Electromagnetism

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Dedicated to my mother, my father and my sister Carolina, who have always believed in me. In the darkest days of my life, they have always been the light that guides me and the strength to continue no matter the circumstances.

Contents

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Preface

This book started as lecture notes aimed for sophomore undergraduate students who study at Tecnológico de Monterrey, Campus Querétaro, México, who are studying for first time electromagnetism. This book is highly based on references [1], [2], [3], [4], [5], [6], [8] and [9] , and many of the exercises presented in this book are from those references.

In this book is covered the basics of electromagnetism and we mention some applications of electromagnetism. We will introduce as smoothly as possible mathematical formalism. All the material is self contained, just dynamics and kinematics as integral and differential calculus is assumed. We will focus in understanding the physics, in presenting different practice exercises, and the nature of electromagnetism in different situations, rather than a highly formal and mathematical description.

Part I

Electrostatics

Chapter 1

Electric Fields

1.1 Electric Charge

All particles in the universe have certain intrinsic properties that define how they interact with each other. The first example of these properties that is commonly studied in physics is the mass. All massive objects in the Universe interact gravitationally with each other given to the fact that they have mass. The property that defines how electromagnetic interaction takes place is the intrinsic property of *electric charge*. The electric charge, mathematically is represented by a real number and have units of Coulombs (C). However, so far we have found in the Universe that the electric charge is quantized, i.e. it is a multiple of a very specific number.

$$\boxed{q = n \cdot 1.6021765 \times 10^{-19}\mathrm{C} \qquad \text{where } n \in Z} \tag{1.1}$$

The number $1.6021765 \times 10^{-19}$C is the electric charge of the proton and the magnitude of the electric charge of the electron. It is also sometimes called as the *fundamental charge e*. We have not found any particle in the Universe so far that has a different electric charge. In the so called *Standard Model*, that is the theory that unifies three of the four fundamental interactions in Nature includes particles called *quarks*, which have fractional electric charge of e. Even though there is experimental evidence of their existence(1990 Physics Nobel Prize), we have not seen them isolated in Nature. However, even the electric charge is quantized, it is very common to take it just as a real number in our calculations, because the gap between one electron charge, two electron charges, three electron charges, ..., so on and so forth are so close to each other that mostly they create a continuum. Actually, strictly speaking, we should not be able to integrate (as you will see later on) at any time, because the integral is a continuum infinitesimal sum. However, the gaps are so close, that once again we just assume they create a continuum.

Along this course we will be dealing essentially with neutrons,electrons and protons, and their properties of mass and electric charge a re shown in the following table. However, in figure 1.1 all fundamental particles that we have discovered so far are presented. A proton and neutron are not for instance fundamental particles. The proton is constituted by three quarks, two up quarks and one down quark. The neutron is constituted by three quarks also,two down quarks and one up quark.

Particle	Charge (C)	Mass(kg)
electron (e)	$-1.6021765 \times 10^{-19}$	9.1094×10^{-31}
proton(p)	$+1.60217655 \times 10^{-19}$	1.67262×10^{-27}
neutron(n)	0	$1.67493 \times 10{-}27$

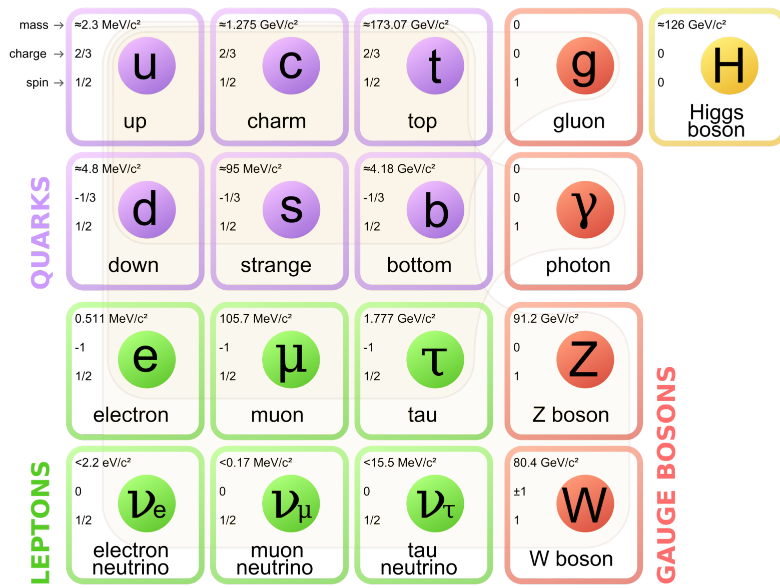


Figure 1.1: Fundamental particles that constitute everything we know so far. They are the particles of the Standard Model. The electric charge is represented in fundamental charge units. Original figure from [10]

1.2 Coulomb's Law

To determine the electric force that one particle at rest in a certain inertial reference frame exerts on other is given by

$$\boxed{\vec{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r}_{12}} \tag{1.2}$$

which we call as Coulomb's Law, where $k = 9 \times 10^9 \frac{\mathrm{Nm}^2}{\mathrm{C}^2}$, r is the distance between the two particles, and $\hat{r}_{12}$ is a unitary vector that mentions the direction of the force. This vector points from electric charge q_1 to electric charge q_2.(See Figures 1.2a, 1.2b). In equation 1.2, we label **the first index as the particle that is exerting a force and the second index to the particle that is feeling a force**, i.e. it should be read as "*Electric charge q_1 exerts a force $\vec{F}_{12}$ on particle q_2*". However, $\vec{F}_{21}$ is the force that

(a) Attracting charges with opposite sign charges

(b) Repealing charges with equal sign charges

Figure 1.2

particle q_2 is exerting on q_1.

