frac{A_3}{D_3} + B_3\frac{G_1 + B_1\delta_2 + A_1\delta_2^2}{D_1D_3} + G_3\frac{B_2D_1 + G_2G_1 + (A_2D_1 + G_2B_1)\delta_2 + G_2A_1\delta_2^2}{D_1D_2D_3}, \\
k_{33} &= \frac{A_3}{D_3} + B_3\frac{G_1 + B_1\delta_3 + A_1\delta_3^2}{D_1D_3} + G_3\frac{G_2G_1 + B_2D_1 + (A_2D_1 + G_2B_1)\delta_3 + G_2A_1\delta_3^2}{D_1D_2D_3}.
\end{aligned}$$

In Eqs. (3.27), (3.28), (3.29) and (3.30) the constants $k_{ij},\ i,j = 1,2,3$ are known. However $c_1$, $c_2$ and $c_3$ need to be found. On the case where $\max(f_1, f_0) < F < f_1 + f_0$ we obtain from Eqs. (3.20) that $\rho(0, 0+) = 0$, and then we obtain

$$c_3 = -c_1 - c_2. \tag{3.31}$$

Using the second and third expressions from Eqs. (3.20) we can eliminate $\rho[3, 0]$ and obtain

$$c_2 = c_{12}c_1 \quad \text{and} \quad c_3 = c_{13}c_1,$$

where

$$\begin{aligned} c_{12} &= -\frac{F(\mu_0(-k_{11}+k_{13})+\mu_1(k_{21}-k_{23}))+f_0\mu_0(k_{11}-k_{13})+f_1\mu_1(-k_{21}+k_{23})}{F(\mu_0(-k_{12}+k_{13})+\mu_1(k_{22}-k_{23}))+f_0\mu_0(k_{12}-k_{13})+f_1\mu_1(-k_{22}+k_{23})}, \\ c_{13} &= \frac{F(\mu_0(-k_{11}+k_{12})+\mu_1(k_{21}-k_{22}))+f_0\mu_0(k_{11}-k_{12})+f_1\mu_1(-k_{21}+k_{22})}{F(\mu_0(-k_{12}+k_{13})+\mu_1(k_{22}-k_{23}))+f_0\mu_0(k_{12}-k_{13})+f_1\mu_1(-k_{22}+k_{23})}. \end{aligned}$$

The constant $c_1$ can be calculated from the normalization condition

$$\int_{\mathbf{W}} \rho(\mathrm{d}w) = 1. \tag{3.32}$$

$c_1$ is a factor of every term in the continuous part of the stationary distribution, $\rho(\cdot)$. Now, we need to find an expression for the atoms of the stationary distribution in terms of the solutions in the continuous part. Using the last expression of Eqs. (3.20) we can see that

$$\rho[3, 0] = \frac{-(F - f_0 - f_1)}{\mu_0 + \mu_1}\rho(3, 0+), \tag{3.33}$$

for $\max(f_1, f_0) < F < f_1 + f_0$.

Let us say $\rho[3, 0] = K_{33}\rho(3, 0+)$, where the constant $K_{31}$ is known. For the rest of the atoms we need to solve the system of equations (3.21). We can use the first, second and third expressions of (3.21) to find

$$\begin{aligned} \rho[0, V] &= \frac{(\lambda_0+\mu_1)(\lambda_1+\mu_0)F\rho(0, V-)+\mu_0(\lambda_0+\mu_1)(F-f_0)\rho(1, V-)}{\lambda_0\lambda_1(\lambda_0+\lambda_1+\mu_0+\mu_1)}, \\ &\quad +\frac{\mu_1(\lambda_1+\mu_0)(F-f_1)\rho(2, V-)}{\lambda_0\lambda_1(\lambda_0+\lambda_1+\mu_0+\mu_1)}, \end{aligned} \tag{3.34}$$

$$\begin{aligned} \rho[1, V] &= \frac{(\lambda_0+\mu_1)F\rho(0, V-)+(\lambda_0+\lambda_1+\mu_1)(F-f_0)\rho(1, V-)}{\lambda_1(\lambda_0+\lambda_1+\mu_0+\mu_1)}, \\ &\quad +\frac{\mu_1(F-f_1)\rho(2, V-)}{\lambda_1(\lambda_0+\lambda_1+\mu_0+\mu_1)}, \end{aligned} \tag{3.35}$$

$$\rho[2,V] = \frac{(\lambda_1+\mu_0)F\rho(0,V-)+\mu_0(F-f_0)\rho(1,V-)}{\lambda_0(\lambda_0+\lambda_1+\mu_0+\mu_1)} + \frac{(\lambda_0+\lambda_1+\mu_0)(F-f_1)\rho(2,V-)}{\lambda_0(\lambda_0+\lambda_1+\mu_0+\mu_1)}. \tag{3.36}$$

As a check, we can substitute these results into the last expression of Eqs. (3.21). We obtain

$$F\rho(0,V)+(F-f_0)\rho(1,V)+(F-f_1)\rho(2,V)+(F-f_0-f_1)\rho(3,V)=0,$$

which also comes after Eq. (3.22). So, these results seem to be correct.

Just as we did for Eq. (3.33), let us say,

$$\begin{aligned}
\rho[0,V] &= K_{00}\rho(0,V-)+K_{01}\rho(1,V-)+K_{02}\rho(2,V-),\\
\rho[1,V] &= K_{10}\rho(0,V-)+K_{11}\rho(1,V-)+K_{12}\rho(2,V-),\\
\rho[2,V] &= K_{20}\rho(0,V-)+K_{21}\rho(1,V-)+K_{22}\rho(2,V-),
\end{aligned}$$

where the constants $K_{ij}$, $i,j=0,1,2$ are known.

Now, we can use Eq. (3.32) to find $c_1$.

For the case $\max(f_0,f_1) < F < f_0+f_1$ we have

$$\int_0^V \{\rho(0,v)+\rho(1,v)+\rho(2,v)+\rho(3,v)\}\,\mathrm{d}v + \rho[0,V]+\rho[1,V]+\rho[2,V]+\rho[3,0]=1. \tag{3.37}$$

Writing Eq. (3.37) in more detail and solving for $c_1$ we have

$$\begin{aligned}
c_1^{-1} = {} & \frac{(1+k_{11}+k_{21}+k_{31})(e^{\delta_1 V}-1)}{\delta_1} + \frac{c_{12}(1+k_{12}+k_{22}+k_{32})(e^{\delta_2 V}-1)}{\delta_2}\\
&+\frac{c_{13}(1+k_{13}+k_{23}+k_{33})(e^{\delta_3 V}-1)}{\delta_3} + K_{33}(k_{31}+c_{12}k_{32}+c_{13}k_{33})\\
&+[K_{00}+K_{10}+K_{20}+k_{11}(K_{01}+K_{11}+K_{21})+k_{21}(K_{02}+K_{12}+K_{22})]e^{\delta_1 V}\\
&+c_{12}[K_{00}+K_{10}+K_{20}+k_{12}(K_{01}+K_{11}+K_{21})+k_{22}(K_{02}+K_{12}+K_{22})]e^{\delta_2 V}\\
&+c_{13}[K_{00}+K_{10}+K_{20}+k_{13}(K_{01}+K_{11}+K_{21})+k_{23}(K_{02}+K_{12}+K_{22})]e^{\delta_3 V}.
\end{aligned}$$

## 3.5 Numerical Results

With the expression of $c_1$, it is now possible to evaluate the complete expression of the stationary distribution. For example, on the case $f_0 = 3/2$, $f_1 = 1$, $\lambda_0 = 3/10$, $\lambda_1 = 2/10$, $\mu_0 = 1/10$, $\mu_1 = 1/15$, $V = 100$, $F = 7/4$ ($\max(f_0, f_1) < F < f_0 + f_1$) we obtain from Eq. (3.27),

$$\rho(0,v) = \frac{\sqrt{2}}{5} \frac{-2 + e^{16(-4+\sqrt{2})v/105} + e^{-16(4+\sqrt{2})v/105}}{-676\sqrt{2} + (2\sqrt{2}-1)e^{-(320\sqrt{2}+1280)/21} + (2\sqrt{2}+1)e^{(320\sqrt{2}-1280)/21}}.$$

We can define the following stationary distribution

$$\rho(v) = \begin{cases} \rho(0,v) + \rho(1,v) + \rho(2,v), & 0 < v < V \\ \rho[3,0] & v = 0 \\ \rho[0,V] + \rho[1,V] + \rho[2,V] & v = V \end{cases}.$$

We can plot both this analytical result and simulation results to illustrate some common cases, see Figure 3.2. On these plots $F = 7/4$, $F = 2$ are considered, and

$$F = \frac{f_0\lambda_0}{\lambda_0 + \mu_0} + \frac{f_1\lambda_1}{\lambda_1 + \mu_1} = \frac{15}{8}, \tag{3.38}$$

i.e., when $F$ is equal to the expected average demand of the two customers.

Some sort of perturbation close to $v = 100$ can be noticed in the curves from the simulation in Figure 3.2. That perturbation comes from the system functionality for the case $\max(f_0, f_1) < F < f_0 + f_1$, i.e., if any of the two customers is active, the level of the buffer can be increased. However, the filling rate is expected to stop when the level of the buffer reaches its maximum. The result is that, if any of the two customers is active, the level of the buffer can be increased to its maximum, then the filling rate is turned off and the level starts to decrease. At any moment that the level is sensed not to be at its maximum again, then the filling rate is restored. On that scenario, for the time that this single customer is active, the level in the buffer swings between its maximum and some close point below. This is the reason why some small peaks can be observed in the computer simulation at some point close to $V$.

For the sake of the analytical solution, the atoms $\rho[1,V]$ and $\rho[2,V]$ were considered as a more steady approximation of the real system behavior. It is worth saying that this behavior is not observed for other choices of $F$. For example, if we choose $F < \min(f_1, f_0)$ we obtain the exact analytical solution.

Now, we can recall the function $f(w)$ and show some plots regarding the efficiency parameter $K$ from Eq. (3.14). Let us write this expression in more detail for this

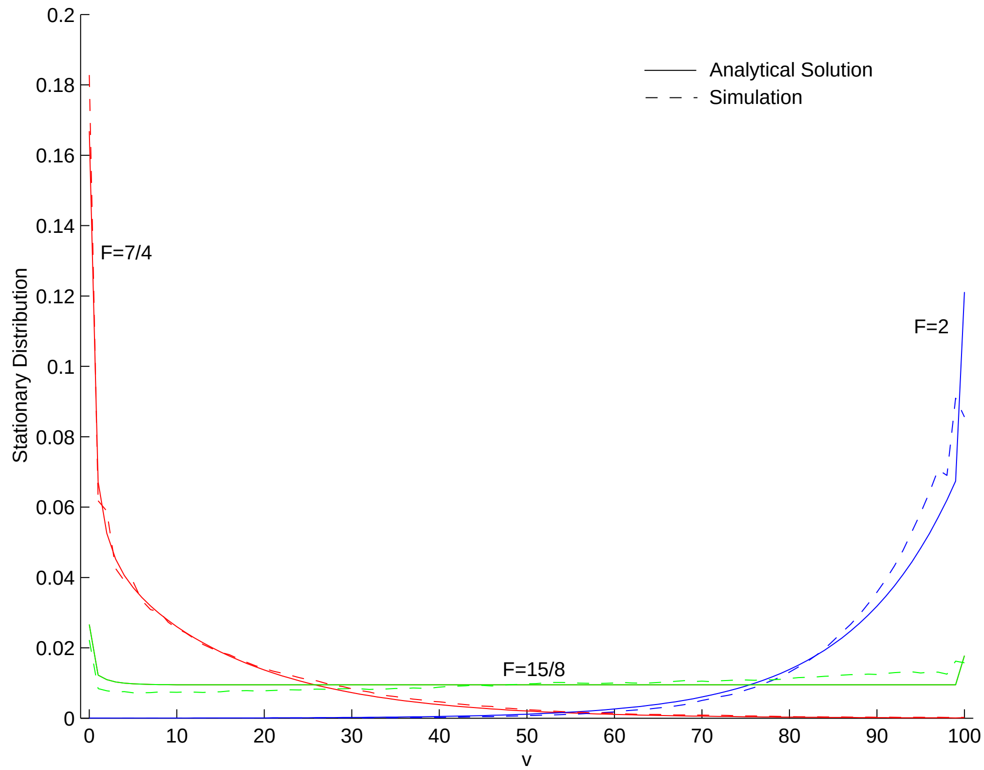


Figure 3.2: Stationary distribution of the buffer for the case $\max(f_1, f_2) < F < f_1 + f_2$ for three different values of $F$.

system with $\max(f_0, f_1) < F < f_0 + f_1$,

$$\begin{aligned} K &= \int_0^V \{f_0\rho(1,v) + f_1\rho(2,v) + (f_0+f_1)\rho(3,v)\}\,\mathrm{d}v + f_0\rho[1,V] + f_1\rho[2,V], \\ &= c_1\left(\frac{e^{\delta_1 V}-1}{\delta_1}\right)(f_0 + f_1K_{11} + f_2K_{21} + f_3K_{31}) \\ &\quad + c_2\left(\frac{e^{\delta_2 V}-1}{\delta_2}\right)(f_0 + f_1K_{11} + f_2K_{21} + f_3K_{31}) \\ &\quad + c_3\left(\frac{e^{\delta_3 V}-1}{\delta_3}\right)(f_0 + f_1K_{11} + f_2K_{21} + f_3K_{31}). \end{aligned}$$

$K(V)$ is shown in Figure 3.3, i.e., $K$ as a function of the maximum capacity of the buffer, for the same three cases of $F$ from Figure 3.2.

As it can be seen from Figure 3.3, the case of $F$ that satisfies condition in Eq. (3.38) keeps growing while each of the other two $F$-equally-spaced cases converge faster to a different value. Notice also that after some value of $V$ the difference between the two highest values of $K$ is relatively small.

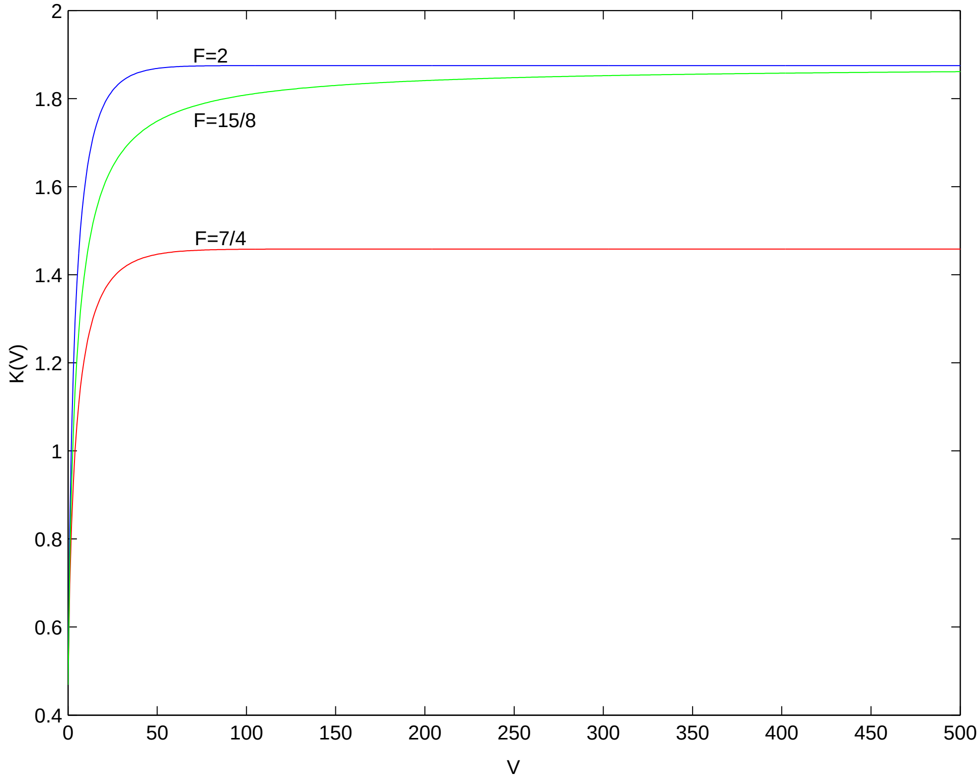


Figure 3.3: Efficiency parameter as a function of the buffer capacity for the case $\max(f_0, f_1) < F < f_1 + f_2$ and three different values of $F$.

It can be proved that in every case

$$\lim_{V\to\infty}\Big[K(V)\Big]_{F \geq \frac{f_1\lambda_1}{\lambda_1+\mu_1}+\frac{f_0\lambda_0}{\lambda_0+\mu_0}} = \frac{f_1\lambda_1}{\lambda_1+\mu_1}+\frac{f_0\lambda_0}{\lambda_0+\mu_0}. \quad (3.39)$$

In addition, it can be proved that in every case

$$\lim_{V\to\infty}\Big[K(V)\Big]_{F < \frac{f_1\lambda_1}{\lambda_1+\mu_1}+\frac{f_0\lambda_0}{\lambda_0+\mu_0}} < F < \frac{f_1\lambda_1}{\lambda_1+\mu_1}+\frac{f_0\lambda_0}{\lambda_0+\mu_0}. \quad (3.40)$$

That is, if the buffer is big enough, no $F$ larger than the average system demand is required to meet the system maximum efficiency. On the other hand, if the incoming stream $F$ is smaller than the expected long term average system demand, the system efficiency $K(V)$ is even smaller than the incoming stream.

These results are the same even for other choices of $F$ besides $\max(f_0, f_1) < F < f_0 + f_1$. For example, if we choose $F < \min(f_0, f_1)$, we obtain the same results as those on (3.39) and (3.40).

According to Section 2, the *phase merging algorithm* can be used to obtain a random evolution $\bar{v}(t)$ in an approximated Markov environment from an evolution $v(t)$ in a semi-Markov environment. Then, some plots regarding the semi-Markov case can be displayed. As a first example, we can consider a $m$-Erlang scenario where both active and inactive sojourn times are considered with such distribution. This is, for the actual sojourn time distributions we have

$$\begin{aligned}
F_0^{(1)}(u) &= \int_0^u \frac{\lambda_0 e^{-\lambda_0 x}(\lambda_0 x)^{m_0^{(1)}-1}}{(m_0^{(1)}-1)!}\,\mathrm{d}x, \\
F_1^{(1)}(u) &= \int_0^u \frac{\mu_0 e^{-\mu_0 x}(\mu_0 x)^{m_1^{(1)}-1}}{(m_1^{(1)}-1)!}\,\mathrm{d}x, \\
F_0^{(2)}(u) &= \int_0^u \frac{\lambda_1 e^{-\lambda_1 x}(\lambda_1 x)^{m_0^{(2)}-1}}{(m_0^{(2)}-1)!}\,\mathrm{d}x, \\
F_1^{(2)}(u) &= \int_0^u \frac{\mu_1 e^{-\mu_1 x}(\mu_1 x)^{m_1^{(2)}-1}}{(m_1^{(2)}-1)!}\,\mathrm{d}x.
\end{aligned}$$

Here, $m_j^{(i)}$ ($i \in \{1,2\}$, $j \in \{0,1\}$), stands for the number of exponentials that form the $m$-Erlang distribution of the $i$ subsystem in the $j$ state. As we know, if we make $m_j^{(i)} = 1$ we get an exponential distribution.