The constant k also is

$$k = \frac{1}{4\pi\epsilon_0} \tag{1.3}$$

where $\epsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{NC}^2}{\mathrm{m}^2}$ called as the *vacuum permittivity*. Its name will make much more sense in chapter 4 when we study dielectric materials. For now, let's take it just as a constant.

Now, how is the force direction when electric charges have equal signs or opposite signs?. So, suppose, we have two electric charged particles $q_1 < 0$ and $q_2 > 0$, then

$$\vec{F}_{12} = -k\frac{q_1 q_2}{r^2}\hat{r}_{12} = k\frac{q_1 q_2}{r^2}\left(-\hat{r}_{12}\right) \tag{1.4}$$

The minus sign means that the force will have an opposite direction to the unitary vector $\hat{r}_{12}$. Therefore, the force vector $\vec{F}_{12}$ is pointing towards q_1. So, qualitatively we see that the q_1 is attracting q_2.(See Figure 1.2a)

However, suppose now that q_1 and q_2 have the same charge sign (could be both charges positive or both charges negative). So

$$\vec{F}_{12} = +k\frac{q_1 q_2}{r^2}\hat{r}_{12} = k\frac{q_1 q_2}{r^2}(+\hat{r}_{12}) \tag{1.5}$$

where the explicit + sign tells us that the force that the charge q_1 exerts on q_2 is such that the force vector has the same direction as $\hat{r}_{12}$. So, qualitatively we see that the q_1 is repealing q_2.(See Figure 1.2b). In general, we have the following rule.

$$\boxed{\text{equal sign charges repel} \qquad \text{opposite sign charges attract}} \tag{1.6}$$

As, a common trick that we will apply during this course (and also as you will notice in many other introductory electromagnetism texts) is that we place our reference frame on the particle that we want to analyze. Then, in order to not be handling with the vector $\hat{r}_{12}$, we use the magnitude of the force

$$|\vec{F}| = k\frac{|q_1||q_2|}{r^2} \tag{1.7}$$

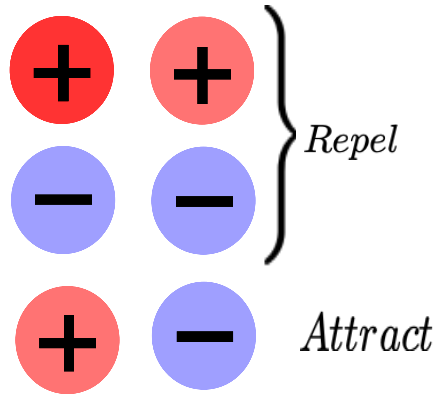


Figure 1.3

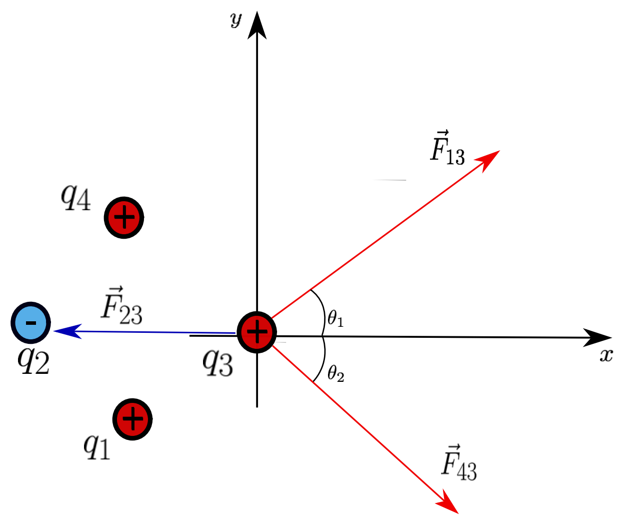


Figure 1.4

then analyze if the particles are attracting or repealing,and give the corresponding sign according to our reference frame. For example, suppose you have the following four electric charges and we place our reference frame in charge q_3

So, to analyze the forces that involve charge q_3 , we notice that electric charge q_2 attracts q_3, while q_1 and q_4 repel electric charge q_3. So, the direction of the forces must be so that we are consistent to the repealing or attraction of the particles. Do not pay attention to what happens to the other particles, now we are analyzing q_3. *Coulomob's Law* describes phenomena where the charges that exerts electric forces are static. It is a common confusion of the students to try to involve all the system, because if you think that all particles should be moving given that all of them are feeling a force... you are right! The thing is that we are just analyzing now q_3. So, writing down the components of the forces we have

$$\sum F_x = -|\vec{F}_{23}| + |\vec{F}_{13}| \cos\theta_1 + |\vec{F}_{43}| \cos\theta_2 \tag{1.8}$$

where we have included in the sign the direction of the force. And for the y component we have.

$$\sum F_y = |\vec{F}_{13}| \sin\theta_1 - |\vec{F}_{43}| \sin\theta_2 \tag{1.9}$$

Once again, the sign in the forces tell us what is the direction in the reference frame that we have chosen.