We use Eqs. (3.8) and (3.9) to calculate the transition probabilities of the embedded Markov chain. We obtain

$$P = \begin{bmatrix}
0 & \frac{m_0^{(2)}\lambda_0}{m_0^{(2)}\lambda_0 + m_0^{(1)}\lambda_1} & \frac{m_0^{(1)}\lambda_1}{m_0^{(2)}\lambda_0 + m_0^{(1)}\lambda_1} & 0 \\
\frac{m_0^{(2)}\mu_0}{m_1^{(1)}\lambda_1 + m_0^{(2)}\mu_0} & 0 & 0 & \frac{m_1^{(1)}\lambda_1}{m_1^{(1)}\lambda_1 + m_0^{(2)}\mu_0} \\
\frac{m_0^{(1)}\mu_1}{m_1^{(2)}\lambda_0 + m_0^{(1)}\mu_1} & 0 & 0 & \frac{m_1^{(2)}\lambda_0}{m_1^{(2)}\lambda_0 + m_0^{(1)}\mu_1} \\
0 & \frac{m_1^{(1)}\mu_1}{m_1^{(2)}\mu_0 + m_1^{(1)}\mu_1} & \frac{m_1^{(2)}\mu_0}{m_1^{(2)}\mu_0 + m_1^{(1)}\mu_1} & 0
\end{bmatrix}.$$

After that, we use Eq. (3.11) to calculate the mean sojourn times and consequently the sojourn time intensities. Let us remember that $q_\theta = (m(h_1,h_2))^{-1}$ with the equivalence $(h_1,h_2) = \{00,01,10,11\} \Leftrightarrow \{0,1,2,3\} = \Theta \ni \theta$. Also let us remember that $q = [q_i\delta_{ij}; i,j \in \{0,1,2,3\}]$ is a diagonal matrix. Then we obtain

$$q = \begin{bmatrix}
\frac{\lambda_0}{m_0^{(1)}} + \frac{\lambda_1}{m_0^{(2)}} & 0 & 0 & 0 \\
0 & \frac{\lambda_1}{m_0^{(2)}} + \frac{\mu_0}{m_1^{(1)}} & 0 & 0 \\
0 & 0 & \frac{\lambda_0}{m_0^{(1)}} + \frac{\mu_1}{m_1^{(2)}} & 0 \\
0 & 0 & 0 & \frac{\mu_0}{m_1^{(1)}} + \frac{\mu_1}{m_1^{(2)}}
\end{bmatrix}.$$

Now, we can calculate the generating operator $Q = q[P - I]$ and solve for the continuous part and atoms of a stationary distribution just as we did in Section 3 for an evolution in a Markov media.

We can consider the inactive sojourn time with an exponential distribution ($m_0^{(i)} = 1$, $i = 1, 2$) and plot several $m$-Erlang cases for the active sojourn time. This example is illustrated in Figures 3.4 and 3.5.

We considered as before

$$F = \frac{f_0 \lambda_0}{\lambda_0 + \mu_0} + \frac{f_1 \lambda_1}{\lambda_1 + \mu_1}, \tag{3.41}$$

$$\max(f_0, f_1) < F < f_0 + f_1. \tag{3.42}$$

We also used $f_0 = 3/2$, $f_1 = 1$, $\lambda_0 = 3/10$, $\lambda_1 = 2/10$, $\mu_0 = 1/10$, $\mu_1 = 1/15$ and $V = 100$.

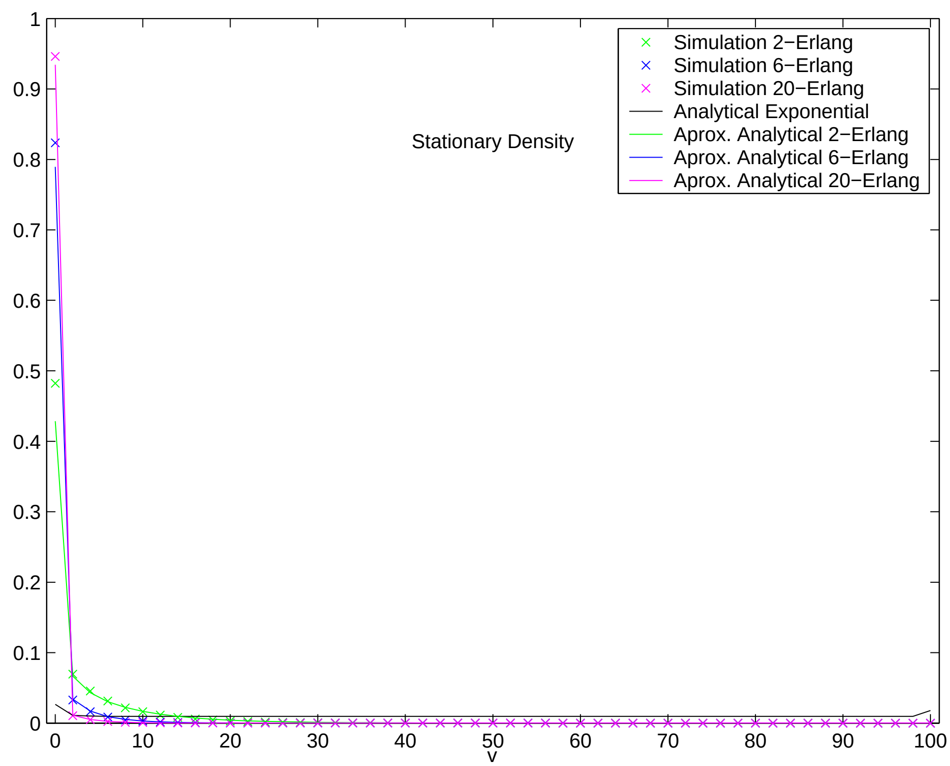


Figure 3.4: Stationary distribution of the buffer with $m$-Erlang distributed active sojourn time and exponentially distributed inactive sojourn time.

In Figures 3.4 and 3.5 we can see a good match between the analytical solutions and simulations for every $m$-Erlang case. In this case we see that the behavior of the curves is modified as the number of exponentials in the $m$-Erlang distributions is increased. We can also observe that for more exponentials in the $m$-Erlang distribution of the active sojourn time the expected value of the active sojourn times for subsystems $S_1$ and $S_2$

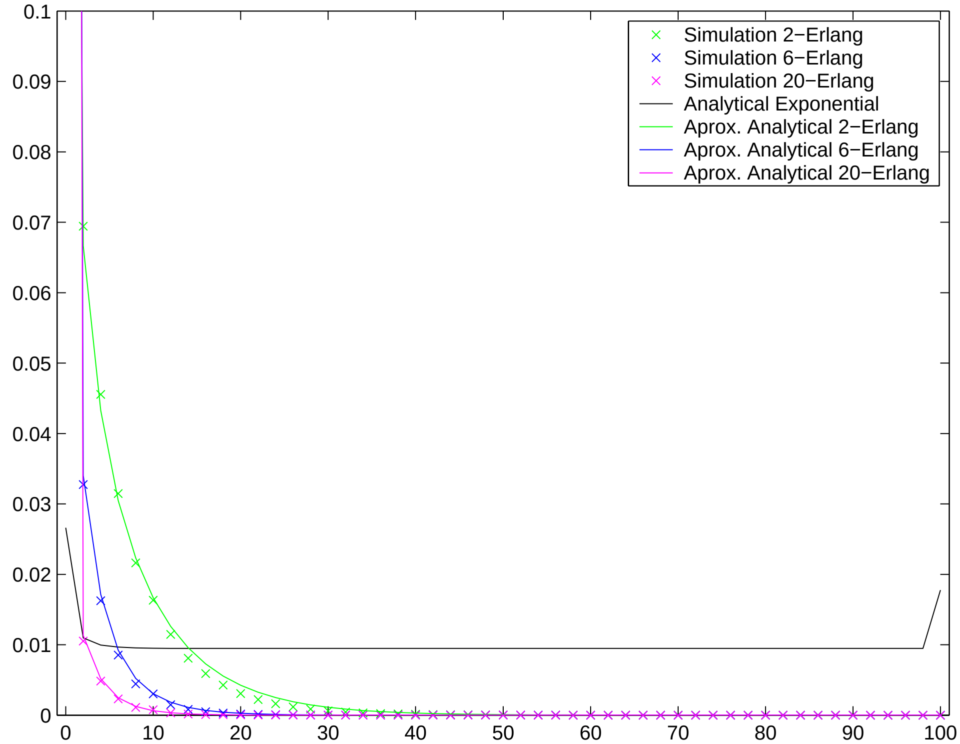


Figure 3.5: A closer view of the stationary distribution of the buffer with $m$-Erlang distributed active sojourn time and exponentially distributed inactive sojourn time.

is longer. Therefore we can see on Figures 3.4 and 3.5 that this causes the stationary distribution to be biased to the empty side of the buffer.

Finally, we can introduce the hyper-exponential semi-Markov case as another example. For this case the distribution of the active and inactive sojourn times were taken as

$$\begin{aligned}
F_0^{(1)}(u) &= 1 - p\exp(-\lambda_0 u) - (1-p)\exp(-\lambda_{0b} u),\\
F_1^{(1)}(u) &= 1 - p\exp(-\mu_0 u) - (1-p)\exp(-\mu_{0b} u),\\
F_0^{(2)}(u) &= 1 - p\exp(-\lambda_1 u) - (1-p)\exp(-\lambda_{1b} u),\\
F_1^{(2)}(u) &= 1 - p\exp(-\mu_1 u) - (1-p)\exp(-\mu_{1b} u).
\end{aligned}$$

The following choices were taken as an instance

$$\begin{aligned}
\lambda_{0b} &= n\lambda_0,\\
\lambda_{1b} &= n\lambda_1,\\
\mu_{0b} &= n\mu_0,
\end{aligned}$$

$$\mu_{1b} \;=\; n\mu_1,$$

$n > 0$. We use Eqs. (3.8) and (3.9) to calculate the transition probabilities of the embedded Markov chain. We obtain

$$P = \begin{bmatrix} 0 & \dfrac{\lambda_0}{\lambda_1+\lambda_0} & \dfrac{\lambda_1}{\lambda_1+\lambda_0} & 0 \\ \dfrac{\mu_0}{\lambda_1+\mu_0} & 0 & 0 & \dfrac{\lambda_1}{\lambda_1+\mu_0} \\ \dfrac{\mu_1}{\mu_1+\lambda_0} & 0 & 0 & \dfrac{\lambda_0}{\mu_1+\lambda_0} \\ 0 & \dfrac{\mu_1}{\mu_0+\mu_1} & \dfrac{\mu_0}{\mu_0+\mu_1} & \mu_0 \end{bmatrix}.$$

This is the same transition probability matrix of the Markov case that does not depend on the choice of $n$.

After that we use Eq. (3.11) to calculate the mean sojourn times and consequently the sojourn time intensities. The result is

$$q = \begin{bmatrix} \dfrac{n(\lambda_0+\lambda_1)}{np+1-p} & 0 & 0 & 0 \\ 0 & \dfrac{n(\lambda_1+\mu_0)}{np+1-p} & 0 & 0 \\ 0 & 0 & \dfrac{n(\lambda_0+\mu_1)}{np+1-p} & 0 \\ 0 & 0 & 0 & \dfrac{n(\mu_0+\mu_1)}{np+1-p} \end{bmatrix}.$$

Now, we can calculate the generating operator $Q = q[P-I]$ and solve for the continuous part and atoms of the stationary distribution just as we did in Section 3 for the evolution in a Markov media.

We can make some plots for this semi-Markov example. In Figure 3.6, the behavior of the approximation can be appreciated along with some plots from simulations for this semi-Markov case. In Figure 3.6 we choose $n = 2$ as an instance. Besides, we took again $f_0 = 3/2$, $f_1 = 1$, $\lambda_0 = 3/10$, $\lambda_1 = 2/10$, $\mu_0 = 1/10$, $\mu_1 = 1/15$ and $V = 100$, as well as $F$ from conditions (3.41) and (3.42).

It can be noticed that we can use the stationary probability density obtained before for the Markov case to obtain the approximated stationary density for these semi-Markov cases. Meaning that for the $m$-Erlang case we may only use the substitutions: $\lambda_0 \to \dfrac{\lambda_0}{m_0^{(1)}}$, $\lambda_1 \to \dfrac{\lambda_1}{m_0^{(2)}}$, $\mu_0 \to \dfrac{\mu_0}{m_1^{(1)}}$ and $\mu_1 \to \dfrac{\mu_1}{m_1^{(2)}}$ directly on the Eqs. (3.27), (3.28), (3.29) and (3.30) as well as in the expressions for the atoms to obtain the approximated stationary density for this semi-Markov case. For the hyper-exponential case we should

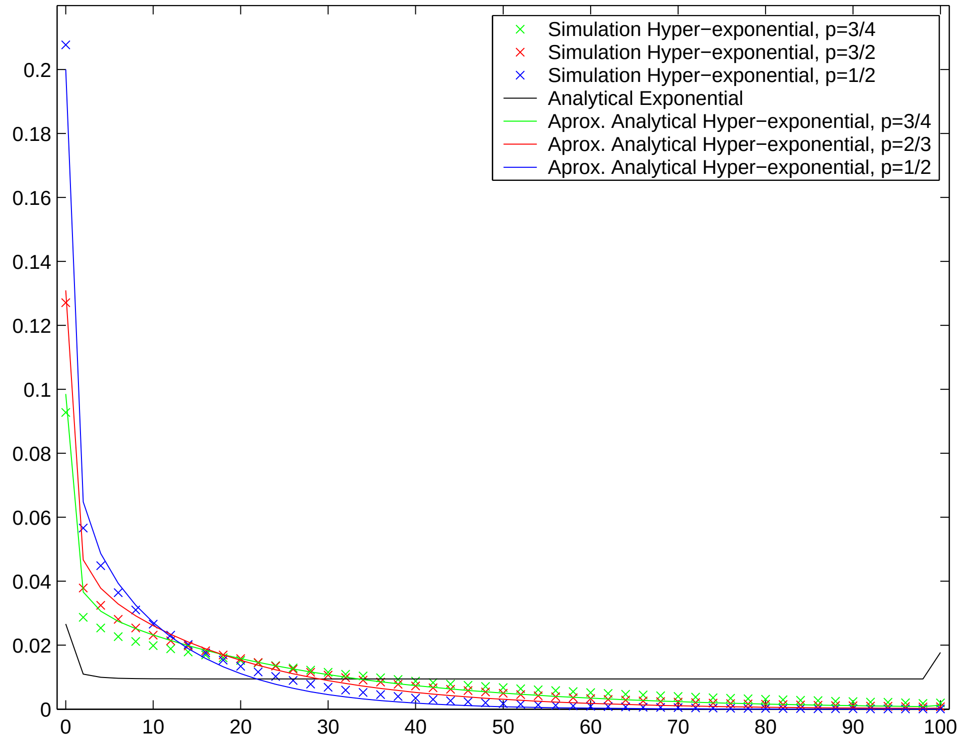


Figure 3.6: Stationary distribution of the Hyper-exponential semi-Markov case for the active and inactive sojourn time.

use the substitutions $\lambda_0 \to \dfrac{\lambda_0}{np+1-p}$, $\lambda_1 \to \dfrac{\lambda_1}{np+1-p}$, $\mu_0 \to \dfrac{\mu_0}{np+1-p}$ and $\mu_1 \to \dfrac{\mu_1}{np+1-p}$.

Also, we can use the same substitutions in the expression (3.38) to obtain the condition that leads to the best usage of the buffer in terms of the stationary efficiency of the system.

# Chapter 4

# Two Equal Customers

## 4.1 Chapter Summary

In this paper we study the stationary efficiency of a system consisting of a finite capacity buffer connected to two equal customers with bursty on-off demands. System functionality is similar to that presented on Chapter 3. Nevertheless, we considered this problem as a previous step before finding a general result for the superposition of any number $N$ of processes. Also, it can be easily proved that the result shown in the previous Chapter is not reducible for the case subsystems $S_1$ and $S_2$ have the same parameters. This regards undefined limits in the solution of stationary probability density the as $f_0 \to f_1$, $\lambda_0 \to \lambda_1$ and $\mu_0 \to \mu_1$. The formulation to this problem is similar to that one shown on Chapter 3, except for the fact that it is necessary to state from the beginning that $S_1$ and $S_2$ have the same parameters. In spite of this consideration, the results of the phase merging algorithm does not show important modifications from those shown on Chapter 3 and then we will use, as well, some of the formulas obtained there.

Since it has already been proved that the PMA is helpful in finding and approximated solution for evolutions in semi-Markov media, we start our formulation in section 2 with the Markov case. Nevertheless, we do consider the semi-Markov formulation in section 4 to show some results that will help us find a general result for the PMA for any number of superposed processes.

The system functionality is as follows:

When active, each customer demands information at a ratio $f$. When both customers are active, information is required at a ratio $2f$. In each of these cases, if the container is empty ($v = 0$), an unproductive situation is considered. When no customer is active, then no product is required. The filling aggregate provides the container with product at a constant ratio $F$. This aggregate is active as long as the amount of information is below the maximum capacity of the buffer ($V$), see Figure 4.1.

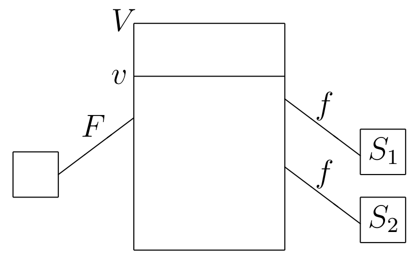


Figure 4.1: A system of two independent random state switching customers and one buffer filled up at a constant rate.

## 4.2 Markov Mathematical Model

Let us introduce the following stochastic process $\{\bar{\chi}(t)\}$ such that

$$\bar{\chi}(t) = \begin{cases} 0, & \text{if no customer is active;} \\ 1, & \text{if } S_1 \text{ is active;} \\ 2, & \text{if } S_2 \text{ is active;} \\ 3, & \text{if customers } S_1 \text{ and } S_2 \text{ are active;} \end{cases}$$

The stochastic process $\bar{\chi}$ is a Markov process on the phase space (or states) $\Theta = \{0, 1, 2, 3\}$. Hence, the generating operator (or matrix) of $\bar{\chi}(t)$ can be written as [21]

$$Q = q[\bar{P} - I] = \begin{bmatrix} -2\lambda & \lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & 0 & \lambda \\ \mu & 0 & -(\lambda+\mu) & \lambda \\ 0 & \mu & \mu & -2\mu \end{bmatrix},$$

where $q = [q_i\delta_{ij}; i, j \in \{0, 1, 2, 3\}]$ is a diagonal matrix of sojourn times intensities of different states and $q_0 = 2\lambda$, $q_1 = \lambda + \mu$, $q_2 = \lambda + \mu$ and $q_3 = 2\mu$. Here, the Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

We should notice that $q_\theta = (m(h_1, h_2))^{-1}$, with the equivalence $(h_1h_2) = \{00, 01, 10, 11\}$ $\Leftrightarrow \{0, 1, 2, 3\} = \Theta \ni \theta$.

The elements of the matrix $\bar{P}$ are the transition probabilities of the Markov chain embedded in the Markov process $\bar{\chi}(t)$, i.e.,

$$\bar{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{\mu}{\lambda+\mu} & 0 & 0 & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & 0 & 0 & \frac{\lambda}{\lambda+\mu} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

If we consider that customers $S_1$ and $S_2$ are equal we can construct a birth-and-death process $\{\chi^{(2)}(t)\}$ as a simplification of process $\bar{\chi}(t)$.

We consider the superposition of two on-off Markov processes as the *birth-and-death* process $\{\chi^{(2)}(t)\}$ of the following form:

$$\chi^{(2)}(t) = \begin{cases} 0, & \text{if no customer is active} \\ 1, & \text{if one customer is active} \\ 2, & \text{if two customers are active} \end{cases}. \tag{4.1}$$

The stochastic process $\chi^{(2)}$ is a Markov process on the phase space (or states) $\Theta^{(2)} = \{0, 1, 2\}$ with the state diagram shown in Fig. 4.2

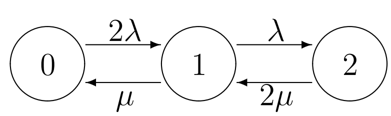


Figure 4.2: A system of two independent customers that is a birth-and-death process with three states.