Now, suppose you would like to analyze, the forces that are exerted on q_2. Then you would place your reference frame in q_2 and obtain a diagram of forces as the following one

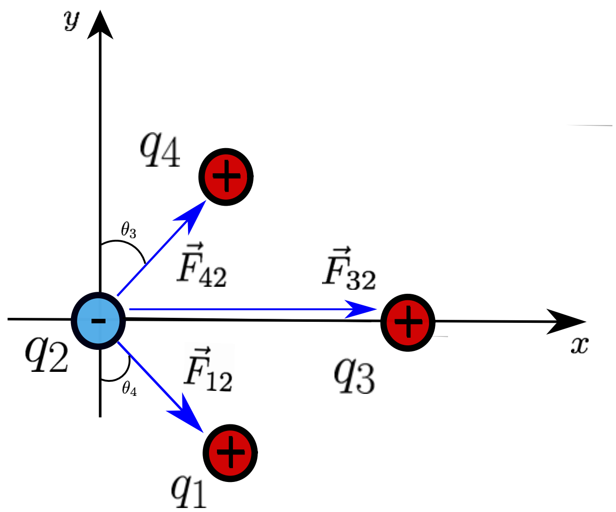


Figure 1.5

Notice that to draw the forces once again you use the general rules in 1.6. The electric charge q_2 will be attracted by q_1, q_3 and q_4, reason why the forces directions are towards those charges. So, the sum of the components of the forces will be

$$\sum F_x = |\vec{F}_{42}| \sin\theta_3 + |\vec{F}_{12}| \sin\theta_4 + |\vec{F}_{32}| \tag{1.10}$$

while for the y component we will have

$$\sum F_y = |\vec{F}_{42}| \cos\theta_3 - |\vec{F}_{12}| \cos\theta_4 \tag{1.11}$$

You could continue with the forces of the other two charges and you would follow the same procedure. Notice that we are assuming that the charges that exert a force on the particle that we are analyzing are static. However, this is not always true (if you have somehow fixed an object with electric charge you can take as static), for example in the cases shown all particles move, because all of them feel a force. That is why 1.2 is also called as *electrostatic force*, because we are assuming no movement of the particles that exert a force on the one that we are analyzing.

1.2.1 The Greatness of Electric Force

Picture in your mind an hydrogen atom. This atom, is composed by a proton and electron, where these two particles are electrically charged and have also mass. So, as we have discussed, they must exert an electrical and a gravitational force to each other. How bigger or lower is the electric force compared to the gravitational force?

To answer this question, lets calculate the fraction

$$\frac{F_E}{F_G} \tag{1.12}$$

where F_G is the gravitational force magnitude, and F_E is the electrical force magnitude. If the fraction is much bigger than 1, then F_E is much bigger than F_G. If they are roughly 1, then they are almost of the same magnitude. If it turned out to be that is lower than 1, then F_G is bigger. So,

$$\frac{F_E}{F_G} = \frac{k\frac{|q_e||q_p|}{r^2}}{G\frac{m_e m_p}{r^2}} = \frac{k|q_e||q_p|}{G m_e m_p} \tag{1.13}$$

Plugging the respective magnitudes

$$\frac{F_E}{F_G} = \frac{(9\times10^9\mathrm{Nm}^2/\mathrm{C}^2)(1.6\times10^{-19}\mathrm{C})(1.6\times10^{-19}\mathrm{C})}{(6.67\times10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2})(9.1\times10^{-31}\mathrm{kg})(1.6\times10^{-27}\mathrm{kg})} = 2.37\times10^{39} \tag{1.14}$$

where we have used the electron charge $q_e = -1.6\times10^{-19}C$, the proton charge $q_p = 1.6\times10^{-19}C$. Notice that the calculation is independent of the distance between the proton and the electron. At any distance, the force ratio will remain the same. The last calculation talks by itself. We have found that $F_E = 2.37\times10^{39}F_G$, a tremendous number of magnitude greater is the electric force than the gravitational force. To give you a better grasp, consider the following quantities.

$$\begin{aligned} \text{Speed of light} \quad & c \approx 3\times10^8\mathrm{m/s} \\ \text{Radius of the Earth} \quad & r_{earth} \approx 6.96\times10^8\mathrm{m} \\ \text{Mass of the Sun} \quad & M_\odot \approx 1.98\times10^{30}\mathrm{kg} \end{aligned} \tag{1.15}$$

So, the electric force ratio to the gravitational force is 31 orders of magnitude greater than the speed of light and radius of the Earth! It is 9 orders of magnitude greater than the mass of Sun!

Example 1: Electric Spheres in Equilibrium

Three identical charges, each with mass $m = 0.1$kg hang from rods as shown in the figure 1.6a. The lengths of the left and right rod are identical L and the angle is 30°. The charges are identical with values $q = 9$nC. Determine the length of the rods.

Solution:

We place our reference frame in the left electric charge. You could actually place your reference frame at any of the charges and the solution will be exactly the same. Since the spheres are in equilibrium, the sum of the forces in x and y component must be zero.

$$\begin{aligned} \sum F_x = -|\vec{F}_{21}| - |\vec{F}_{31}| + F_{Tx} = 0 \\ \sum F_y = F_{Ty} - mg = 0 \end{aligned} \tag{1.16}$$

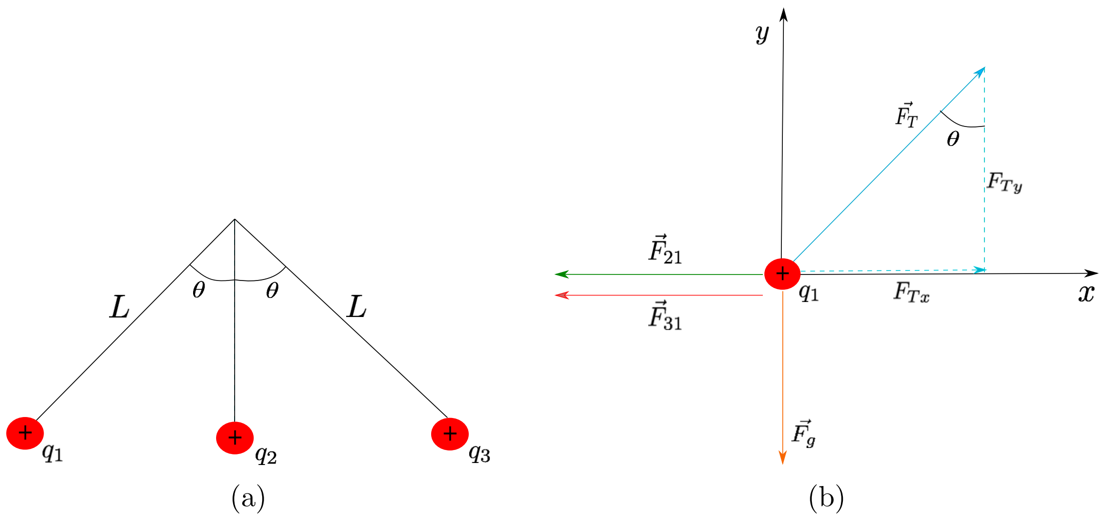


Figure 1.6

Therefore, isolating the components of the tension from the previous two equations, we are left with

$$\begin{aligned} |\vec{F}_T|\sin\theta &= |\vec{F}_{21}| + |\vec{F}_{31}| \\ |\vec{F}_T|\cos\theta &= mg \end{aligned} \tag{1.17}$$

where we have already included $F_{Tx} = |\vec{F}_T|\sin\theta$ and $F_{Ty} = |\vec{F}_T|\cos\theta$ (See Figure 1.6b). So, we divide the first equation over the second in 1.17, and we cancel out the tension force, that we did not know from the beginning. Recalling that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have

$$\tan\theta = \frac{|\vec{F}_{21}| + |\vec{F}_{31}|}{mg} \Rightarrow \quad |\vec{F}_{21}| + |\vec{F}_{31}| = mg\tan\theta \tag{1.18}$$

We plug in the formula of Coulomb's Law of the electric forces

$$k\frac{|q_2||q_1|}{r_{21}^2} + k\frac{|q_3||q_1|}{r_{31}^2} = mg\tan\theta \tag{1.19}$$

where notice that the distance between the sphere 2 and sphere 1 was labelled as r_{21} (do not confuse with unitary vector $\hat{r}_{21}$); and the distance between sphere 3 and sphere 1 was labelled as r_{31}. Finally,see figure 1.6a and notice that we can find the distances r_{21} and r_{31} by using angle θ with

$$\sin\theta = \frac{r_{21}}{L} \Rightarrow r_{21} = L\sin\theta \quad , \quad r_{31} = 2L\sin\theta \tag{1.20}$$

So, plugging r_{21} and r_{31} in equation 1.19 and factorizing common terms, we obtain

$$k\frac{q^2}{L^2}\Big(\frac{1}{\sin^2\theta} + \frac{1}{4\sin^2\theta}\Big) = mg\tan\theta \tag{1.21}$$

where since $q_1 = q_2$, we just called them as q. Therefore, by isolating L and factorizing $\sin^2\theta$

$$L = \sqrt{\frac{kq^2}{mg\tan\theta\sin^2\theta}\cdot\frac{5}{4}} \tag{1.22}$$

So, finally plugging in the last formula the values given by the exercise , we have

$$L = \sqrt{(\frac{9\times10^{9})\frac{\text{Nm}}{\text{C}}(9\times10^{-9}\text{C})^2}{(0.1\text{kg})(9.81\frac{\text{m}}{\text{s}^2})\tan 45\sin^2 45}\cdot\frac{5}{4}} = 1.36\times10^{-3}\text{m} \tag{1.23}$$

Example 2: Charge in Equilibrium

Three point charges are in the x axis . The positive charge $q_1 = 80\mu$C is fixed at $x = 2m$. The positive charge $q_2 = 20\mu$C is fixed at the origin. If we know that q_3 is negative and it is in equilibrium. Where is q_3 located ?

Solution: Since the electric charge q_3 is in equilibrium, the sum of forces must be zero, so we have

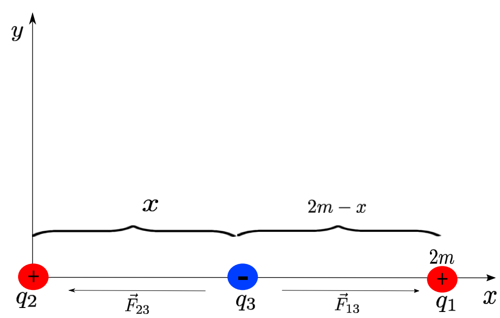


Figure 1.7

$$\begin{aligned}\sum F_x &= -|\vec{F}_{23}| + |\vec{F}_{13}| = 0\\ &= -k\frac{|q_2||q_3|}{r_{23}^2} + k\frac{|q_3||q_1|}{r_{13}^2} = 0\end{aligned} \tag{1.24}$$

Eliminating common factors k, $|q_3|$ and rearranging the last equation we have

$$\frac{r_{13}}{r_{23}} = \sqrt{\frac{|q_1|}{|q_2|}} \tag{1.25}$$

Notice, from figure 1.7 that the distance $r_{13} = 2m - x$ and $r_{23} = x$, where x is the position that we want to find. So the last equation becomes

$$\frac{2\text{m} - x}{x} = \sqrt{\frac{|q_1|}{|q_2|}} \Rightarrow \frac{2\text{m}}{x} = \sqrt{\frac{|q_1|}{|q_2|}} + 1 \Rightarrow x = \frac{2\text{m}}{\sqrt{\frac{|q_1|}{|q_2|}} + 1} \tag{1.26}$$

Substituing the values, we obtain

$$x = \frac{2\text{m}}{\sqrt{\frac{80\times10^{-6}\text{C}}{20\times10^{-6}\text{C}}} + 1} = 0.66\text{m} \tag{1.27}$$