Then, for this system we have the following matrix of sojourn times intensities

$$q^{(2)} = \begin{bmatrix} 2\lambda & 0 & 0 \\ 0 & \lambda+\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix}. \tag{4.2}$$

Also, we have a transition probability matrix given by

$$P^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} \\ 0 & 1 & 0 \end{bmatrix}. \tag{4.3}$$

Hence, the generating operator (or matrix) can be written as [21]

$$Q^{(2)} = q^{(2)}[P^{(2)} - I] = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix}. \tag{4.4}$$

Then, we consider a function $C^{(2)}(w)$ on the space $\mathbf{W}^{(2)} = \{0,1,2\} \times [0,V]$ defined as

$$C^{(2)}(w) = \begin{cases} F & w = \{0,v\},\ 0 < v < V; \\ F - f & w = \{1,v\},\ 0 < v < V; \\ F - 2f & w = \{2,v\},\ 0 < v < V; \\ 0 & \text{other cases} \end{cases}. \tag{4.5}$$

Denote as $v(t)$ the amount of information in the buffer at time $t$. It can be easily verified that $v(t)$ satisfies the following equation

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = C^{(2)}(\chi^{(2)}(t), v(t)), \tag{4.6}$$

with the initial condition $v(0) = v_0 \in [0,V]$. Eq. (4.6) determines the random evolution of the system in the Markov medium $\chi(t)$ [17].

The sojourn time distribution functions, say $F_\theta^{(2)}(t)$, have the following form for the different states: $F_0^{(2)}(t) = 1 - e^{-2\lambda t}$, $F_1^{(2)}(t) = 1 - e^{-(\lambda+\mu)t}$, and $F_2^{(2)}(t) = 1 - e^{-2\mu t}$. Now, denote as $f_\theta^{(2)} = \dfrac{\mathrm{d}F_\theta^{(2)}(t)}{\mathrm{d}t}$ and $r_\theta^{(2)} = \dfrac{f_\theta(t)}{1 - F_\theta(t)}$ for all $\theta \in \Theta^{(2)}$, i.e., $r_0^{(2)} = 2\lambda$, $r_1^{(2)} = \lambda + \mu$, $r_2^{(2)} = 2\mu$. Then, $\xi^{(2)}(t) = (\chi^{(2)}(t), v(t))$ is a Markov process with the generator [17, 10]

$$A^{(2)}\phi(\theta,v) = C^{(2)}(\theta,v)\frac{\partial}{\partial v}\phi(\theta,v) + r_\theta^{(2)}[P^{(2)}\phi(\theta,v) - \phi(\theta,v)], \tag{4.7}$$

where $P^{(2)}\phi(\theta,v) = \sum_{y\in\Theta} p_{\theta y}\phi(y,v)$, or equivalently,

$$A^{(2)}\phi(\theta,v) = C^{(2)}(\theta,v)\frac{\partial}{\partial v}\phi(\theta,v) + Q^{(2)}\phi(\theta,v). \tag{4.8}$$

Denote by $\rho$ the stationary distribution of process $\xi^{(2)}(t)$. Then, for every function $\phi(\cdot)$ belonging to the domain of the operator $A$ we have

$$\int_{\mathbf{W}^{(2)}} A^{(2)}\phi(w)\rho(\mathrm{d}w) = 0. \tag{4.9}$$

The analysis of the properties of process $\xi^{(2)}(t)$ leads up to the conclusion that, for case $f < F < 2f$, the stationary distribution $\rho$ has atoms at points $(2,0)$, $(0,V)$ and $(1,V)$. We denote them as $\rho[2,0]$, $\rho[0,V]$ and $\rho[1,V]$. The continuous part of $\rho$ is denoted by $\rho(\theta,v)$.

Let us write Eq. (4.9) in more detail.

$$\begin{aligned}\int_{\mathbf{W}^{(2)}} A^{(2)}\phi(w)\rho(\mathrm{d}w) \quad = \quad & \int_{0+}^{V-}\Bigg\{\Big[F\frac{\partial}{\partial v}\phi(0,v) - 2\lambda\phi(0,v) + 2\lambda\phi(1,v)\Big]\rho(0,v) \\ & + \Big[(F-f)\frac{\partial}{\partial v}\phi(1,v) + \mu\phi(0,v) - (\mu+\lambda)\phi(1,v) + \lambda\phi(2,v)\Big]\rho(1,v) \\ & + \Big[(F-2f)\frac{\partial}{\partial v}\phi(2,v) + 2\mu\phi(1,v) - 2\mu\phi(2,v)\Big]\rho(2,v)\Bigg\}\mathrm{d}v \\ & + [2\mu\phi(1,0) - 2\mu\phi(2,0)]\rho[2,0] \\ & + [-2\lambda\phi(0,V) + 2\lambda\phi(1,V)]\rho[0,V] \\ & + [\mu\phi(0,V) - (\mu+\lambda)\phi(1,V) + \lambda\phi(2,V)]\rho[1,V] = 0.\end{aligned}$$

Let $A^{(2)*}$ be the conjugate operator of $A^{(2)}$. By changing the order of integration in Eq. (4.9) we can obtain the following expression for the continuous part of $A^{(2)*}\rho$.

$$\left\{\begin{array}{rcl} -2\lambda\rho(0,v) + \mu\rho(1,v) & = & F\frac{\partial}{\partial v}\rho(0,v) \\ 2\lambda\rho(0,v) - (\mu+\lambda)\rho(1,v) + 2\mu\rho(2,v) & = & (F-f)\frac{\partial}{\partial v}\rho(1,v) \\ \lambda\rho(1,v) - 2\mu\rho(2,v) & = & (F-2f)\frac{\partial}{\partial v}\rho(2,v) \end{array}\right. . \tag{4.10}$$

Eqs. (4.10) can also be stated in the following form:

$$Q^{(2)T}\begin{bmatrix}\rho(0,v)\\ \rho(1,v)\\ \rho(2,v)\end{bmatrix} = \begin{bmatrix} F & 0 & 0 \\ 0 & (F-f) & 0 \\ 0 & 0 & (F-2f)\end{bmatrix}\frac{\partial}{\partial v}\begin{bmatrix}\rho(0,v)\\ \rho(1,v)\\ \rho(2,v)\end{bmatrix}. \tag{4.11}$$

We obtain the expression for the atoms for the case $f < F < 2f$,

$$\left\{\begin{array}{rcl} -F\rho(0,0+) & = & 0 \\ -(F-f)\rho(1,0+) + 2\mu\rho[2,0] & = & 0 \\ -(F-2f)\rho(2,0+) - 2\mu\rho[2,0] & = & 0 \end{array}\right. \tag{4.12}$$

and

$$\left\{\begin{array}{rcl} F\rho(0,V-) - 2\lambda\rho[0,V] + \mu\rho[1,V] & = & 0 \\ (F-f)\rho(1,V-) + 2\lambda\rho[0,V] - (\mu+\lambda)\rho[1,V] & = & 0 \\ (F-2f)\rho(2,V-) + \lambda\rho[1,V] & = & 0 \end{array}\right. . \tag{4.13}$$

Similarly, we obtain expression for atoms for the case $F < f$,

$$\left\{\begin{array}{rcl} -F\rho(0,0+) + \mu\rho[1,0] & = & 0 \\ -(F-f)\rho(1,0+) - (\lambda+\mu)\rho[1,0] + 2\mu\rho[2,0] & = & 0 \\ -(F-2f)\rho(2,0+) + \lambda\rho[1,0] - 2\mu\rho[2,0] & = & 0 \end{array}\right. \tag{4.14}$$

and

$$\begin{cases} F\rho(0,V-) - 2\lambda\rho[0,V] &= 0 \\ (F-f)\rho(1,V-) + 2\lambda\rho[0,V] &= 0 \\ (F-2f)\rho(2,V-) &= 0 \end{cases} . \tag{4.15}$$

In these equations we have defined the notation

$$\rho(\theta, 0+) := \lim_{v \downarrow 0} \rho(\theta, v)$$

and

$$\rho(\theta, V-) := \lim_{v \uparrow 0} \rho(\theta, v)$$

It follows from Eqs. (4.10) that

$$F\rho(0,v) + (F-f)\rho(1,v) + (F-2f)\rho(2,v) = c = \text{cte}. \tag{4.16}$$

From Eqs. (4.12), (4.13), (4.14) and (4.15) we get $c = 0$ in Eq. (4.16) for both cases of $F$.

Using Eqs. (4.10) and (4.16) we can get

$$\begin{aligned} \frac{\partial\rho(0,v)}{\partial v} &= K_{00}\rho(2,v) + K_{01}\rho(1,v), \\ \partial\rho(1,v)\partial v &= K_{10}\rho(0,v) + K_{11}\rho(1,v) \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} K_{00} &= \frac{-2\lambda}{F} \\ K_{01} &= \frac{\mu}{F} \\ K_{10} &= \frac{-1}{(F-f)}\left[\frac{2\mu F}{F-2f} - 2\lambda\right] \\ K_{11} &= -\left[\frac{\lambda+\mu}{F-f} + \frac{2\mu}{F-2f}\right]. \end{aligned}$$

Solving the set of Eqs. (4.17) we get

$$\rho(1,v) = c_0 e^{\delta_0 v} + c_1 e^{\delta_1 v} \tag{4.18}$$

$$\rho(0,v) = \frac{c_0}{K_{10}}(\delta_0 - K_{11})e^{\delta_0 v} + \frac{c_1}{K_{10}}(\delta_1 - K_{11})e^{\delta_1 v}, \tag{4.19}$$

where,

$$\delta_0 = \frac{K_{11} + K_{00} + \sqrt{(K_{11} + K_{00})^2 - 4(K_{11}K_{00} - K_{01}K_{10})}}{2}$$
$$\delta_1 = \frac{K_{11} + K_{00} - \sqrt{(K_{11} + K_{00})^2 - 4(K_{11}K_{00} - K_{01}K_{10})}}{2}$$

We can evaluate expressions for $K_{00}$, $K_{01}$, $K_{10}$ and $K_{11}$ and we find

$$\begin{aligned} \delta_0 &= \frac{-(\mu + \lambda)}{F - f} \\ \delta_1 &= \frac{-2(\mu F + \lambda(F - 2f))}{F(F - 2f)} \end{aligned}$$

Rewriting expression (4.16) we find that

$$\begin{aligned} \rho(2, v) &= \frac{-F}{F - 2f}\rho(0, v) - \frac{F - f}{F - 2f}\rho(1, v) \\ &= c_0 \left( \frac{-F(\delta_0 - K_{11})}{(F - 2f)K_{10}} - \frac{F - f}{F - 2f} \right) e^{\delta_0 v} + c_1 \left( \frac{-F(\delta_1 - K_{11})}{(F - 2f)K_{10}} - \frac{F - f}{F - 2f} \right) e^{\delta_1 v} \end{aligned}$$

On the case that $f < F < 2f$ we know from Eqs. (4.12) that $\rho(0, 0+) = 0$. Thus, we get that

$$c_0(\delta_0 - K_{11}) + c_1(\delta_1 - K_{11}) = 0.$$

That is,

$$c_1 = \frac{\delta_0 - K_{11}}{K_{11} - \delta_1} c_0 = D_1\, c_0.$$

For the case $F < f$ we know from Eqs. (4.15) that $\rho(2, V-) = 0$. Then

$$c_0 \left( \frac{-F(\delta_0 - K_{11})}{K_{10}} - (F - f) \right) e^{\delta_0 V-} + c_1 \left( \frac{-F(\delta_1 - K_{11})}{K_{10}} - (F - f) \right) e^{\delta_1 V-} = 0.$$

And we have

$$c_1 = -\frac{F(\delta_0 - K_{11}) + K_{10}(F - f)}{F(\delta_1 - K_{11}) + K_{10}(F - f))} c_0 e^{(\delta_0 - \delta_1)V-} = D_2\, c_0.$$

The constant $c_0$ can be calculated using the normalization equation.

$$\int_{\mathbf{W}^{(2)}} \rho(\mathrm{d}w) = 1. \tag{4.20}$$

But, first we have to find the expressions for the atoms of the stationary distribution $\rho$.

Solving system (4.13) and (4.12), we get the expressions for the atoms for the case $f < F < 2f$.

$$\begin{aligned}
\rho[0,V] &= -\frac{F-f}{2\lambda}\rho(1,V-) - (\mu+\lambda)\frac{F-2f}{2\lambda^2}\rho(2,V-) \\
\rho[1,V] &= -\frac{F-2f}{\lambda}\rho(2,V-) \\
\rho[2,0] &= -\frac{F-2f}{2\mu}\rho(2,0+)
\end{aligned}$$

That is,

$$\begin{aligned}
\frac{\rho[0,V]}{c_0} &= \frac{1}{2\lambda^2}\left(\frac{F(\mu+\lambda)(\delta_0-K_{11})}{K_{10}} + \mu(F-f)\right)e^{\delta_0 V-} \\
&\quad -\frac{\delta_0-K_{11}}{2\lambda^2}\left(\frac{F(\mu+\lambda)}{K_{10}} + \frac{\mu(F-f)}{\delta_1-K_{11}}\right)e^{\delta_1 V-} \\
\frac{\rho[1,V]}{c_0} &= \frac{1}{\lambda}\left(\frac{F(\delta_0-K_{11})}{K_{10}} + (F-f)\right)e^{\delta_0 V-} \\
&\quad -\frac{\delta_0-K_{11}}{\lambda}\left(\frac{F}{K_{10}} + \frac{F-f}{\delta_1-K_{11}}\right)e^{\delta_1 V-} \\
\frac{\rho[2,0]}{c_0} &= \frac{1}{2\mu}\left(\frac{F(\delta_0-K_{11})}{K_{10}} + (F-f)\right) \\
&\quad -\frac{\delta_0-K_{11}}{2\mu}\left(\frac{F}{K_{10}} + \frac{F-f}{\delta_1-K_{11}}\right)
\end{aligned}$$

Then, we can get from Eq. (4.20)

$$c_0^{-1} = \frac{a_0 e^{\delta_0 V-} + a_1 e^{\delta_1 V-} + a_2}{2\delta_0\delta_1 K_{10}\lambda^2\mu(F-2f)(\delta_1-K_{11})}$$

where

$$\begin{aligned}
a_0 &= \delta_1\mu(\delta_1-K_{11})\{-2f\lambda^2[K_{10}+2(\delta_0-K_{11})] \\
&\quad +\delta_0(F-2f)[K_{10}(F-f)(\mu+2\lambda)+F(\mu+3\lambda)(\delta_0-K_{11})]\} \\
a_1 &= -\delta_0\mu(\delta_0-K_{11})\{-2f\lambda^2[K_{10}+2(\delta_1-K_{11})] \\
&\quad +\delta_1(F-2f)[K_{10}(F-f)(\mu+2\lambda)+F(\mu+3\lambda)(\delta_1-K_{11})]\} \\
a_2 &= -\lambda^2(\delta_0-\delta_1)\{\delta_0\delta_1 K_{10}(F-f)(F-2f) \\
&\quad +2\mu f[2\delta_1\delta_0+(K_{10}-2K_{11})(\delta_0+\delta_1-K_{11})]\}
\end{aligned}$$

For the case $F < f$ we solve system (4.14) and (4.15). Then we find,

$$\begin{aligned}
\frac{\rho[0,V]}{c_0} &= \frac{F}{2\lambda}\left[\frac{\delta_0-K_{11}}{K_{10}}e^{\delta_0 V-}+D_2\frac{\delta_1-K_{11}}{K_{10}}e^{\delta_1 V-}\right]\\
\frac{\rho[1,0]}{c_0} &= \frac{F}{\mu}\left[\frac{\delta_0-K_{11}}{K_{10}}+D_2\frac{\delta_1-K_{11}}{K_{10}}\right]\\
\frac{\rho[2,0]}{c_0} &= \frac{\mu K_{10}(F-f)+F(\mu+\lambda)(\delta_0-K_{11})}{2\mu^2K_{10}}\\
&\quad +D_2\frac{\mu K_{10}(F-f)+F(\mu+\lambda)(\delta_1-K_{11})}{2\mu^2K_{10}}
\end{aligned}$$

Then, using Eq. (4.20) we get.

$$c_0^{-1}=\frac{b_0e^{\delta_0V-}+b_1e^{(\delta_0-\delta_1)V-}+b_2}{2\delta_0\delta_1\lambda\mu^2K_{10}(F-2f)[K_{10}(F-f)+F(\delta_1-K_{11})]}$$

where

$$\begin{aligned}
b_0 &= \mu^2(\delta_0-\delta_1)\{K_{10}(F-2f)[F\delta_0\delta_1(F-f)+f\lambda(K_{10}-2K_{11})]\\
&\quad +Ff\lambda[K_{10}+2(\delta_0-K_{11})][K_{10}+2(\delta_1-K_{11})]\};\\
b_1 &= -\delta_0\lambda[K_{10}(F-f)+F(\delta_0-K_{11})]\{2f\mu^2[K_{10}+2(\delta_1-K_{11})]\\
&\quad +\delta_1(F-2f)[\mu K_{10}(F-f)+F(\lambda+3\mu)(\delta_1-K_{11})]\};\\
b_2 &= \delta_1\lambda[K_{10}(F-f)+F(\delta_1-K_{11})]\{2f\mu^2[K_{10}+2(\delta_0-K_{11})]\\
&\quad +\delta_0(F-2f)[\mu K_{10}(F-f)+F(\lambda+3\mu)(\delta_0-K_{11})]\};
\end{aligned}$$

Now, let us define the function $f(w)$ on $\mathbf{W}^{(2)}$ as follows:

$$f(w):=\begin{cases} f, & if \quad w=\{1,v\},\ 0<v\le V;\\ f, & if \quad w=\{2,v\},\ 0<v\le V;\\ 2f, & if \quad w=\{3,v\},\ 0<v\le V;\\ 0 & \text{in other cases.}\end{cases} \tag{4.21}$$

Let us assume the joint stochastic process with a two-dimensional phase space $\xi(t)=\left(\chi^{(2)}(t),v(t)\right)$. Denote as $I(T)$ the amount of information delivered to customers $S_1$ and $S_2$, in a time interval $[0,T]$. Then, we can state the following equality

$$K=\lim_{T\to\infty}\frac{V(T)}{T}=\lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(\xi(t)\right)\,\mathrm{d}t. \tag{4.22}$$

It follows from ergodic theory [21] that if the process $\xi(t)$ has a stationary distribution $\rho(\cdot)$, then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(\xi(t)\right)\,\mathrm{d}t=\int_{\mathbf{W}^{(2)}}f(w)\mathrm{d}\rho(w). \tag{4.23}$$

Hence, by using Eq. (4.22) we obtain

$$K=\int_{\mathbf{W}^{(2)}}f(w)\mathrm{d}\rho(w)=\int_{\mathbf{W}^{(2)}}f(w)\rho(\mathrm{d}w). \tag{4.24}$$

## 4.3 Numerical Results and Stationary Efficiency

With the expression of $c_0$, now it is possible to evaluate for the complete expression of the stationary distribution. For example, on the case $f = 3/2$, $\lambda = 3/10$, $\mu = 1/10$, $V = 100$ and $F = 9/4$ ($f < F < 2f$) we get, from Eq. (4.19),

$$\rho(0, v) = -\frac{1}{10}\frac{1 - e^{8v/15}}{169 - e^{-160/3}}$$

We can plot this analytical result with others as well as results from a computer simulation to illustrate some common cases. In Figure 4.3 we show plots for the stationary distribution $\rho(v)$. That is, for

$$\rho(v) = \begin{cases} \rho(0, v) + \rho(1, v) + \rho(2, v), & 0 < v < V; \\ \rho[2, 0], & v = 0; \\ \rho[0, V] + \rho[1, V] & v = V; \end{cases}$$

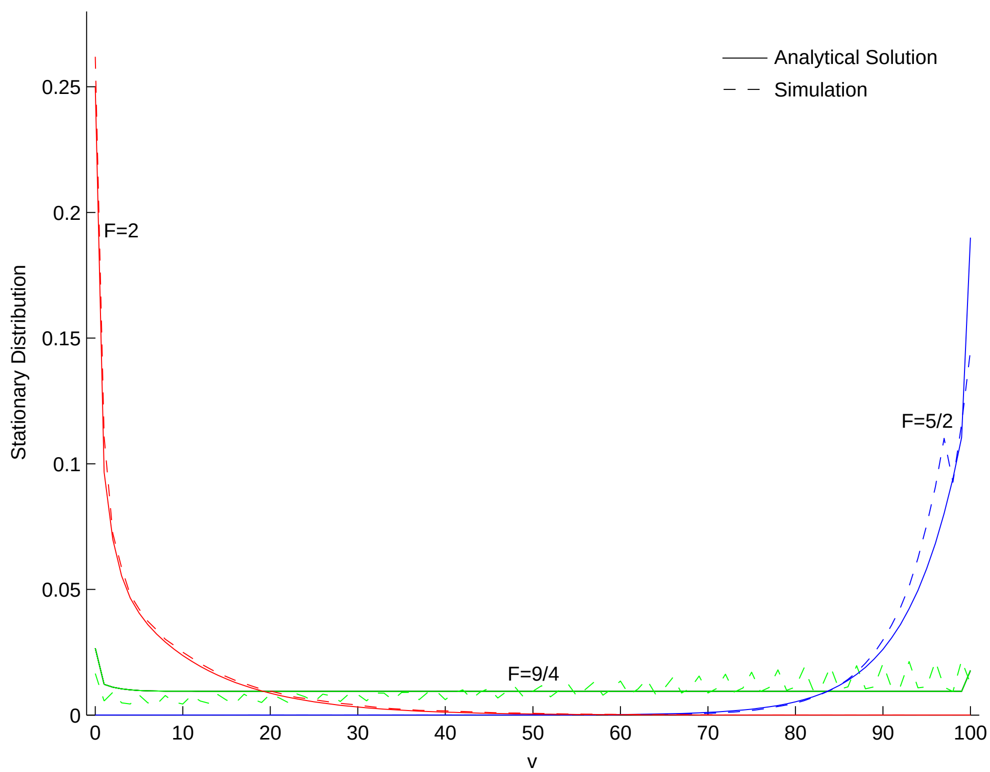


Figure 4.3: Stationary distribution of the buffer for the case $f < F < 2f$ for three different values of $F$.