1.2.2 Discovering Coulomb's Law

Coulomb's Law is a remarkable achievement for science. The french physicist ***Charles Augustin de Coulomb's*** experiment to determine the electric force was a beautiful, smart and also sensible experiment. The original drawing of the experiment of Coulomb is shown in figure 1.8a. The experiment consists of charging two metal spheres.When the spheres get electrically charged they repulse each other, making the balance to twist (See figure 1.8b). When this happens, since there is rotational motion, a torque is produced by the force of the electric charge. Recall that a torque is given by $\tau = \vec{r} \times \vec{F}$. Therefore, for this particular case the torque produced by the force $\vec{F}_{21}$

$$\vec{\tau}_e = \vec{r} \times \vec{F}_{12} \tag{1.28}$$

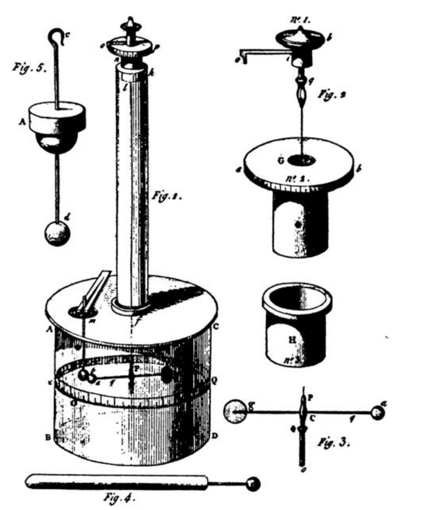
where we labelled this torque with e just to denote that it is the torque produced by the electric force. Now, the magnitude of the torque is given by

$$|\vec{\tau}_e| = |\vec{r}||\vec{F}_{12}| \sin \phi \tag{1.29}$$

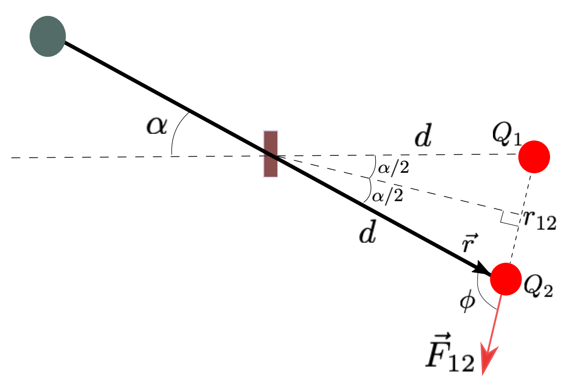
where ϕ is the angle between the force vector and the position vector $\vec{r}$ where the force is applied . From the sum of the angles in one triangle formed in figure 1.8b we have that

$$\alpha/2 + 90^{\circ} + (180^{\circ} - \phi) = 180^{\circ} \Longrightarrow \phi = \alpha/2 + 90^{\circ} \tag{1.30}$$

Using that for any angle θ, $\sin(\theta + 90^{\circ}) = \cos\theta$, and that $|\vec{r}| = d$ (See figure 1.8b), we have that the magnitude of the electric torque is



(a) Coulomb torsion balance experiment. Original figure taken from [11]



(b)

Figure 1.8

$$\tau_e = |\vec{F}_{12}| \cos\frac{\alpha}{2} d \tag{1.31}$$

Now, when *Coulomb* was doing his experiments, he wanted to figure out how was the electrical force related to the distance between the charges. By Coulomb's time, it was already accepted by physicists that gravitational force followed from Newton's Law as

$$F_G \propto \frac{m_1 m_2}{r^2} \tag{1.32}$$

Hence, many physicists of the time claimed that the electric force followed a similar inverse square law as

$$F_E \propto \frac{|Q_1||Q_2|}{r^2} \tag{1.33}$$

So, *Coulomb* in few words tested if the electric force was proportional to $1/r^2$ So, let's write by now the magnitude of $\vec{F}_{21}$ as

$$|\vec{F_{12}}| = k\frac{|Q_1||Q_2|}{r_{12}^n} \tag{1.34}$$

and let's find out what is n by using *Coulomb's* experiment and the data he published. We use the information published in the paper in reference [12]. Plugging equation 1.34 in equation 1.31

$$|\vec{\tau}_e| = k\frac{|Q_1||Q_2|}{r_{12}^n}\cos\frac{\alpha}{2}d \tag{1.35}$$

so this torque will be the responsible to make a circular motion of Q_2. However, there is certain moment when the sphere stops moving, due to the torque produced by the wire (resisting torque). It is analogue to the restoration force of a spring. When a spring moves from its equilibrium position, it tends to return to equilibrium. Similar happens when you twist a wire, the wire will tend to return to equilibrium. So, there is certain angle α at which the sphere stays in equilibrium. By several experiments, Coulomb was capable to obtain the following relationship for the resisting torque produced by the wire

$$\vec{\tau}_R = \frac{wD^4}{l}(\alpha_m + \alpha) \tag{1.36}$$

where w is constant characteristic of the metal of the wire, D the diameter of the wire, l the length of the wire, α_m the angle of twist and α is the angle of separation between the centers of the spheres. Therefore, when the sphere stopped moving we have

$$\sum \vec{\tau} = \vec{\tau}_e - \vec{\tau}_R = 0 \tag{1.37}$$

Therefore, equating the magnitudes of the torques

$$k\frac{|Q_1||Q_2|}{r_{12}^n}\cos\frac{\alpha}{2}d = \frac{wD^4}{l}(\alpha_m + \alpha) \tag{1.38}$$

If we isolate r_{12}^n

$$r_{12}^n = kl\frac{|Q_1||Q_2|}{wD^4(\alpha_m + \alpha)}\cos\frac{\alpha}{2}d \tag{1.39}$$

And here comes the magic. Suppose you have made the experiment such that you have found the equilibrium at angle α and α_m. You maintain fixed the system except for one thing, you twist the micrometer by an angle β_m, and so the movable sphere approaches to the stationary sphere by a new angle β. So, you will obtain this time that

$$r_{12}^{\prime n} = kl\frac{|Q_1||Q_2|}{wD^4(\beta_m + \beta)}\cos\frac{\beta}{2}d \tag{1.40}$$

where the label $r_{12}^{'}$ is just to denote that is the distance between the two charges when the second measurement takes place.