On these plots it is considered $F = 2$, $F = 5/2$, and

$$F = \frac{2f\lambda}{\lambda + \mu}. \tag{4.25}$$

For this case that means $F = 15/8$.

Some sort of perturbation close to $v = 100$ can be noticed in the curves from the simulation. This effect comes from the system functionality for the case $f < F < 2f$. From this condition of $F$, it can be said that, if any of the two customers is active, the level of the buffer can be increased. That is because the rate $F$, filling the buffer, is larger than the demanding information rate of any customer. However, the filling rate is expected to stop when the level of the buffer reaches its maximum. The result is that, if any of the two customers is active, the level of the buffer can be increased up to its maximum, then the filling rate is turned off and the level starts to decrease. At any moment the level is sensed not to be at its maximum again, the filling rate is restored. On that scenario, for the time that this single customer is active the level of the buffer swings between its maximum and some close point below. That is why some small peaks can be seen in the simulation at some point close to $V$.

For sake of the analytical solution, the atom $\rho[1, V]$ was considered as a more steady approximation of the real system behavior.

No such approximation is taken for the case $F < f$. Plots from analytical solutions as well as from simulations for this case are shown in figure (4.4). The parameters considered for this case are $f = 3/2$, $\lambda = 1/20$, $\mu = 1/10$, and $V = 100$, somehow similar to the ones used for plots on Figure 4.3. Three different cases of F are considered, including

$$F = \frac{2f\lambda}{\lambda + \mu}.$$

This is $F$=1. The cases $F = 3/4$ and $F = 5/4$ are also considered.

We can now recall the function $f(w)$ and show some plots regarding the efficiency parameter $K$ from Eq. (4.24). Let us write this expression in more detail for this system with $f < F < 2f$,

$$K = f \int_0^V \{\rho(1, v) + 2\rho(2, v)\}\mathrm{d}v + f\rho[1, V].$$

We show on Figure 4.5 K(V), i.e., $K$ as a function of the maximum capacity of the buffer, for the same three cases of $F$ from Figure 4.3. On Figure 4.6 we show $K(V)$ for the three cases of $F$ from Figure 4.4.

As it can be seen from Figure 4.5, the case of $F$ that satisfies condition in Eq. (4.25) keeps growing while the other two F-equally-spaced cases converge faster to a value. Notice also that after some value of $V$ the difference between the two higher values of $K(V)$ is relatively small. The same behavior can be observed on Figure 4.6.

Actually, it can be seen that the limit of the efficiency parameter $K$ as $V$ tends to infinity, for the cases where

$$F \geq \frac{2f\lambda}{\lambda + \mu},$$

is

$$\lim_{V\to\infty} K(V) = \frac{2f\lambda}{\lambda + \mu}.$$

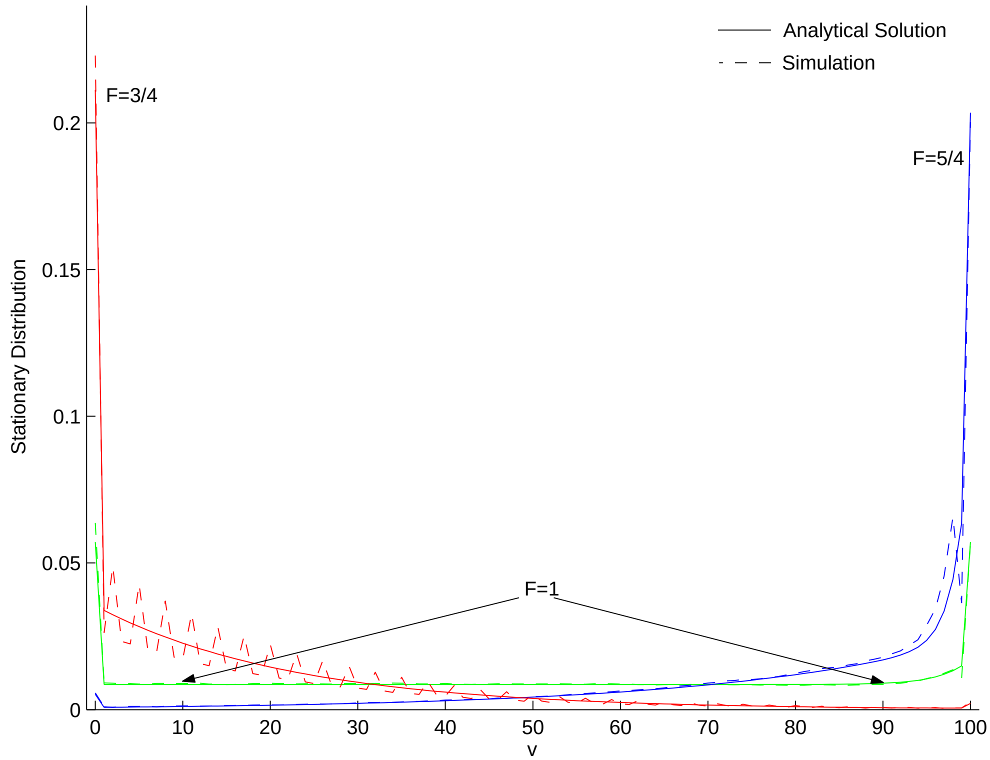


Figure 4.4: Stationary distribution of the buffer for the case $F < f$ for three different values of $F$.

That means that, if the buffer is big enough, no $F$ larger than $\frac{2f\lambda}{\lambda+\mu}$ is required to meet the maximum efficiency of the system.

In some other case, if the buffer is not big enough, some efficiency close to the maximum of the system can be reached using

$$F > \frac{2f\lambda}{\lambda+\mu}$$

This holds for both analyzed cases of $F$, i.e., $f < F < 2f$ and $F < f$.

## 4.4 Semi-Markov Mathematical Model

Consider the semi-Markov process $\{\chi(t)\}$ which is the superposition of two independent alternating semi-Markov processes ($N = 2$) with the phase space

$$\mathbf{Z} = \left\{(h, x^i) : h \in \mathbf{H}, x^i \in \mathbf{R}_+^2\right\},$$

where

$$\mathbf{H} = \{h : h = (h_1, h_2), \quad h_i = 0, 1; \, i = 1, 2\},$$

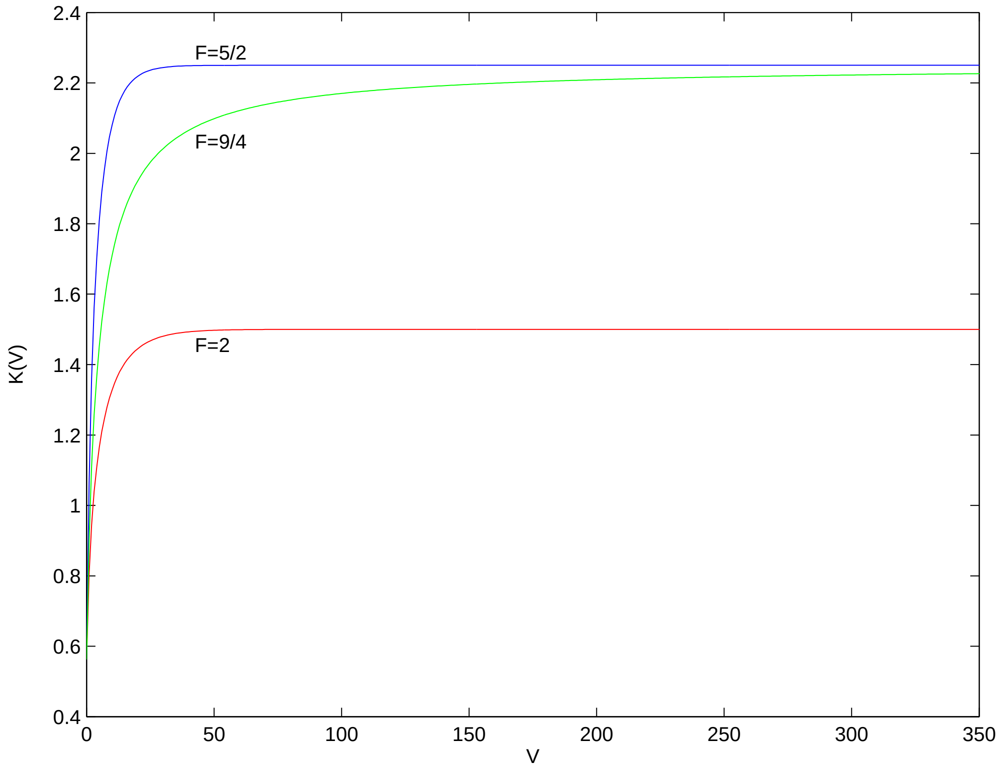


Figure 4.5: Efficiency parameter as a function of the buffer capacity for the case $f < F < 2f$ and three different values of $F$.

and $\mathbf{R}_+^2 = \{\vec{x} : \vec{x} = (x, 0), x \geq 0\} \bigcup \{\vec{x} : \vec{x} = \ (0, x), x \geq 0\}$. We have defined $h_i$ as

$$h_i = \begin{cases} 1, & \text{if } S_i \text{ is active;} \\ 0, & \text{if } S_i \text{ is not active,} \end{cases}$$

where $S_i$ stands for subsystem $i$. The component $x$ of the vector $(x, 0)$ (respectively $(0, x)$) is the residual life from the last state change of $S_1$ (respectively $S_2$). The initial distribution of $\chi(t)$ is $P\{\chi(0) = (1, 1; 0, 0)\} = 1$.

Let us write this in more detail:

$(1, 1; 0, x)$ - subsystem $S_1$ starts to be active and subsystem $S_2$ has been active for the time $x$,

$(1, 1; x, 0)$ - subsystem $S_2$ starts to be active and subsystem $S_1$ has been active for the time $x$,

$(1, 0; 0, x)$ - subsystem $S_1$ starts to be active and subsystem $S_2$ has been inactive for the time $x$,

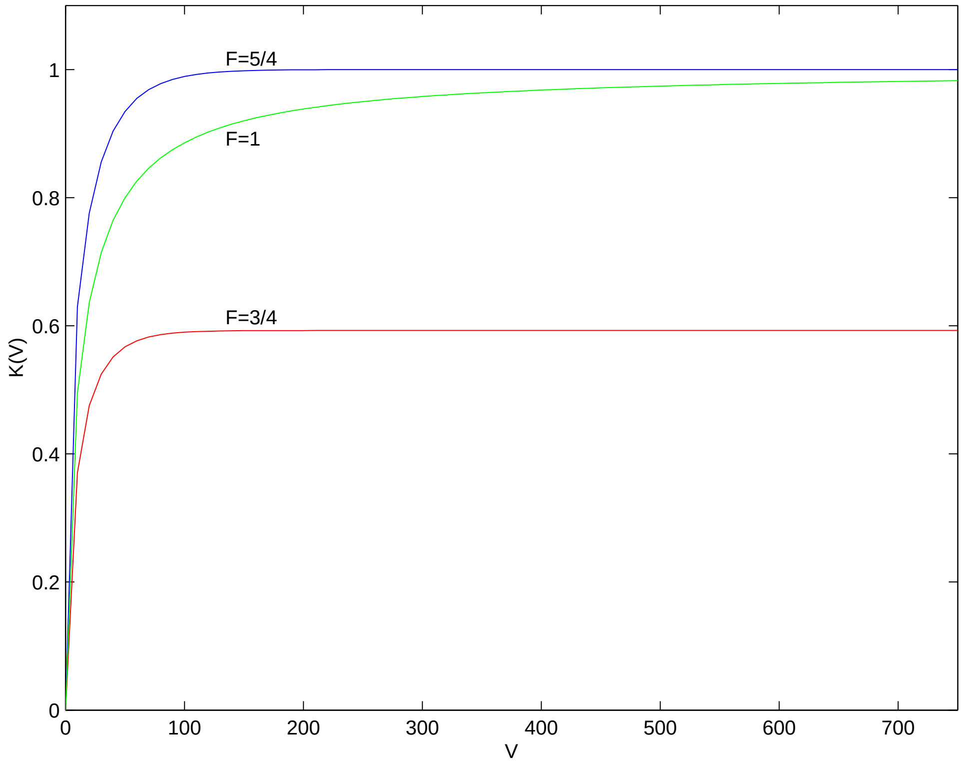


Figure 4.6: Efficiency parameter as a function of the buffer capacity for the case $F < f$ and three different values of $F$.

$(1, 0; x, 0)$ - subsystem $S_2$ starts to be inactive and subsystem $S_1$ has been active for the time $x$,

$(0, 1; 0, x)$ - subsystem $S_1$ starts to be inactive and subsystem $S_2$ has been active for the time $x$,

$(0, 1; x, 0)$ - subsystem $S_2$ starts to be active and subsystem $S_1$ has been inactive for the time $x$,

$(0, 0; 0, x)$ - subsystem $S_1$ starts to be inactive and subsystem $S_2$ has been inactive for the time $x$,

$(0, 0; x, 0)$ - subsystem $S_2$ starts to be inactive and subsystem $S_1$ has been inactive for the time $x$.

Let $v(t)$ be the amount of information in the buffer at time $t$. It was shown in [19] that we can use the PMA to reduce the random evolution $v(t)$ in the semi-Markov medium $\chi(t)$ to the Markov evolution $\bar{v}(t)$ in the Markov medium $\bar{\chi}(t)$. Then we have the following transition probabilities of the embedded Markov chain.

$$P\left\{(h_1, h_2)\left(\bar{h}_1, h_2\right)\right\} =$$

$$\frac{\int_0^\infty \int_0^\infty \bar{F}^{(2)}_{h_2}(x+u)\mathrm{d}F^{(1)}_{\bar{h}_1}(u)\mathrm{d}x + \int_0^\infty \int_0^\infty \bar{F}^{(2)}_{h_2}(u)\mathrm{d}uF^{(1)}_{\bar{h}_1}(x+u)\mathrm{d}x}{\int_0^\infty \bar{F}^{(1)}_{h_1}(x)\mathrm{d}x + \int_0^\infty \bar{F}^{(1)}_{h_2}(x)\mathrm{d}x}, \quad (4.26)$$

$$P\left\{(h_1,h_2)\left(h_1,\bar{h}_2\right)\right\} =$$

$$\frac{\int_0^\infty \int_0^\infty \bar{F}^{(1)}_{h_1}(x+u)\mathrm{d}F^{(2)}_{\bar{h}_2}(u)\mathrm{d}x + \int_0^\infty \int_0^\infty \bar{F}^{(1)}_{h_1}(u)\mathrm{d}uF^{(2)}_{\bar{h}_2}(x+u)\mathrm{d}x}{\int_0^\infty \bar{F}^{(1)}_{h_1}(x)\mathrm{d}x + \int_0^\infty \bar{F}^{(1)}_{h_2}(x)\mathrm{d}x}. \quad (4.27)$$

where $\bar{h}_i = 1 - h_i$, $\bar{F}(x) = 1 - F(x)$, and $F(x)$ is the cumulative distribution function.

The mean sojourn times of the process $\bar{\chi}(t)$ in states from $\mathbf{X} = \{00, 01, 10, 11\}$ are given by

$$\begin{aligned} m(h_1,h_2) &= \\ & \int_0^\infty \rho(h_1,h_2;x,0)m(h_1,h_2;x,0)\mathrm{d}x + \int_0^\infty \rho(h_1,h_2;0,x)m(h_1,h_2;0,x)\mathrm{d}x, \quad (4.28) \\ &= c_{s0}\left(\int_0^\infty \int_0^\infty \bar{F}^{(1)}_{h_1}(y)\bar{F}^{(2)}_{h_2}(x+y)\mathrm{d}y\mathrm{d}x + \int_0^\infty \int_0^\infty \bar{F}^{(1)}_{h_1}(x+y)\bar{F}^{(2)}_{h_2}(y)\mathrm{d}y\mathrm{d}x\right). \end{aligned}$$

As a first example we can consider some $m$-Erlang scenario where both, active and inactive sojourn times are considered with such distribution. This is, for the actual sojourn time distributions we have

$$\begin{aligned} F^{(1)}_0(u) &= \int_0^u \frac{\lambda e^{-\lambda x}(\lambda x)^{m_0-1}}{(m_0-1)!}\mathrm{d}x, \\ F^{(1)}_1(u) &= \int_0^u \frac{\mu e^{-\mu x}(\mu x)^{m_1-1}}{(m_1-1)!}\mathrm{d}x, \\ F^{(2)}_0(u) &= \int_0^u \frac{\lambda e^{-\lambda x}(\lambda x)^{m_0-1}}{(m_0-1)!}\mathrm{d}x, \\ F^{(2)}_1(u) &= \int_0^u \frac{\mu e^{-\mu x}(\mu x)^{m_1-1}}{(m_1-1)!}\mathrm{d}x. \end{aligned}$$

Here, $m_1$ and $m_0$ stand for the $m$ parameter of the distribution for the active and inactive subsystems states.