Finally, let's take the ratio

$$\left(\frac{r_{12}^{'}}{r_{12}}\right)^n = \frac{kl\frac{|Q_1||Q_2|}{wD^4(\beta_m+\beta)}\cos\frac{\beta}{2}d}{kl\frac{|Q_1||Q_2|}{wD^4(\alpha_m+\alpha)}\cos\frac{\alpha}{2}d} \quad \Rightarrow \left(\frac{r_{12}^{'}}{r_{12}}\right)^n = \frac{(\alpha_m + \alpha)\cos\frac{\beta}{2}}{(\beta_m + \beta)\cos\frac{\alpha}{2}} \tag{1.41}$$

Now, by using $\sin\frac{\alpha}{2}$, (See figure 1.8b) we have that

$$r_{12} = 2d\sin\frac{\alpha}{2} \quad \text{and also} \quad r_{12}^{'} = 2d\sin\frac{\beta}{2} \tag{1.42}$$

Therefore, the fraction in equation 1.41 becomes

$$\left(\frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha}{2}}\right)^n = \frac{(\alpha_m + \alpha)\cos\frac{\beta}{2}}{(\beta_m + \beta)\cos\frac{\alpha}{2}} \tag{1.43}$$

Recalling that in general the natural logarithm has the property $\ln(f^n) = n\ln(f)$ where f is a function, we apply natural logarithm in both sides of the last equation and isolate n

$$n = \frac{\ln\left(\frac{(\alpha_m+\alpha)\cos\frac{\beta}{2}}{(\beta_m+\beta)\cos\frac{\alpha}{2}}\right)}{\ln\left(\frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha}{2}}\right)} \tag{1.44}$$

Beautiful! There are no other words to describe it. We have found n, which tell us how the electric force will be dependant to the distance. When Coulomb, published his results , he reported the following three data pairs (information from paper in reference [12])

1. *First Trial. Having charged the two balls with the head of a pin with the micrometer index set at O, the ball a of the needle is separated from the ball t by 36 degrees.*

2. *Second Trial. Turning the suspension thread through 126 degrees by means of the knob O of the micrometer, the two balls are found separated and at rest at 18 degrees from one another.*

3. *Third Trial. After turning the suspension thread through 567 degrees, the two balls are separated by 8 degrees and a half*

So , we have the two pairs of α and β angles.

1. $\alpha_m = 0$, $\alpha = 36$, $\beta_m = 126$, $\beta = 18$. Therefore, using equation 1.44, $n = 1.981$
2. $\alpha_m = 126$, $\alpha = 18$, $\beta_m = 567$, $\beta = 8$ Therefore, using equation 1.44, $n = 1.842$

Therefore the average exponent n that Coulomb found experimentally was

$$n = 1.911 \tag{1.45}$$

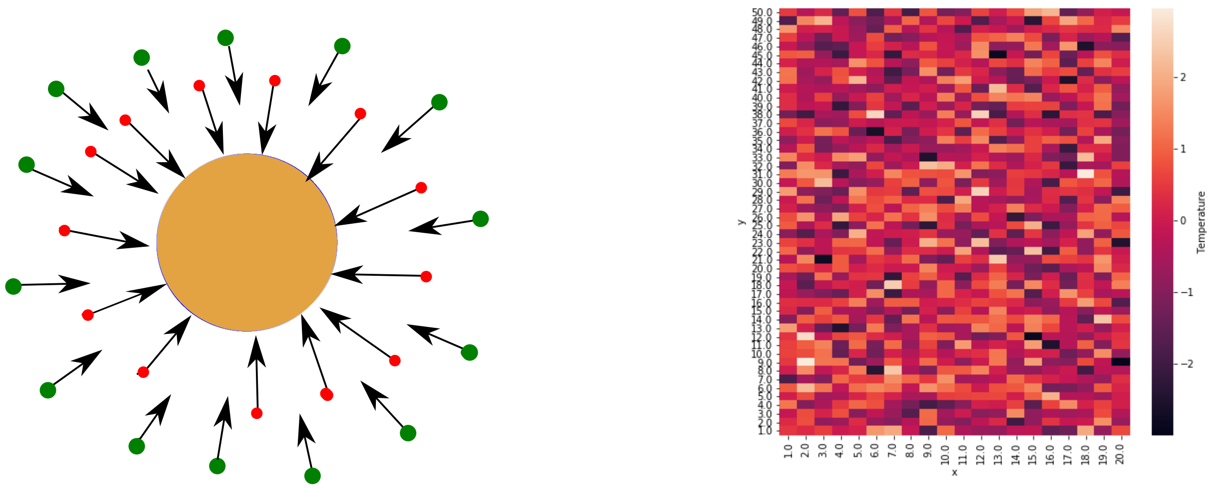
This result error is 4.45% to the inverse squared law , a remarkable result obtained in an era with no high technology, without the possibility of isolating the experiment of air resistance and isolating the electric charge in the spheres. Coulomb concluded that the electric force is proportional to $1/r^2$, the result 1.911 is so close to the inverse squared law that it can be concluded that the deviation was just experimental error. Notice, how with an experiment that could be considered simple (compared to today experiments as the Large Hadron Collider or detector for Gravitational waves) leads to a such fundamental law of Nature.