We use Eqs. (4.26) and (4.27) to calculate the transition probabilities of the embedded

Markov chain. We obtain

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{m_0\mu}{m_0\mu + m_1\lambda} & 0 & 0 & \frac{m_1\lambda}{m_0\mu + m_1\lambda} \\ \frac{m_0\mu}{m_0\mu + m_1\lambda} & 0 & 0 & \frac{m_1\lambda}{m_0\mu + m_1\lambda} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}. \quad (4.29)$$

After that we use Eq. (4.28) to calculate the mean sojourn times and consequently the sojourn time intensities defined as $q_\theta = (m(h_1, h_2))^{-1}$. Let us consider the equivalence $(h_1, h_2) = \{00, 01, 10, 11\} \Leftrightarrow \{0, 1, 2, 3\} = \Theta \ni \theta$ to simplify notation. Also let us remember that $q = [q_i \delta_{ij}; i, j \in \{0, 1, 2, 3\}]$ is a diagonal matrix. Then we obtain

$$q = \begin{bmatrix} \frac{2\lambda}{m_0} & 0 & 0 & 0 \\ 0 & \frac{\lambda}{m_0} + \frac{\mu}{m_1} & 0 & 0 \\ 0 & 0 & \frac{\lambda}{m_0} + \frac{\mu}{m_1} & 0 \\ 0 & 0 & 0 & \frac{2\mu}{m_1} \end{bmatrix}. \quad (4.30)$$

Now, since there is no distinction between subsystems $S_1$ and $S_2$, we can construct one birth-and-death process just to simplify the solution of the system. We consider the process

$$\chi_r(t) = \begin{cases} 0, & \text{if no customer is active,} \\ 1, & \text{if one customer is active,} \\ 2, & \text{if both customers are active.} \end{cases}$$

We obtain from Eq. (4.29) the transition probabilities for the simplified process as

$$P_r = \begin{bmatrix} 0 & 1 & 0 \\ \frac{m_0\mu}{m_0\mu + m_1\lambda} & 0 & \frac{m_1\lambda}{m_0\mu + m_1\lambda} \\ 0 & 1 & 0 \end{bmatrix}. \quad (4.31)$$

Also, we can obtain from Eq. (4.30) the sojourn time intensities for the simplified process as

$$q_r = \begin{bmatrix} \dfrac{2\lambda}{m_0} & 0 & 0 \\ 0 & \dfrac{\lambda}{m_0} + \dfrac{\mu}{m_1} & 0 \\ 0 & 0 & \dfrac{2\mu}{m_1} \end{bmatrix}. \tag{4.32}$$

The stochastic process $\chi_r$ is a Markov process on the phase space (or states) $\Theta_r = \{0, 1, 2\}$. Hence, the generating operator (or matrix) of the process can be written as $Q_r = q_r[P_r - I]$ [21].

At this point it is not difficult to see that the approximated semi-Markov process corresponds to a Markov process with the escalated parameters in the form $\lambda \to \dfrac{\lambda}{m_0}$ and $\mu \to \dfrac{\mu}{m_1}$. Then, we may wonder if this construction holds for approximations different from $N = 2$. The first choice obvious inspection is the case $N = 1$.

We consider the process

$$\chi_r^{(1)}(t) = \begin{cases} 0, & \text{if customer is not active,} \\ 1, & \text{if customer is active.} \end{cases}$$

For this case, we can write directly the transition probability matrix as well as the sojourn time intensity matrix in the form $P_r^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $q_r^{(1)} = \begin{bmatrix} \dfrac{\lambda}{m_0} & 0 \\ 0 & \dfrac{\mu}{m_1} \end{bmatrix}$.

Again, the approximated semi-Markov process corresponds to a Markov process with the escalated parameters of the form $\lambda \to \dfrac{\lambda}{m_0}$ and $\mu \to \dfrac{\mu}{m_1}$.

As a second example for the case $N = 2$, we introduce the hyper-exponential semi-Markov case. For this case, the distribution of the active and inactive sojourn times are taken as

$$\begin{aligned} F_0^{(1)}(u) &= 1 - p\exp(-\lambda u) - (1-p)\exp(-\lambda_b u), \\ F_1^{(1)}(u) &= 1 - p\exp(-\mu u) - (1-p)\exp(-\mu_b u), \\ F_0^{(2)}(u) &= 1 - p\exp(-\lambda u) - (1-p)\exp(-\lambda_b u), \\ F_1^{(2)}(u) &= 1 - p\exp(-\mu u) - (1-p)\exp(-\mu_b u). \end{aligned}$$

The following choices can be taken as an instance:

$$\begin{aligned}\lambda_b &= k\lambda,\\ \mu_b &= k\mu,\end{aligned}$$

$k > 0$. We use Eqs. (4.26) and (4.27) to calculate the transition probabilities of the embedded Markov chain and we obtain

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{\mu}{\lambda+\mu} & 0 & 0 & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & 0 & 0 & \frac{\lambda}{\lambda+\mu} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}. \tag{4.33}$$

After that we use Eq. (4.28) to calculate the mean sojourn times and consequently the sojourn time intensities. The result is

$$q = \begin{bmatrix} \frac{2k\lambda}{1+(k-1)p} & 0 & 0 & 0 \\ 0 & \frac{k(\lambda+\mu)}{1+(k-1)p} & 0 & 0 \\ 0 & 0 & \frac{k(\lambda+\mu)}{1+(k-1)p} & 0 \\ 0 & 0 & 0 & \frac{2k\mu}{1+(k-1)p} \end{bmatrix} \tag{4.34}$$

Again, since there is no distinction between subsystems $S_1$ and $S_2$, we can construct one birth-and-death process just as we did previously to simplify the solution of the system. We obtain from Eq. (4.33) the transition probabilities matrix for the simplified process as

$$P_r = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} \\ 0 & 1 & 0 \end{bmatrix}.$$

Also, we can obtain from Eq. (4.34) the sojourn time intensities of the simplified process as

$$q_r = \begin{bmatrix} \frac{2k\lambda}{1+(k-1)p} & 0 & 0 \\ 0 & \frac{k(\lambda+\mu)}{1+(k-1)p} & 0 \\ 0 & 0 & \frac{2k\mu}{1+(k-1)p} \end{bmatrix} \tag{4.35}$$

The generating operator of this Markov process is also $Q_r = q_r[P_r - I]$. Once again, we can see that the approximated semi-Markov process corresponds to a Markov process with the parameters escalated, this time in the form $\lambda \rightarrow \dfrac{\lambda}{1+(k-1)p}$ and $\mu \rightarrow \dfrac{\mu}{1+(k-1)p}$

Now, let us consider the semi-Markov process $\chi_r^{(1)}(t)$ for the hyper-Exponential case $N = 1$. We can write directly the transition probability matrix as well as the sojourn time instensity matrix in the form $P_r^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and

$$q_r^{(1)} = \begin{bmatrix} \dfrac{\lambda}{1+(k-1)p} & 0 \\ 0 & \dfrac{\mu}{1+(k-1)p} \end{bmatrix}.$$

Again, the approximated semi-Markov process corresponds to a Markov process with the escalated parameters of the form $\lambda \rightarrow \dfrac{\lambda}{1+(k-1)p}$ and $\mu \rightarrow \dfrac{\mu}{1+(k-1)p}$.

Then, for now on, we can generalize the $m$-Erlang approximation for any $N$ with a Markov process with the parameters escalated in the form $\lambda \rightarrow \dfrac{\lambda}{m_0}$ and $\mu \rightarrow \dfrac{\mu}{m_1}$.

For the Hyper-Exponential approximation we can escalate the Markov process parameters in the form $\lambda \rightarrow \dfrac{\lambda}{1+(k-1)p}$ and $\mu \rightarrow \dfrac{\mu}{1+(k-1)p}$ for any other $N$ as well.

We can make some plots for the $m$-Erlang case. In Figure 4.7 the case

$$F = \frac{2f\lambda}{\lambda+\mu}, \tag{4.36}$$

$$F < f, \tag{4.37}$$

was considered. Again, we choose $f = 3/2$, $\lambda = /20$, $\mu = 1/10$, and $V = 100$. For each one of the curves, shown on Figure 4.7 we used the same "$m$" number for every $m$-Erlang distribution of sojourn time.

As it can be seen, from Figure 4.7, the stationary distribution for every $m$-Erlang case is very similar in the continuous part. That is due to the fact that, when the active sojourn time was added with another exponential distribution i.e., we increase the $m$ number of the $m$-Erlang distribution, the same thing happened to the inactive sojourn time. Then, some proportion of the "unscaled" exponential case is kept and that caused the results to be very similar. Some flat behavior is maintained in the approximated analytical solution and also in the simulated system. However the area under the continuous part tends to be smaller as we increase the number of exponentials, then the atoms tend to be higher. This tendency is more drastic in the approximated solution than in the simulation and that caused that, for a greater number of exponentials added, the less the analytical solution approximates the simulation. As expected, the best match

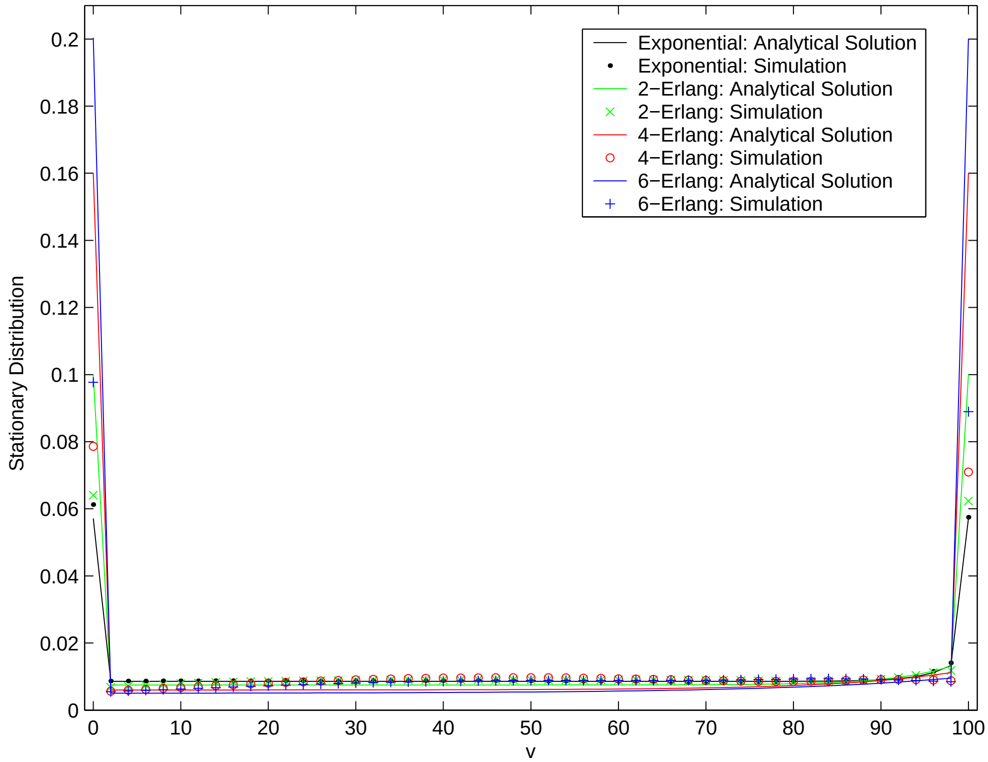


Figure 4.7: Stationary distribution of the $m$-Erlang semi-Markov case for the active and inactive sojourn time.

between analytical solution and simulation is the merely exponential case. We can show some other examples where a good match between approximated analytical solution and simulation is maintained for different $m$-Erlang cases.

We can consider the inactive sojourn time with an exponential distribution and plot several $m$-Erlang cases for the active sojourn time. This example is illustrated in Figures 4.8 and 4.9. Again, for these figures we used $F$ based on conditions stated in Eqs. (4.36) and (4.37). We also used $f = 3/2$, $\lambda = /20$, $\mu = 1/10$, and $V = 100$.

In Figures 4.8 and 4.9 we can see a good match between the analytical solutions and simulations for every $m$-Erlang case. In this case we see that the behavior of the curves is modified as we increase the number of exponentials in the $m$-Erlang distributions. For more exponentials in the $m$-Erlang distribution of the active sojourn time expect subsystems $S_1$ and $S_2$ to be active longer. Therefore we can see on Figures 4.8 and 4.9 that this causes the stationary distribution to be biased to the empty side of the buffer.

Also, we can make some plots for the hyper-exponential case. In Figure 4.10, the behavior of the approximation can be appreciated along with some plots from simulations for this semi-Markov case. We choose In Figures 4.10 and we choose $n = 2$ as an instance. Besides, we took again $f = 3/2$, $\lambda = 1/20$, $\mu = 1/10$, and $V = 100$, as well as $F$ from conditions (4.36) and (4.37).

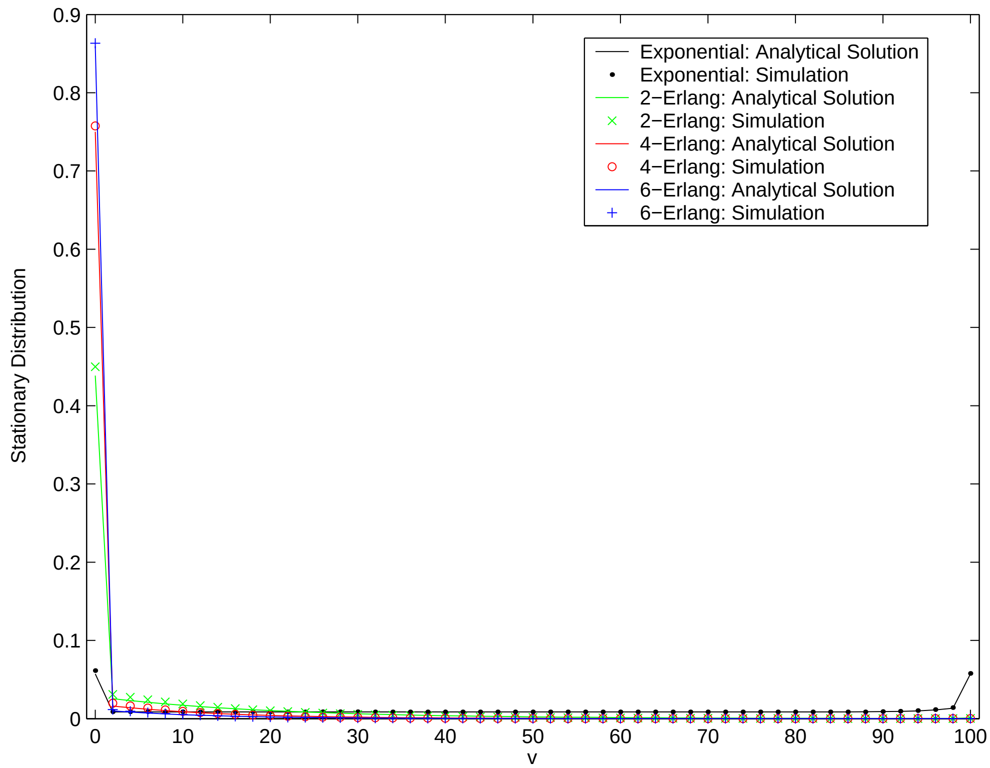


Figure 4.8: Stationary distribution of the buffer with $m$-Erlang distributed active sojourn time and exponentially distributed inactive sojourn time.

As it can be seen, the behavior of the system for different values of $p$ is very similar. That is again, because active and inactive were modified simultaneously.

This is, subsystem $S_1$ has a probability $p$ to have a expected sojourn time $1/\lambda_0$ every time it switches to the inactive state, also has a probability $1-p$ to have a expected sojourn time twice as large ($n = 2$). The same thing happens to the active sojourn time. It has a probability $p$ to have a expected value of $1/\mu_0$ and a probability $1-p$ to have a expected value twice as large. So, as the proportion $p$ of short inactive sojourn times corresponds to the proportion $p$ of short active sojourn times and the proportion $1-p$ corresponds to the active and inactive long sojourn times, in the long term a very similar behavior to the exponential case if observed in the curves. For subsystem $S_2$ it is the same.

It is worth to say that Figure 4.10 has a rather small scale. For example a little “jump” can be appreciated for the atoms of the exponential case between the analytical solution and the simulation, but this approximation is just as good as the one shown for the exponential case in Figure 4.8. It can be seen in Figure 4.10 that the approximated analytical solution is good for every value of $p$.

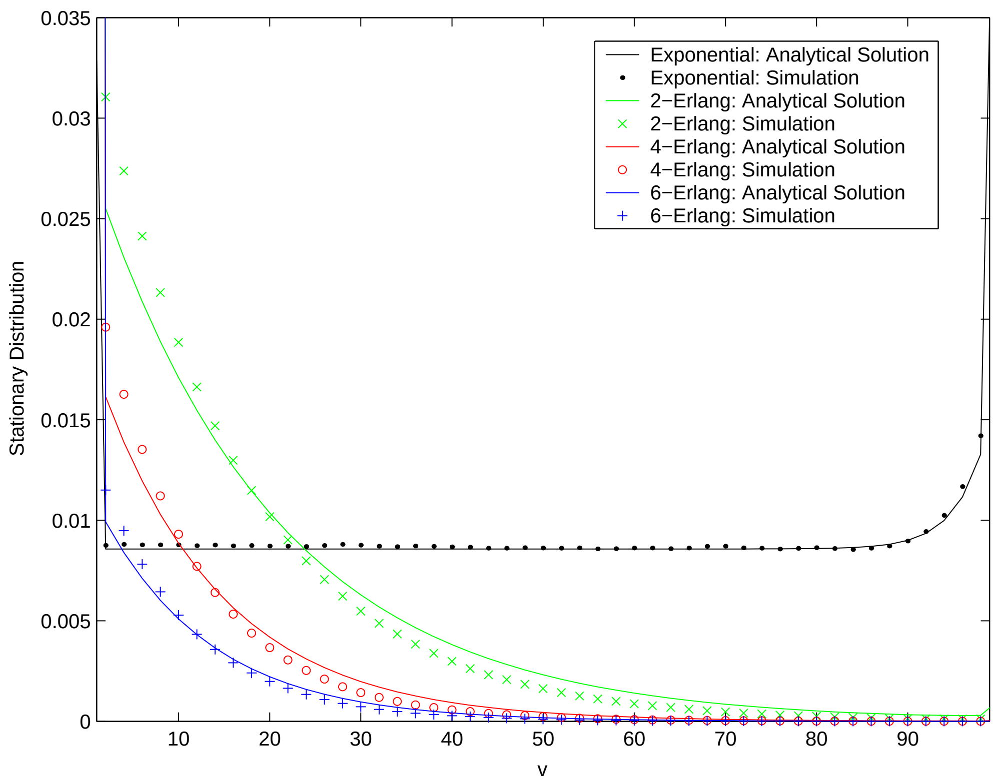


Figure 4.9: Detail: Stationary distribution of the buffer with $m$-Erlang distributed active sojourn time and exponentially distributed inactive sojourn time.