1.3 A grasp about fields

Let's start our discussion by asking ourselves why is it possible that electric charges can feel a force even though they are not in contact? Think about this, when you push a table, obviously you are interacting with the table and exerting a force. However, why is it possible that a charged body two meters away from another particle feels a force? The answer is that, whenever there is an electrically charged body, it creates an entity that we call as $electric\ field$. But, what do we mean with a $field$? The concept of field is complex and goes beyond the scope of this book. Actually, the quantum behaviour of fields is a complete research area and has lead us to know how even matter has mass or not. However, lets give a general picture about what a field means.

A field mathematically is defined as a $function$ that designates to every single point in space and time a quantity. If you think about it, in previous courses we have already learned physical variables that can be modelled as fields. As a first example for instance is temperature. Every single point in space has a particular temperature, and also, it can change with respect time. As second example, pressure. These two examples designate a quantity to every single point in space and time that is just a $scalar$. Therefore these fields that associate to every single point in space and time a scalar are called $scalar fields$. However, there are also another kind of fields that designates to every single point in space and time a $vector$. And you already know an example, the $gravitational field$. If you place yourself in any point in space, there is a certain vector pointing towards to the source of that gravitational field (See Figure 1.9a). And, of course that field also designates a vector for each possible time. The fields that designate a vector to every single point in space and time are called $vector fields$.

It is of huge importance to mention that there is a certain difference between the temperature field and gravitational field. And not just the fact that one is scalar and the



(a) Gravitational Field created by a mass source. Every point in space has a vector, which also depends on time

(b) Temperature Field. Every point in space has certain temperature, which also has a value depending on time.

Figure 1.9

other is vector. The gravitational field is an entity. And what we mean about that, is that it is not a variable associated to something. For example , the pressure field of a gas is associated to the force that exerts in every single point a gas in infinitesimal areas. However, the gravitational field is that something (as the gas for example or an object) that has energy, and it is the one that interacts with any massive body. The gravitational field is not just a mathematical model of a physical variable, it is a real physical entity.

Therefore, as those fields that are familiar to us now, we have a new field that also is an entity, *the electric field*. This field is a vector field as the gravitational field and it is created whenever there is a charged particle. So, when there is an electric charge, it interacts with the electric field of other electric charged particles and that is why there can exist a distant force between electric charges. The electric field is not just a mathematical description of distance forces, it transports energy and it is so real that light is an electromagnetic wave (it has a electric field component and magnetic field component).

1.4 Electric Field of point charges

We have mentioned before that the property of particles that tells us how two bodies interact electrically is defined by the electric charge. Also, we have mentioned that the responsible for the distance force is the electric field, so how are these two ideas related? To answer this, let's start defining the electric field .

The electric field of a point electric charged particle is given by

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}_0}{q_0} \tag{1.46}$$

$\vec{F}_0$ is the force exerted on a particle q_0. This charge q_0 we call it as *test charge*. This electric charge q_0 we define it as positive, and it is a charge that will only help us to define the electric field in a first instance. So, if we substitute the force in equation 1.46 we have

$$\vec{E} = \lim_{q_0 \to 0} \frac{kQq_0/r^2}{q_0}\hat{r} \tag{1.47}$$

So, notice that the electric field is therefore independent of the test charge q_0

$$\boxed{\vec{E} = k\frac{Q}{r^2}\hat{r}} \tag{1.48}$$

The last equation is crucial, and it is the electric field created by a point electric charge Q. So, now multiply both sides of last equations for another electric charge q and we obtain

$$q\vec{E} = k\frac{Qq}{r^2}\hat{r} \tag{1.49}$$

So notice that the right hand side looks exactly as equation 1.2. Therefore, we have another important equation

$$\boxed{\vec{F} = q\vec{E}} \tag{1.50}$$

so going back to our initial question , we know therefore that the electric force that a electric charge q feels given by an electric field created by another electric charge Q is given by 1.50 ; and there can exist a distance force because there is an electric field.

Now, the electric field is a vector field. So, the electric field vectors at each point in space and time has a direction. The direction of the field vectors are completely determined by the electric charge sign. To see this, let's start by picking a negative electric charge $-Q$. Therefore

$$\vec{E} = k\frac{(-Q)}{r^2}\hat{r} = k\frac{Q}{r^2}(-\hat{r}) \tag{1.51}$$

Therefore, the direction is opposite to the unitary vector $\hat{r}$. Since the vector $\hat{r}$ points outwards to the electric charge $-Q$, then the vector $\vec{E}$ will go towards to the electric charge $-Q$. For the case that the electric charge is $+Q$, then

$$\vec{E} = k\frac{(+Q)}{r^2}\hat{r} = k\frac{Q}{r^2}(+\hat{r}) \tag{1.52}$$

where we explicitly wrote the + sign of the electric charge so that we see that the direction of the electric field is the same as the vector $\hat{r}$. Now, let's take this similar idea but for any point around the electric charge. When drawing this, since we don't take carefulness of drawing each vector with its corresponding magnitude, we just draw lines as arrows. Even though of course this lines are not carefully drawn to the magnitude of the electric field at each point, we call them as *electric field lines.* These lines gives us the behaviour of the electric field in every single point in space. Now, so picking all points around the electric charge $+Q$ and $-Q$ we see that the behaviour of the electric field is as shown in .