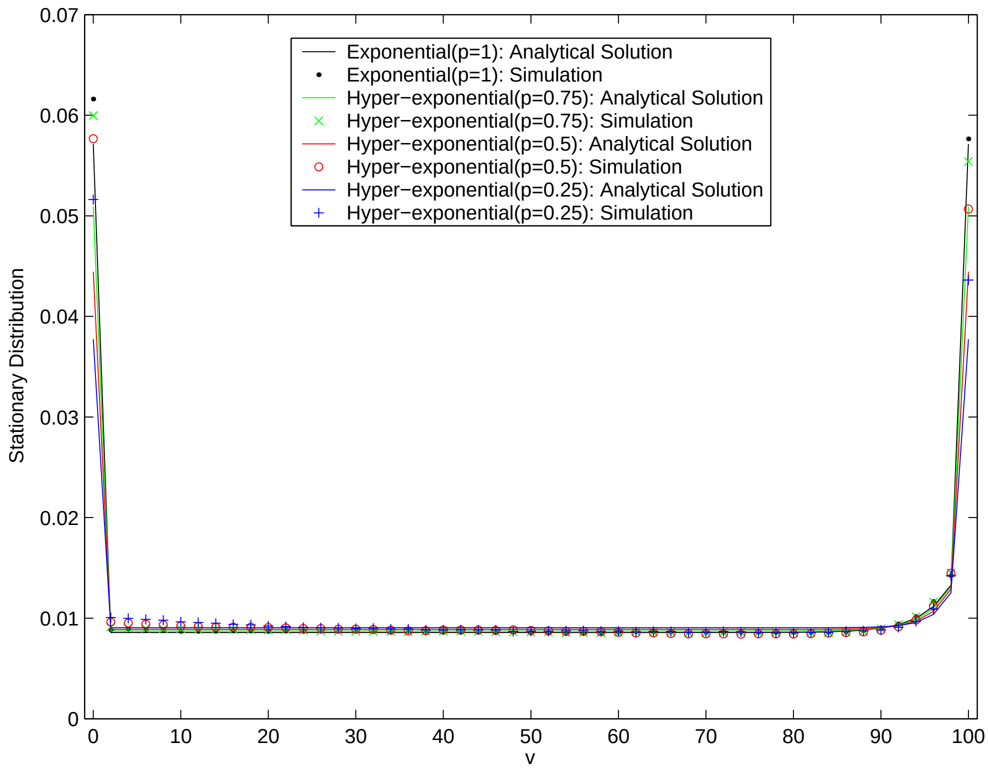


Figure 4.10: Stationary distribution of the Hyper-exponential semi-Markov case for the active and inactive sojourn time.

# Chapter 5

# N Equal Customers

## 5.1 Chapter Summary

In this chapter we study the stationary probability density of a system consisting of a finite capacity buffer connected to $N$ equal customers with bursty on-off demands. We assume that the buffer is filled up at a constant rate and we analyze the case when this filling rate satisfies an optimization condition according to the customer demands. We will also show that we can generalize the results for the stationary efficiency $K$.

During the active state, each customer demands information at a rate $f$. Hence, when $n$ customers are active, information is demanded at a rate $n \times f$. On the other hand, an unproductive situation is considered if the buffer is empty, i.e., $v = 0$. No product is required when all customers are inactive. The filling rate of the buffer $F$ is considered a constant. The buffer is filled as long as the amount of information is below its maximum capacity $V$, see Figure 5.1.

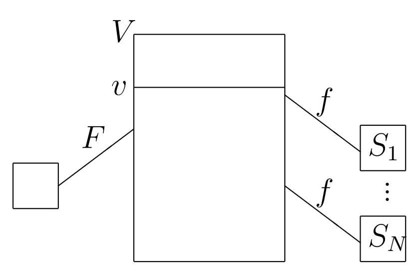


Figure 5.1: A system of $N$ independent customers and one buffer filled up at a constant rate $F$.

## 5.2 Markov Mathematical process to $N$ customers

We consider the superposition of $N$ on-off Markov processes as the *birth-and-death* process $\{\chi^{(N)}\}$ of the following form:

$$\chi^{(N)}(t) = \begin{cases} 0, & \text{if no customer is active} \\ 1, & \text{if one customer is active} \\ 2, & \text{if two customers are active} \\ \vdots & \vdots \\ N, & \text{if } N \text{ customers are active} \end{cases}.$$

The stochastic process $\chi^{(N)}$ is a Markov process on the phase space (or states) $\Theta^{(N)} = 0, 1, 2, ..N$ with the state diagram shown in Fig. 5.2.

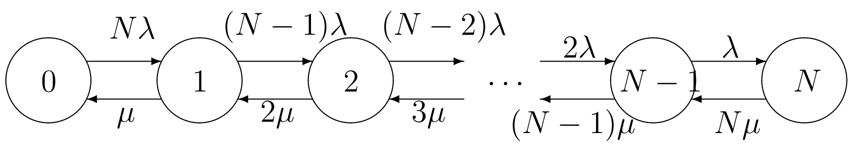


Figure 5.2: A system of $N$ independent customers that is a birth-and-death process with $N+1$ states.

Then, for this system we have the following matrix of sojourn time intensities

$$q^{(N)} = \begin{bmatrix} N\lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu+(N-1)\lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2\mu+(N-2)\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (N-1)\mu+\lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & N\mu \end{bmatrix}.$$

Also, we have a transition probability matrix given by

$$P^{(N)} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \dfrac{\mu}{\mu+(N-1)\lambda} & 0 & \dfrac{(N-1)\lambda}{\mu+(N-1)\lambda} & \cdots & 0 & 0 \\ 0 & \dfrac{2\mu}{2\mu+(N-2)\lambda} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \dfrac{\lambda}{\lambda+(N-1)\mu} \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Hence, the generating operator (or matrix) can be written as [21]

$$Q^{(N)} = q^{(N)}[P^{(N)} - I] = \tag{5.1}$$
$$\begin{bmatrix} -N\lambda & N\lambda & 0 & 0 & \dots & 0 & 0 \\ \mu & -(N-1)\lambda-\mu & (N-1)\lambda & 0 & \dots & 0 & 0 \\ 0 & 2\mu & -(N-2)\lambda-2\mu & (N-2)\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2\lambda & 0 \\ 0 & 0 & 0 & 0 & \dots & -\lambda-(N-1)\mu & \lambda \\ 0 & 0 & 0 & 0 & \dots & N\mu & -N\mu \end{bmatrix}.$$

Then, we consider a function $C^{(N)}(w)$ on the space $\mathbf{W}^{(N)} = \{0, 1, 2, \dots, N\} \times [0, V]$ defined as

$$C^{(N)}(w) = \begin{cases} F & w = \{0, v\},\ 0 < v < V; \\ F - f & w = \{1, v\},\ 0 < v < V; \\ F - 2f & w = \{2, v\},\ 0 < v < V; \\ \vdots & \vdots \\ F - (N-1)f & w = \{N-1, v\},\ 0 < v < V; \\ F - Nf & w = \{N, v\},\ 0 < v < V; \\ 0 & \text{other cases.} \end{cases}$$

Denote by $v(t)$ the amount of information in the buffer at time $t$. It can be easily verified that $v(t)$ satisfies the following equation

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = C^{(N)}(\chi^{(N)}(t), v(t)), \tag{5.2}$$

with the initial condition $v(0) = v_0 \in [0, V]$. Eq. (5.2) determines the random evolution of the system in the Markov medium $\chi^{(N)}(t)$ [17].

The sojourn time probability distribution functions, say $F_\theta^{(N)}(t)$, have the following form for the different states:

$$\begin{cases} F_0^{(N)}(t) &=& 1 - e^{-N\lambda t}, \\ F_1^{(N)}(t) &=& 1 - e^{-((N-1)\lambda+\mu)t}, \\ F_2^{(N)}(t) &=& 1 - e^{-((N-2)\lambda+2\mu)t}, \\ \vdots & \vdots & \vdots \\ F_{N-1}^{(N)}(t) &=& 1 - e^{-(\lambda+(N-1)\mu)t}, \\ F_N^{(N)}(t) &=& 1 - e^{-N\mu t}. \end{cases}$$

Now, denote as $f_\theta^{(N)} = \dfrac{\mathrm{d}F_\theta}{\mathrm{d}t}$ and $r_\theta^{(N)} = \dfrac{f_\theta(t)}{1 - F_\theta(t)}$ for all $\theta \in \Theta^{(N)}$, i.e.,

$$\begin{cases} r_0^{(N)} & = & N\lambda, \\ r_1^{(N)} & = & (N-1)\lambda + \mu, \\ r_2^{(N)} & = & (N-2)\lambda + 2\mu, \\ \vdots & \vdots & \vdots \\ r_{N-1}^{(N)} & = & \lambda + (N-1)\mu, \\ r_N^{(N)} & = & N\mu. \end{cases}$$

Then, $\xi^{(N)}(t) = \Big(\chi^{(N)}(t), v(t)\Big)$ is a Markov process with generator [17, 10]

$$A^{(N)}\phi(\theta, v) = C^{(N)}(\theta, v)\frac{\partial}{\partial v}\phi(\theta, v) + r_\theta^{(N)}[P^{(N)}\phi(\theta, v) - \phi(\theta, v)], \tag{5.3}$$

where $P^{(N)}\phi(\theta, v) = \sum_{y\in\Theta} p_{\theta y}\phi(y, v)$, or equivalently,

$$A^{(N)}\phi(\theta, v) = C^{(N)}(\theta, v)\frac{\partial}{\partial v}\phi(\theta, v) + Q^{(N)}\phi(\theta, v).$$

Denote by $\rho$ the stationary distribution of the process $\xi^{(N)}(t)$. Then, for every function $\phi(\cdot)$ belonging to the domain of the operator $A$ we have

$$\int_{\mathbf{W}^{(N)}} A^{(N)}\phi(w)\rho(dw) = 0. \tag{5.4}$$

Let $A^{(N)*}$ be the conjugate operator of $A^{(N)}$. By changing the order of integration in Eq. (5.4) we can obtain the following expression for the continuous part of $A^{(N)*}\rho$.

$$Q^{(N)T}\rho^{(N)} = \begin{bmatrix} F & 0 & 0 & \dots & 0 & 0 \\ 0 & F-f & 0 & \dots & 0 & 0 \\ 0 & 0 & F-2f & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & F-(N-1)f & 0 \\ 0 & 0 & 0 & \dots & 0 & F-Nf \end{bmatrix} \frac{\partial}{\partial v}\rho^{(N)}, \tag{5.5}$$

where $\rho^{(N)} = \begin{bmatrix} \rho(0,v) \\ \rho(1,v) \\ \rho(2,v) \\ \vdots \\ \rho(N-2,v) \\ \rho(N-1,v) \\ \rho(N,v) \end{bmatrix}$ and $\rho(\theta, v)$ denote the continuous part of $\rho$.

Let us write Eqs. (5.5) in more detail,

$$\begin{aligned} -N\lambda\rho(0,v) + \mu\rho(1,v) &= F\frac{\partial}{\partial v}\rho(0,v) \\ N\lambda\rho(0,v) - [(N-1)\lambda + \mu]\rho(1,v) + 2\mu\rho(2,v) &= (F-f)\frac{\partial}{\partial v}\rho(1,v) \\ (N-1)\lambda\rho(1,v) - [(N-2)\lambda + 2\mu]\rho(2,v) + 3\mu\rho(3,v) &= (F-2f)\frac{\partial}{\partial v}\rho(2,v) \\ (N-2)\lambda\rho(2,v) - [(N-3)\lambda + 3\mu]\rho(3,v) + 4\mu\rho(4,v) &= (F-3f)\frac{\partial}{\partial v}\rho(3,v) \\ \vdots &= \vdots \\ \lambda\rho(N-1,v) - N\mu\rho(N,v) &= (F-Nf)\frac{\partial}{\partial v}\rho(N,v). \end{aligned} \tag{5.6}$$

Then, we can write a general equation for any $n$ as

$$\begin{aligned} (F-(n-1)f)\frac{\partial}{\partial v}\rho(n-1,v) \quad = \quad & (N-(n-2))\lambda\rho(n-2,v) \\ & -[(N-(n-1))\lambda + (n-1)\mu]\rho(n-1,v) \\ & +n\mu\rho(n,v), \end{aligned} \tag{5.7}$$

such that $1 < n \leq N$.

Solving for $\rho(n,v)$ we have

$$\begin{aligned} \rho(n,v) \quad = \quad & \frac{(F-(n-1)f)}{n\mu}\frac{\partial}{\partial v}\rho(n-1,v) + \frac{[(N-(n-1))\lambda + (n-1)\mu]}{n\mu}\rho(n-1,v) \\ & -\frac{(N-(n-2))\lambda}{n\mu}\rho(n-2,v), \end{aligned} \tag{5.8}$$

for $1 < n \leq N$. We also know that

$$\rho(1,v) = \frac{F}{\mu}\frac{\partial}{\partial v}\rho(0,v) + \frac{N\lambda}{\mu}\rho(0,v). \tag{5.9}$$

## 5.3 Stationary Distribution

We can use Eqs. (5.8) and (5.9) to solve the continuous part of the stationary distribution of the system with two customers ($N = 2$).

As a first step we can evaluate Eqs. (5.8) and (5.9) for $N = 2$. Then we obtain

$$\rho(1, v) = \frac{F}{\mu}\frac{\partial}{\partial v}\rho(0, v) + \frac{2\lambda}{\mu}\rho(0, v) \tag{5.10}$$

and

$$\begin{aligned}\rho(n, v) &= \frac{(F-(n-1)f)}{n\mu}\frac{\partial}{\partial v}\rho(n-1, v) + \frac{((3-n)\lambda + (n-1)\mu)}{n\mu}\rho(n-1, v)\\ &\quad -\frac{(4-n)\lambda}{n\mu}\rho(n-2, v),\end{aligned} \tag{5.11}$$

for $n = N = 2$. Then, we obtain

$$\rho(2, v) = \frac{(F-f)}{2\mu}\frac{\partial}{\partial v}\rho(1, v) + \frac{(\lambda+\mu)}{2\mu}\rho(1, v) - \frac{\lambda}{\mu}\rho(0, v). \tag{5.12}$$

So far, we have from Eq.(5.10) an expression for $\rho(1, v)$ in terms of $\rho(0, v)$. If we use this expression into Eq. (5.12) we can also obtain an expression of $\rho(2, v)$ in terms of $\rho(0, v)$ as follows

$$\rho(2, v) = \frac{F(F-f)}{2\mu^2}\frac{\partial^2}{\partial v^2}\rho(0, v) + \frac{(3F\lambda - 2f\lambda + \mu F)}{2\mu^2}\frac{\partial}{\partial v}\rho(0, v) + \frac{\lambda^2}{\mu^2}\rho(0, v). \tag{5.13}$$

Then, we can use Eq. (4.16) and evaluate expressions (5.10) and (5.13) for $\rho(1, v)$ and $\rho(2, v)$. We obtain

$$\begin{aligned}&\frac{F(F-f)(F-2f)}{2\mu^2}\frac{\partial^2}{\partial v^2}\rho(0, v)\\ &+\left(\frac{F(F-f)}{\mu} + \frac{(F-f)(F-2f)\lambda}{\mu^2} + \frac{F(F-2f)(\lambda+\mu)}{2\mu^2}\right)\frac{\partial}{\partial v}\rho(0, v)\\ &+\left(F + \frac{2\lambda(F-f)}{\mu} + \frac{(F-2f)(\lambda+\mu)\lambda}{\mu^2} - \frac{(F-2f)\lambda}{\mu}\right)\rho(0, v) = 0.\end{aligned} \tag{5.14}$$

If we solve this equation we get the following result:

$$\rho(0,v) = C_{01} \exp\left\{-2\frac{F(\lambda+\mu)-2f\lambda}{F(F-2f)}\,v\right\} + C_{02}\exp\left\{\frac{(\lambda+\mu)}{F-f}\,v\right\}. \tag{5.15}$$

If we consider $F(\lambda+\mu)-2f\lambda = 0$ and we solve for $F$, we obtain

$$F = \frac{2f\lambda}{\lambda+\mu}, \tag{5.16}$$

which is a consistent condition for systems with $N$ customers. Actually, this condition can also be constructed if we make $F$ equal to the expected average demand of a two-customer system. We start considering the long-term proportion of time that one customer is active, i.e., $\frac{\lambda}{\lambda+\mu}$. If we multiply this proportion by $f$ then we obtain the long-term average customer demand. Let us remember that each customer process is independent. Then, if we make $F$ equal to the average demand of the two-customer system, we have $F = \frac{2f\lambda}{\lambda+\mu}$, which could be considered as an optimizing condition.

By considering condition (5.16) in Eq. (5.14) we can express this equation in the following form:

$$\frac{\partial^2}{\partial v^2}\rho(0,v) + \frac{(\lambda+\mu)^2}{f(\lambda-\mu)}\frac{\partial}{\partial v}\rho(0,v) = 0, \tag{5.17}$$

which has the following solution:

$$\rho(0,v) = C_{01} + C_{02}\exp\left\{-\frac{(\lambda+\mu)^2}{f(\lambda-\mu)}\,v\right\}. \tag{5.18}$$

If we look at Eqs. (5.10) and (5.13) it is not difficult to see that

$$\begin{aligned} \rho(1,v) &= C_{11} + C_{12}\exp\left\{-\frac{(\lambda+\mu)^2}{f(\lambda-\mu)}\,v\right\}, \\ \rho(2,v) &= C_{21} + C_{22}\exp\left\{-\frac{(\lambda+\mu)^2}{f(\lambda-\mu)}\,v\right\}. \end{aligned} \tag{5.19}$$

Now, we can think again about the system with three customers. If we recall Eqs. (5.8) and (5.9), and we evaluate them for $N=3$. Then, we obtain

$$\rho(1,v) = \frac{F}{\mu}\frac{\partial}{\partial v}\rho(0,v) + \frac{3\lambda}{\mu}\rho(0,v) \tag{5.20}$$

and

$$\begin{aligned} \rho(n,v) \;=\; & \frac{F-(n-1)f}{n\mu}\frac{\partial}{\partial v}\rho(n-1,v) + \frac{(4-n)\lambda+(n-1)\mu}{n\mu}\rho(n-1,v) \\ & -\frac{(5-n)\lambda}{n\mu}\rho(n-2,v) \end{aligned} \tag{5.21}$$

for $1 < n \leq N$. Then we may evaluate this equation for $n = 2$ and $n = 3$ giving

$$\rho(2,v) = \frac{(F-f)}{2\mu}\frac{\partial}{\partial v}\rho(1,v) + \frac{2\lambda+\mu}{2\mu}\rho(1,v) + \frac{3\lambda}{2\mu}\rho(0,v) \tag{5.22}$$

and

$$\rho(3,v) = \frac{(F-2f)}{3\mu}\frac{\partial}{\partial v}\rho(2,v) + \frac{\lambda+2\mu}{3\mu}\rho(2,v) + \frac{2\lambda}{3\mu}\rho(1,v). \tag{5.23}$$

It is not difficult to prove that, as an extension of Eq. (4.16) for the two-customer system, the following equality holds for the three-customer system

$$F\rho(0,v) + (F-f)\rho(1,v) + (F-2f)\rho(2,v) + (F-3f)\rho(3,v) = 0. \tag{5.24}$$

Now, we can use expressions (5.20), (5.22) and (5.23) in Eq. (5.24) just as in the case for a two-customer system and we can obtain an expression for $\rho(0,v)$. If we use the condition

$$F = \frac{3f\lambda}{\lambda+\mu}, \tag{5.25}$$

which resembles condition (5.16), then we can express Eq. (5.24) in the following form:

$$f^2(2\lambda-\mu)(\lambda-2\mu)\frac{\partial^3}{\partial v^3}\rho(0,v) + 4f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial^2}{\partial v^2}\rho(0,v) + 2(\lambda+\mu)^4\frac{\partial}{\partial v}\rho(0,v) = 0. \tag{5.26}$$

Eq. (5.25) can be constructed by making $F$ equal to the expected average demand of the three-customer system in the same way we constructed Eq. (5.16) for the two-customer system. Also, we may find Eq. (5.25) as a condition to eliminate one of the roots in the equation for $\rho(0,v)$, just as we found Eq. (5.16). Conditions like these are present ahead in equations for systems with a larger number of customers.

It is not difficult to see that the solution for differential equation (5.26) is of the following manner:

$$\begin{aligned}\rho(0,v) \quad &= \quad C_{01} + C_{02}\exp\left\{-\frac{(2\lambda-2\mu+\sqrt{2\lambda\mu})(\lambda+\mu)^2}{(2\lambda-\mu)(\lambda-2\mu)f}\,v\right\} \\ &\quad + C_{03}\exp\left\{-\frac{(2\lambda-2\mu-\sqrt{2\lambda\mu})(\lambda+\mu)^2}{(2\lambda-\mu)(\lambda-2\mu)f}\,v\right\}.\end{aligned} \tag{5.27}$$

Therefore, solutions for $\rho(1,v)$, $\rho(2,v)$ and $\rho(3,v)$ take the general form of $\rho(0,v)$

$$\rho(i,v) \quad = \quad C_{i1} + C_{i2}\exp\left\{-\frac{(2\lambda-2\mu+\sqrt{2\lambda\mu})(\lambda+\mu)^2}{(2\lambda-\mu)(\lambda-2\mu)f}\,v\right\}$$

$$+C_{i3}\exp\left\{-\frac{(2\lambda-2\mu-\sqrt{2\lambda\mu})(\lambda+\mu)^2}{(2\lambda-\mu)(\lambda-2\mu)f}v\right\}, \tag{5.28}$$

for $i = \{0, 1, 2, 3\}$.

Let us focus on equations for $\rho(0, v)$, which lead to the general form of the rest of the continuous part of the stationary distribution.

We can use Eqs. (5.8), (5.9) along with the $N$-general form equation

$$\sum_{n=0}^{N}(F-nf)\rho(n,v)=0, \tag{5.29}$$

that comes from the generalization of Eqs. (4.16) and (5.24), and the condition

$$F=\frac{Nf\lambda}{\lambda+\mu}, \tag{5.30}$$

that comes from the generalization of conditions (5.16) and (5.25), to find the set of equations for $N = 1, 4, 5, 6, 7, 8$ just as it is shown in Table 5.1.

If we look at the equations in Table 5.1, it is not difficult to realize the pattern governing them. It can be seen that there is a difference between equations for even and odd $N$ according to factor

$$\left(\frac{\partial}{\partial v}+\frac{(\lambda+\mu)^2}{(\lambda-\mu)f}\right). \tag{5.31}$$

The rest of the equations follow the same pattern for every $N$. Then, we can write a tentative general equation for a system with any $N > 2$ customers.

For odd $N > 2$ we may say that

$$\begin{aligned}\Bigg(\prod_{n=1}^{(N-1)/2}&(f^2((N-n)\lambda-n\mu)(n\lambda-(N-n)\mu)\frac{\partial^2}{\partial v^2}\\ &+2(N-n)nf(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}\\ &+(N-n)n(\lambda+\mu)^4)\Bigg)\quad\frac{\partial}{\partial v}\rho(0,v)\quad=0.\end{aligned} \tag{5.32}$$

For even $N > 2$ we may say that

$$\begin{aligned}\Bigg(\prod_{n=1}^{(N/2)-1}&(f^2((N-n)\lambda-n\mu)(n\lambda-(N-n)\mu)\frac{\partial^2}{\partial v^2}\\ &+2(N-n)nf(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}\\ &+(N-n)n(\lambda+\mu)^4)\Bigg)\quad\left(\frac{\partial}{\partial v}+\frac{(\lambda+\mu)^2}{(\lambda-\mu)f}\right)\frac{\partial}{\partial v}\rho(0,v)\quad=0.\end{aligned} \tag{5.33}$$

| | |
|---|---|
| $N$=1 | $\frac{\partial}{\partial v}\rho(0,v) = 0$ |
| $N$=2 | $\left(\frac{\partial}{\partial v}+\frac{(\lambda+\mu)^2}{(\lambda-\mu)f}\right)\frac{\partial}{\partial v}\rho(0,v) = 0$ |
| $N$=3 | $\left(f^2(2\lambda-\mu)(\lambda-2\mu)\frac{\partial^2}{\partial v^2}+4f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+2(\lambda+\mu)^4\right) \times\frac{\partial}{\partial v}\rho(0,v) = 0$ |
| $N$=4 | $\left(f^2(3\lambda-\mu)(\lambda-3\mu)\frac{\partial^2}{\partial v^2}+6f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+3(\lambda+\mu)^4\right) \times\left(\frac{\partial}{\partial v}+\frac{(\lambda+\mu)^2}{(\lambda-\mu)f}\right)\frac{\partial}{\partial v}\rho(0,v) = 0$ |
| $N$=5 | $\left(f^2(4\lambda-\mu)(\lambda-4\mu)\frac{\partial^2}{\partial v^2}+8f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+4(\lambda+\mu)^4\right) \times\left(f^2(3\lambda-2\mu)(2\lambda-3\mu)\frac{\partial^2}{\partial v^2}+12f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+6(\lambda+\mu)^4\right) \times\frac{\partial}{\partial v}\rho(0,v) = 0$ |
| $N$=6 | $\left(f^2(5\lambda-\mu)(\lambda-5\mu)\frac{\partial^2}{\partial v^2}+10f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+5(\lambda+\mu)^4\right) \times\left(f^2(4\lambda-2\mu)(2\lambda-4\mu)\frac{\partial^2}{\partial v^2}+16f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+8(\lambda+\mu)^4\right) \times\left(\frac{\partial}{\partial v}+\frac{(\lambda+\mu)^2}{(\lambda-\mu)f}\right)\frac{\partial}{\partial v}\rho(0,v) = 0$ |
| $N$=7 | $\left(f^2(6\lambda-\mu)(\lambda-6\mu)\frac{\partial^2}{\partial v^2}+12f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+6(\lambda+\mu)^4\right) \times\left(f^2(5\lambda-2\mu)(2\lambda-5\mu)\frac{\partial^2}{\partial v^2}+20f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+10(\lambda+\mu)^4\right) \times\left(f^2(4\lambda-3\mu)(3\lambda-4\mu)\frac{\partial^2}{\partial v^2}+24f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+12(\lambda+\mu)^4\right) \times\frac{\partial}{\partial v}\rho(0,v) = 0$ |
| $N$=8 | $\left(f^2(7\lambda-\mu)(\lambda-7\mu)\frac{\partial^2}{\partial v^2}+14f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+7(\lambda+\mu)^4\right) \times\left(f^2(6\lambda-2\mu)(2\lambda-6\mu)\frac{\partial^2}{\partial v^2}+24f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+12(\lambda+\mu)^4\right) \times\left(f^2(5\lambda-3\mu)(3\lambda-5\mu)\frac{\partial^2}{\partial v^2}+30f(\lambda-\mu)(\lambda+\mu)^2\frac{\partial}{\partial v}+15(\lambda+\mu)^4\right) \times\left(\frac{\partial}{\partial v}+\frac{(\lambda+\mu)^2}{(\lambda-\mu)f}\right)\frac{\partial}{\partial v}\rho(0,v) = 0$ |

Table 5.1: $\rho(0,v)$ equations for $N=1,2,3,4,5,6,7,$ and 8.

Now we can proceed to find a general solution for (5.32) and (5.33). As it can be seen, equations in Table 5.1 are factored in second order polynomials, therefore it is not difficult to find their roots. We obtain the general solutions as follows:

For odd $N > 2$,

$$\begin{aligned}\rho(i,v) &= C_{i,0} \\ &+ \sum_{n=1}^{(N-1)/2} C_{i,n} \exp\left\{-\frac{\left((N-n)n(\lambda-\mu)+(N-2n)\sqrt{(N-n)n\lambda\mu}\right)(\lambda+\mu)^2}{((N-n)\lambda-n\mu)(n\lambda-(N-n)\mu)f}\,v\right\} \quad (5.34) \\ &+ \sum_{m=1}^{(N-1)/2} C_{i,N-m} \exp\left\{-\frac{\left((N-m)m(\lambda-\mu)-(N-2m)\sqrt{(N-m)m\lambda\mu}\right)(\lambda+\mu)^2}{((N-m)\lambda-m\mu)(m\lambda-(N-m)\mu)f}\,v\right\}.\end{aligned}$$

For even $N \geq 2$,

$$\begin{aligned}\rho(i,v) &= C_{i,0} + C_{i,N/2} \exp\left\{-\frac{(\lambda+\mu)^2}{(\lambda-\mu)f}\,v\right\} \\ &+ \sum_{n=1}^{(N/2)-1} C_{i,n} \exp\left\{-\frac{\left((N-n)n(\lambda-\mu)+(N-2n)\sqrt{(N-n)n\lambda\mu}\right)(\lambda+\mu)^2}{((N-n)\lambda-n\mu)(n\lambda-(N-n)\mu)f}\,v\right\} \quad (5.35) \\ &+ \sum_{m=1}^{(N/2)-1} C_{i,N-m} \exp\left\{-\frac{\left((N-m)m(\lambda-\mu)-(N-2m)\sqrt{(N-m)m\lambda\mu}\right)(\lambda+\mu)^2}{((N-m)\lambda-m\mu)(m\lambda-(N-m)\mu)f}\,v\right\}.\end{aligned}$$

As we know, these solutions are not only valid for $\rho(0,v)$, but also for the general form of the rest of the stationary distribution. Nevertheless we may just consider the constants from $\rho(0,v)$ as $C_{0,n} \Leftrightarrow C_n$, $n \in \{0,1,...,N-1\}$ in order to simplify the solution and we may express the rest of the distributions $\rho(i,v)$ in terms of $\rho(0,v)$ using Eqs. (5.8) and (5.9).

The analysis of the properties of the process $\xi^{(N)}(t)$ leads up to the conclusion that, for the case $af < F < bf$, the stationary distribution $\rho$ has atoms at the points $(b,0),\ (b+1,0),...,(N,0)$, and at the points $(0,V),\ (1,V),...,(a,V)$. We denote them as $\rho[b,0],\ \rho[b+1,0],...,\rho[N,0]$ and $\rho[0,V],\ \rho[1,V],...,\rho[a,V]$.

We get from the discrete part of Eq. (5.4) the following expressions

$$\begin{bmatrix} F & 0 & 0 & \dots & 0 & 0 \\ 0 & F-f & 0 & \dots & 0 & 0 \\ 0 & 0 & F-2f & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & F-(N-1)f & 0 \\ 0 & 0 & 0 & \dots & 0 & F-Nf \end{bmatrix} \begin{bmatrix} \rho(0,0+) \\ \rho(1,0+) \\ \rho(2,0+) \\ \vdots \\ \rho(N-2,0+) \\ \rho(N-1,0+) \\ \rho(N,0+) \end{bmatrix} = Q^{(N)T}\mathbf{R}(0), \tag{5.36}$$

where $\mathbf{R}(0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \rho[b,0] \\ \rho[b+1,0] \\ \vdots \\ \rho[N,0] \end{bmatrix}$. Also,

$$\begin{bmatrix} F & 0 & 0 & \dots & 0 & 0 \\ 0 & F-f & 0 & \dots & 0 & 0 \\ 0 & 0 & F-2f & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & F-(N-1)f & 0 \\ 0 & 0 & 0 & \dots & 0 & F-Nf \end{bmatrix} \begin{bmatrix} \rho(0,V-) \\ \rho(1,V-) \\ \rho(2,V-) \\ \vdots \\ \rho(N-2,V-) \\ \rho(N-1,V-) \\ \rho(N,V-) \end{bmatrix} = -Q^{(N)T}\mathbf{R}(V), \tag{5.37}$$

where $\mathbf{R}(V) = \begin{bmatrix} \rho[0,V] \\ \rho[1,V] \\ \vdots \\ \rho[a,V] \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

If we use $b = a+1$, that is, $a < F < a+1$ it is easy to see from Eq. (5.36) that

$$\begin{cases} \rho(0,0+) &= 0, \\ \rho(1,0+) &= 0, \\ &\vdots \\ \rho(a-1,0+) &= 0. \end{cases} \tag{5.38}$$

Also, from Eq. (5.37) we obtain that

$$\begin{cases} \rho(a+2, V-) &= 0, \\ \rho(a+3, V-) &= 0, \\ &\vdots \\ \rho(N, V-) &= 0. \end{cases} \tag{5.39}$$

It is not difficult to use Eqs. (5.38) and (5.39) to obtain expressions for the constants $C_n$, $n \in \{1, 2, ..., N-1\}$.

Also, by using expressions in Eq. (5.36) not equal to zero we may obtain expressions for the atoms in $\mathbf{R}(0)$ in terms of the continuous part $\rho(\theta, 0+)$. In the same way we may use (5.37) to obtain expressions for the atoms in $\mathbf{R}(V)$ in terms of the continuous part $\rho(\theta, V-)$.

After that it is not difficult to calculate constant $C_0$ using the normalization condition

$$\int_{\mathbf{W}^{(\mathbf{N})}} \rho(\mathrm{d}w) = 1.$$

That completes the calculation of the stationary distribution $\rho$ of the system.

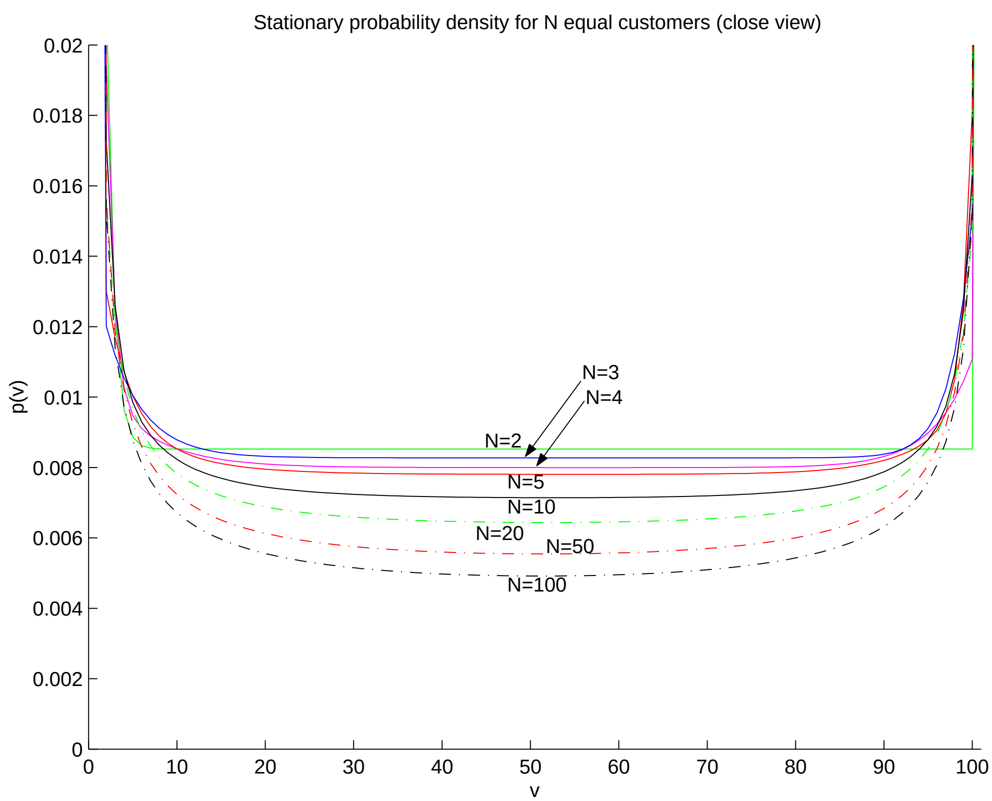


Figure 5.3: Stationary distribution of the buffer for N equal customers.

Figures 5.3 and 5.4 show some plots of the stationary probability density of the level of the buffer for the cases $N = 2, 3, 4, 5, 10, 20, 50$. The first Figure depicts only de stationary distribution and the second one adds some simulations for these $N$ cases.

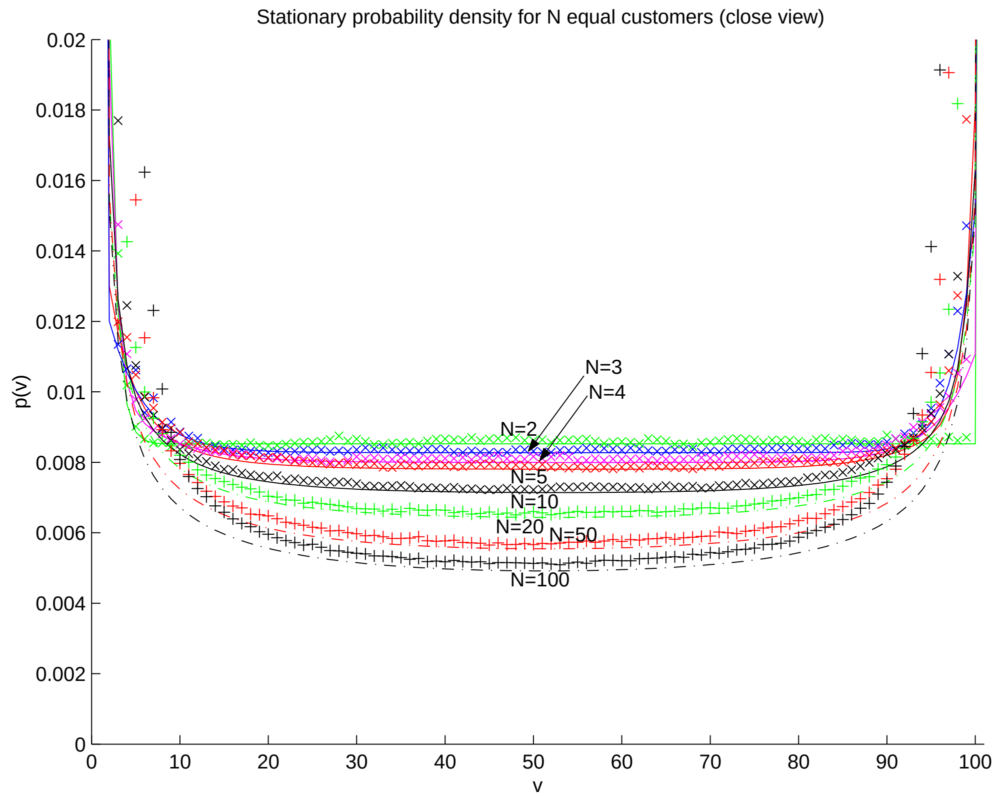


Figure 5.4: Stationary distribution of the buffer for N equal customers and simulation.

## 5.4 Stationary Efficiency

The stationary distribution of this system $\rho$ has already been found for any $N$. Now, we are going to use those results to find generalized results for the stationary efficiency.

Let us define the function $f^{(N)}(w)$, where $w \in \mathbf{W} = \Theta^{(N)} \times [0, V]$, of the following from

$$f(w) := \begin{cases} f, & if \quad w = \{1, v\},\ 0 < v \leq V; \\ 2f, & if \quad w = \{2, v\},\ 0 < v \leq V; \\ 3f, & if \quad w = \{3, v\},\ 0 < v \leq V; \\ \vdots & \qquad\qquad\qquad \vdots \\ Nf, & if \quad w = \{N, v\},\ 0 < v \leq V; \\ 0 & \qquad\qquad \text{in other cases.} \end{cases} \tag{5.40}$$

This is the efficiency of the system.

Let us assume the joint stochastic process with a two-dimensional phase space $\xi(t) = (\bar{\chi}(t), \bar{v}(t))$. Then, we can state the following equality

$$K = \lim_{T\to\infty} \frac{V(T)}{T} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(\xi(t)\right)\, dt. \tag{5.41}$$

It follows from ergodic theory [21] that if the process $\xi(t)$ has a stationary distribution $\rho(\cdot)$, then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(\xi(t)\right)\,\mathrm{d}t=\int_{\mathbf{W}} f(w)\mathrm{d}\rho(w). \tag{5.42}$$

Hence, by using Eq. (5.41) we obtain for the stationary efficiency of the system that

$$K=\int_{\mathbf{W}} f(w)\mathrm{d}\rho(w)=\int_{\mathbf{W}} f(w)\rho(\mathrm{d}w). \tag{5.43}$$

We will begin the generalization of the stationary efficiency of the system with the case of the $N=2$. For this case we have

$$K=f\int_0^V\{\rho_r(1,v)+2\rho_r(2,v)\}\mathrm{d}v+f\rho_r[1,V]. \tag{5.44}$$

We can identify $\dfrac{2f\lambda}{\lambda+\mu}$ as the expected average demand of the two customers. Then, for the stationary efficiency of the system, we can identify three different scenarios respect to the incoming stream $F$. Those are

$$F=F_a=\frac{2f\lambda}{\lambda+\mu},\quad F=F_b<\frac{2f\lambda}{\lambda+\mu}\quad\text{and}\quad F=F_c>\frac{2f\lambda}{\lambda+\mu}.$$

Let us use the choice $f<F<2f$ as an instance to define the existing atoms of the system. Then, for the case $F=F_a$ we have $f<\dfrac{2f\lambda}{\lambda+\mu}<2f$. Then, we can use without loss of generality $f=1$. Also, in order to simplify calculations, we can use the substitution $\mu=\alpha\lambda$, $0<\alpha$. Then, we obtain $1<\dfrac{2}{1+\alpha}<2$. Then, we have $0<\alpha<1$. We calculate the stationary efficiency of the system using Eq. (5.44) as

$$K_a(V)=2\frac{\lambda(1+\alpha)^2V+2\alpha+\alpha(\alpha-1)\exp\left(\dfrac{\lambda(1+\alpha)^2V}{\alpha-1}\right)}{(1+\alpha)(\lambda(1+\alpha)^2V+3\alpha+1)+\alpha(\alpha^2-1)\exp\left(\dfrac{\lambda(1+\alpha)^2V}{\alpha-1}\right)}.$$

For this parameter we should study the case $V\to\infty$. We obtain

$$\lim_{V\to\infty}K_a(V)=\frac{2}{1+\alpha}=\frac{2\lambda}{\lambda+\mu}=F.$$

This is a rather good case where the stationary efficiency of the system equals the expected average demand and equals the incoming stream $F$.

Now we should study the cases $F=F_{b,c}$. For this cases we have $f<F_{b,c}<2f$. Let us use $F(k)=\dfrac{kf\lambda}{\lambda+\mu}$, where $F_b=F(k_b)$, $k_b<2$ and $F_c=F(k_c)$, $k_c>2$ . Then, we can use

$f = 1$ without loss of generality and we have $1 < \dfrac{k}{1+\alpha} < 2$. We obtain the stationary efficiency of the system in the following form.

$$\begin{aligned} K_{b,c}(V) &= \frac{2n\alpha((4k-k^2-4-4\alpha)e_1(V)+2\alpha k e_2(V)+(k-2)(k-2-2\alpha)e_3(V))}{(1+\alpha)(D_{k1})} \\ &+ \frac{2\alpha(2-k)((k-2-2\alpha)^2 e_4(V)-\alpha k^2 e_5(V))}{(1+\alpha)(D_{k2})}, \end{aligned}$$

where

$$\begin{aligned} D_{k1} &= (4\alpha+k^2-4k+4)(k-2-2\alpha)e_1(V)+\alpha^2k^3e_2(V) \\ &+2\alpha(k-2)(k-2-2\alpha)(k-1-\alpha)e_3(V) \end{aligned}$$

and

$$\begin{aligned} D_{k2} &= (4\alpha+4+k^2-4k)(k-2-2\alpha) \\ &+2\alpha(k-2)(k-2-2\alpha)(k-1-\alpha)e_4(V)+\alpha^2k^3e_5(V). \end{aligned}$$

Also,

$$\begin{aligned} e_1(V) &= \exp\left(\frac{V\lambda(1+\alpha)(4\alpha^2+(k^2-4k+8)\alpha+4+(k-2)k)}{k(k-2-2\alpha)(k-1-\alpha)}\right), \\ e_2(V) &= \exp\left(\frac{V\lambda(1+\alpha)(2\alpha^2+(4-k)\alpha+4-k)}{(k-2-2\alpha)(k-1-\alpha)}\right), \\ e_3(V) &= \exp\left(\frac{2\lambda V(1+\alpha)(2+4\alpha+(k+2)\alpha^2)}{k(k-2-2k)(k-1-\alpha)}\right), \\ e_4(V) &= \exp\left(-\frac{\lambda(1+\alpha)^2V}{k-1-\alpha}\right), \\ e_5(V) &= \exp\left(-\frac{2V(1+\alpha)^2(k-2)\lambda}{(k-2-2\alpha)k}\right) \end{aligned}$$

Now, we should study the cases $V \to \infty$ for $K_{b,c}(V)$. First we obtain

$$\lim_{V\to\infty} K_b(V) = \frac{2\alpha k_b}{(1+\alpha)(2+2\alpha-k_b)} < F_b = \frac{k_b}{1+\alpha} < \frac{2\lambda}{\lambda+\mu}.$$

Then, for the region $0 < k < 2$, the incoming stream $F$ is a boundary of the limit of the stationary efficiency $K(V)$ as the buffer size grows to infinity.

It is evident that $K(V)$ is a non-decreasing function of $F$. Also, we know that the maximum stationary efficiency of the system is the long term average customers demand. Then we have for the case $K_c(V)$ that

$$\lim_{V\to\infty} K_c(V) = \frac{2}{1+\alpha} < F_c$$

Then, for the region $2 < k$, the stationary efficiency $K(V)$ is less than the incoming stream $F$.

Let us plot an example of the limit of the stationary efficiency along the different regions of $k$. In Figure, 5.5 we use $\alpha = 1/6$ as an example. This choice gives us $7/6 < k < 14/6$ for $F = \dfrac{6k}{7}$ and for $K(V)$ as well.

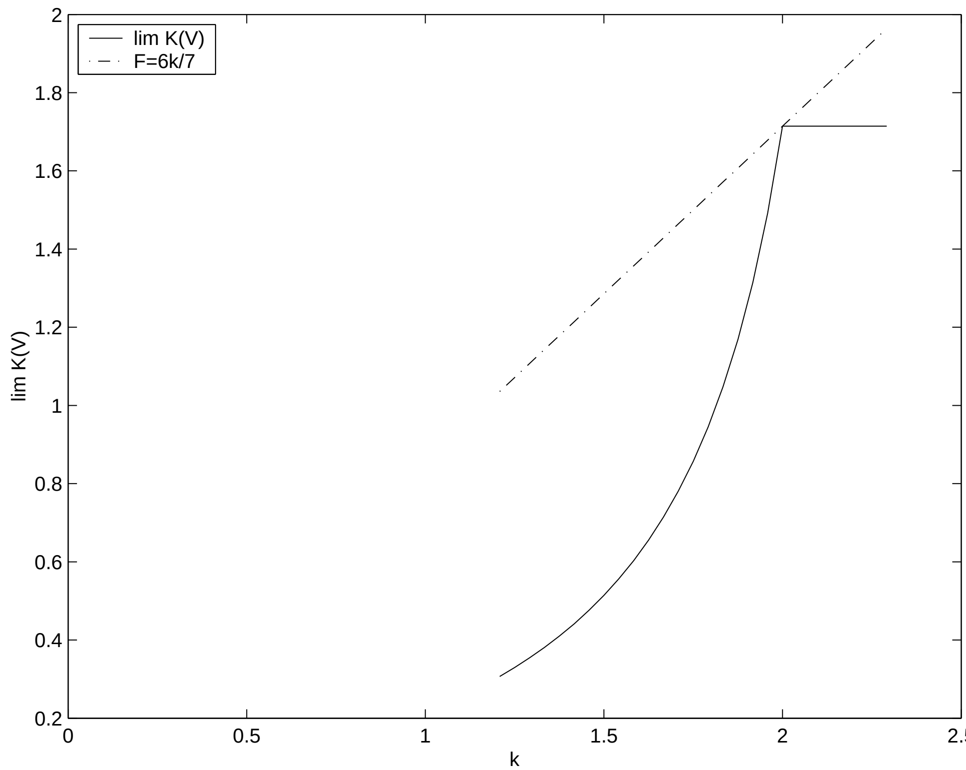


Figure 5.5: Limit of K(V) as V tends to infinity with $\alpha = 1/6$ and $F = 6k/7$.

To show that this results do not depend on the choice $f < F < 2f$, we can show that we can obtain the same results for the case $F < f$. With this choice we have for the case $F = F_a$ that $\dfrac{2f\lambda}{\lambda+\mu} < f$. Then, we can use without loss of generality $f = 1$ and we obtain $\dfrac{2}{1+\alpha} < 1$. We calculate the stationary efficiency of the system as.

$$K_a(V) = 2\alpha \frac{(1+\lambda(1+\alpha)^2V)\exp\left(\dfrac{\lambda(1+\alpha)^2V}{\alpha-1}\right) - 1}{1-\alpha^2+(1+\alpha)\alpha(\lambda(1+\alpha)^2V+\alpha+3)\exp\left(\dfrac{\lambda(1+\alpha)^2V}{\alpha-1}\right)},$$

Also, we obtain that

$$\lim_{V\to\infty} K_a(V) = \frac{2}{1+\alpha} = \frac{2\lambda}{\lambda+\mu} = F.$$

This is a good case where the stationary efficiency of the system equals the expected average demand and equals the incoming stream $F$.

For the cases $F_{b,c}$ we use the substitution $F(k) = \dfrac{kf\lambda}{\lambda+\mu}$, where $F_b = F(k_b)$, $k_b < 2$ and $F_c = F(k_c), k_c > 2$. Then, we can use $f = 1$ without loss of generality and we have $\dfrac{k}{1+\alpha} < 1$. We obtain the stationary efficiency of the system in the following form.

$$K_{b,c}(V) = \frac{2\alpha^2 n}{1+\alpha} \times \frac{(4\alpha + k^2 - 4k + 4)g_1(V) + k(2-k)g_2(V) - 4 + 2k - 4\alpha}{\alpha^2 n(4\alpha + k^2 - 4k + 4)g_1(V) + 2\alpha k(k-2)(k-1-\alpha)g_2(V) + (k-2-2\alpha)^3}, \tag{5.45}$$

where

$$\begin{aligned} g_1(V) &= \exp\left(-\frac{2V(1+\alpha)^2(k-2)\lambda}{(k-2-2\alpha)k}\right) \\ g_2(V) &= \exp\left(-\frac{(1+\alpha)^2(4\alpha + k^2 - 4k + 4)V\lambda}{(k-1-\alpha)(k-2-2\alpha)}\right) \end{aligned}$$

Now, we study the case $V \to \infty$ for $K_{b,c}$. First, given $k_b < 2$, we obtain

$$\lim_{V\to\infty} K_b(V) = \frac{4\alpha^2 k_b}{(k_b - 2 - 2\alpha)^2(1+\alpha)} < F_b = \frac{k_b}{1+\alpha} < \frac{2\lambda}{\lambda+\mu}$$

Then, for this case where the incoming stream is less than the expected long term average customers demand, we obtain that the stationary efficiency of the system is even less than the incoming stream no matter the buffer grows to infinity. Then, for this case we also have that for the region $0 < k < 2$, the incoming stream $F$, is a boundary of the limit of stationary efficiency $K(V)$ as the buffer size grows to infinity.

On the other hand, it is not difficult to show from Eq. (5.45) that

$$\lim_{V\to\infty} K_c(V) = \frac{2}{1+\alpha} < F_c.$$

Then, for the region $2 < k$, the stationary efficiency $K(V)$ is less than the incoming stream $F$. This result was expected because it is evident that $K(V)$ is a non-decreasing function of $F$, i.e., k, and we know that the maximum stationary efficiency of the system is the long term average customers demand.

Let us plot an example of the limit of the stationary efficiency along the different regions of $k$. In Figure (5.6) we use $\alpha = 2$ as an example. This choice gives us $k < 3$ for $F = \dfrac{k}{3}$ and for $K(V)$ as well.

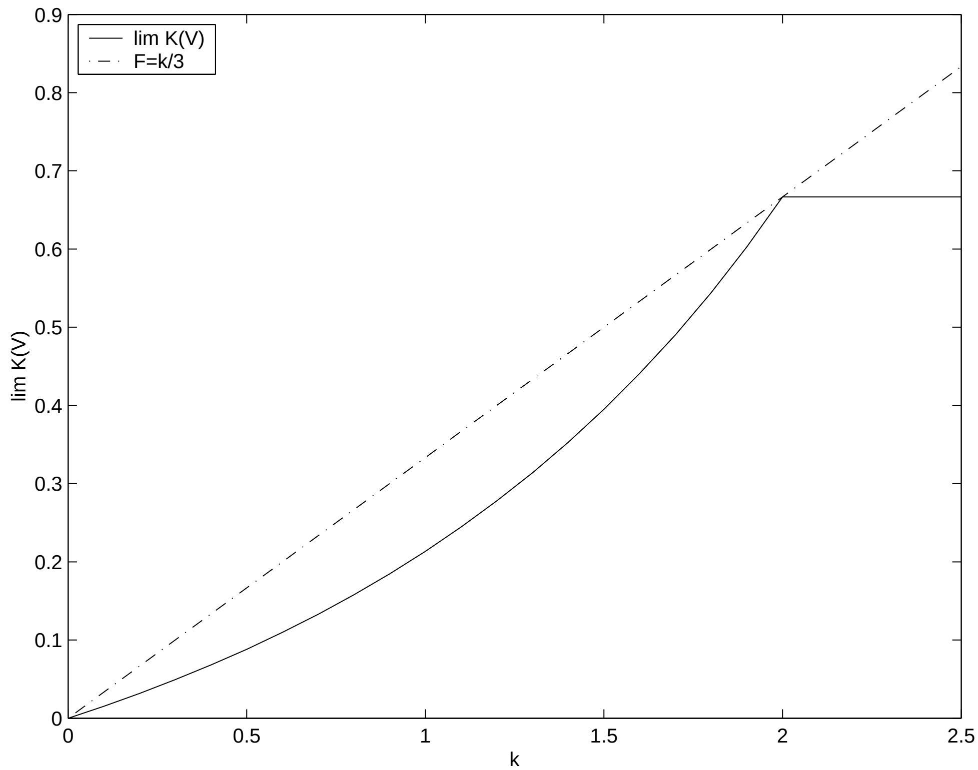


Figure 5.6: Limit of K(V) as V tends to infinity with $\alpha = 2$ and $F = k/3$.

Then as conclusions we can say that the stationary efficiency does not depend directly on $\lambda$ nor $\mu$ but in the ratio $\alpha = \lambda/\mu$. Also, we can say that no matter the buffer grows to infinity, the stationary efficiency of this system is less than the incoming stream $F$ everywhere except in one point that is $F = 2/(1+\alpha)$. Also we can say that the maximum stationary efficiency of the system is the long term average customers demand. The system reaches this efficiency for $F \geq \dfrac{2}{1+\alpha}$ and $V \to \infty$.

Once we know this, it is possible to design the system for a determined efficiency or percentage of the maximum efficiency using expressions for the stationary efficiency shown above.

## 5.5 Generalization for any $N$

Now we can start generalizing results for a system with any $N$. Calculations for the stationary efficiency for any $F$ can be cumbersome, but now we can focus on the case the incoming stream equals the long term average customers demand for the system with any number $N$ of customers. As it was shown we can use $f = 1$ without loss of generality and we have $F = \dfrac{Nf\lambda}{\lambda + \mu} = \dfrac{N}{1+\alpha}$. It is not difficult to prove the following results

$$\begin{aligned}
N = 1, \quad F = \frac{1}{1+\alpha}, \quad \lim_{V\to\infty} K^{(1)}(V) = \frac{1}{1+\alpha};\\
N = 3, \quad F = \frac{3}{1+\alpha}, \quad \lim_{V\to\infty} K^{(3)}(V) = \frac{3}{1+\alpha};\\
N = 4, \quad F = \frac{4}{1+\alpha}, \quad \lim_{V\to\infty} K^{(4)}(V) = \frac{4}{1+\alpha};\\
N = 5, \quad F = \frac{5}{1+\alpha}, \quad \lim_{V\to\infty} K^{(5)}(V) = \frac{5}{1+\alpha}.
\end{aligned}$$

Then, if we add the result we already have for $N = 2$, we can generalize in the following form

$$\lim_{V\to\infty} \left[K^{(N)}(V)\right]_{F \geq \frac{Nf}{1+\alpha}} = \frac{Nf}{1+\alpha} = \frac{Nf\lambda}{\lambda+\mu},$$

for any $a < F < a+1$, $a = 0, 1, 2, ..., c$.

Although it was mention that for this system the incoming stream $F$ is turned off once the buffer reaches its maximum capacity $V$, all the results shown here regarding the stationary efficiency of the system also match those of a system with an overflowed buffer where the incoming stream $F$ is always on. For that system we can consider the stationary overflowed information as

$$H = f\rho[1, V] + 2f\rho[2, V] + \cdots + af\rho[a, V], \tag{5.46}$$

for $a < F < a+1$.

# Chapter 6

# Conclusions

In this thesis we studied the stationary efficiency of a system consisting of a finite capacity buffer connected to $N$ equal customers with bursty on-off demands. We assume that the alternating demands can be modeled by a semi-Markov stochastic process and we assume that the buffer is filled up at a constant rate.

It was shown that it is possible to use the *phase merging algorithm* to reduce a semi-Markov process to an approximated Markov process. Once this is done, it is possible to find some closed-form expression for the stationary distribution of the system. It has been seen that the approximation that the algorithm gives may be good enough for some applications. We showed plots of some analytical results and computer simulations regarding the Markov and semi-Markov cases.

Also, it has been shown that the approximation can also be considered to obtain expressions for the stationary efficiency of the system for some semi-Markov cases. Besides the driving function $C(w)$ typical in the random evolution formulation [14], we introduce an additional function $f(w)$ to capture the functionality of our scheme. Two cases were studied regarding the incoming stream $F$ in terms of the stationary efficiency $K$ and we showed some typical cases for this parameter. Some graphics of the performance parameter $K$ where added and we analyzed an optimization condition which is related to the long-term average demand of the customers.

First, we considered the case of two different customers which may be a system of interest for some applications. Regardless the fact that we only superposed two processes, the number of different parameters makes it difficult to find expressions for the stationary probability density of the system and the stationary efficiency. Nevertheless, these expressions are showed in terms of the system parameters, namely $F$, $f_0$, $f_1$, $\lambda_0$, $\lambda_1$, $\mu_0$ and $\mu_1$.

In the Chapter 4 we considered the case of two equal customers. It is easy to see that the result from the case of two different customers can not be reduced a result for the case of two equal customers. Also, it is important to show this case because it shows how the formulation is simplified by this assumption. The use of the birth and death process is introduced and this is an important previous step before finding a more general solution for the case of the superposition of any number $N$ of different customers, namely, processes.

In Chapter 5 We found that the problem of the single buffer with $N$ equal customers connected to it has a general solution for the stationary probability distribution of the amount of stored information. We found this general solution considering only one condition that could be considered as an optimizing condition. We constructed this condition so that the stream of information is equal to the expected average demand of the system, and it is also present as one of the roots of the general equation to be solved. By having this condition, the general equation is simplified so that the solution may present one less exponential term.

Our buffer is a subsystem aimed to increase the availability of information. It is worth to mention that, even thought it was mentioned as part of the system functionality that the main stream $F$ turns *off* when the buffer reaches its maximum capacity, the results presented here also match to those of a system with an overflowed buffer. That is, a system where the main stream $F$ is always *on* and when the buffer reaches its maximum capacity some data may be thrown away.

We showed that the maximum efficiency of the system with any $N$ customers is precisely the expected average demand of the customers. If the incoming stream is equal or greater than the expected average demand of the customers, the system reaches its maximum efficiency if the buffer size grows to infinity. Nevertheless, only when the incoming stream $F$ equals the expected average demand of the customers the efficiency of the system equals the incoming stream when buffer size grows to infinity. In the other case, the efficiency of the system $K$ is below the incoming stream $F$. These results also match those of system with an overflowed buffer where the incoming stream $F$ is always on.

It is not difficult to see that this model is the same as the one that would be used for a particle that moves in one dimension with absorbing boundaries that retain the particle until it changes direction. For the case of one process (customer), this particle would have only two velocities in different directions. For the case of $N$ processes this particle would have $N$ different velocities and every time it changes state it can either accelerate its velocity or it can decelerate up to the point where it changes direction.

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