Therefore, we have once again a general rule

$$\boxed{\begin{array}{r}\text{negative charges electric field lines point to the charge}\\ \text{positive charges electric field lines point outwards to the charge}\end{array}} \tag{1.53}$$

The last rule is crucial to qualitatively obtain results from our calculations later on. And we can actually make our first qualitative deduction. Suppose you have electric field lines as shown in figure 1.10a and figure 1.10b, and there is a random positive electric charge $+q$ and $-q$ respectively. Where will the electric charge $+q$ and charge $-q$ move? To the right or to the left? The answer is quite easy if you think what kind of electric charge would produce that electric field . A positive charge placed at the left would not produce such electric field because the electric field lines of the positive charge would go to the right (electric field lines of positive charges are outwards the electric charge). So, think as a negative charge $-Q$ placed at the left is producing such electric field lines. Also, we already know that opposite sign charges attract and same sign charges repel. Therefore, the force must be as shown in figures 1.10a , figure 1.10b.

Also you could use the electric force

$$\vec{F} = q\vec{E} \tag{1.54}$$

, if the electric charge is negative then

$$\vec{F} = -q\vec{E} = q(-\vec{E}) \tag{1.55}$$

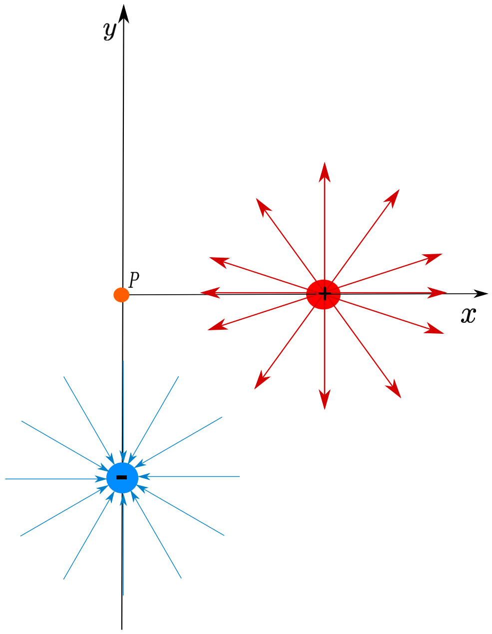
where $-\vec{E}$ indicate us that the direction is opposite to the electric field direction. Similarly if you use the formula of the electric force, you will see that if it is positive the charge, the force vector must have the same direction of the electric field.

So , from just using electric field lines qualitatively we have found the general following rule

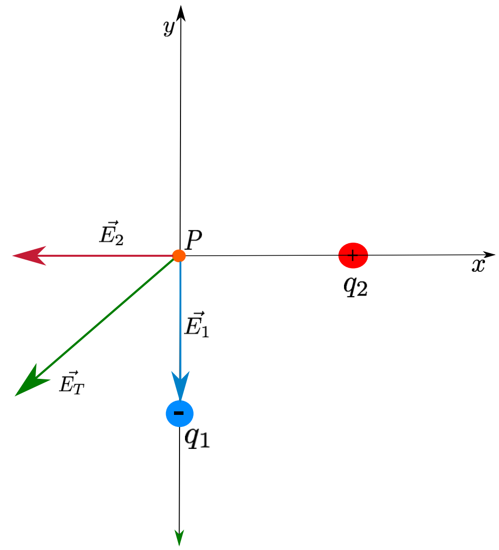
$$\boxed{\begin{array}{r}\text{negative charges move against the direction of an external electric field lines}\\ \text{positive charges move to the same direction of an external electric field lines}\end{array}} \tag{1.56}$$

(a) (b)

Figure 1.10



(a) Electric Field lines of two particles, one negative and one positive.



(b) Electric field vectors at the origin of produced by two particles. Green vector represents the total electric field of both particles

Figure 1.11

Example 3 : Electric Field at the origin given two electric charges

Two point charges $q1 = -7$nC, and $q2 = 10$nC are located at $(0.3, 0)$m , and $(0, -0.2)$m respectively, at the xy plane. (a) Find the electric field vector at the origin due to both charges. (b) If, then a third charge $q_3 = 20$nC is located at the origin. What magnitude of the force would be exerted on it?

Solution: First of all, lets recall how electric field lines behave depending on the electric charge of the particle. The electric field lines produced by the particles are shown in 1.11a. However, we are just interested what happens at the origin. Therefore, by just keeping the direction of the electric field vectors produced by the particles at P, we obtain what we see in figure 1.11b. Now, the electric field vector of the positive charge points to $-x$, while the electric field of the negative charge points to $-y$. Therefore, we simply have

$$\begin{aligned} \sum E_x = -k\frac{|q_2|}{r_2^2} = -9\times 10^9 \mathrm{Nm^2/C^2}\frac{10\times 10^{-9}\mathrm{C}}{(0.3)^2} = -1000\frac{\mathrm{N}}{\mathrm{C}} \\ \sum E_y = -k\frac{|q_1|}{r_2^2} = -9\times 10^9 \mathrm{Nm^2/C^2}\frac{7\times 10^{-9}\mathrm{C}}{(0.2)^2} = -1575\frac{\mathrm{N}}{\mathrm{C}} \end{aligned} \tag{1.57}$$

Now, if we place a third particle at the origin, it will feel a force due to the electric field produced by the particles q_1 and q_2. By using equation 1.50 we find the force q_3 feels

$$\begin{aligned} F_x &= q_3E_x = \left(20\times10^{-9}\text{C}\right)\left(-1000\frac{\text{N}}{\text{C}}\right) = -20\times10^{-6}N \\ F_y &= q_3E_y = \left(20\times10^{-9}\text{C}\right)\left(-1575\frac{\text{N}}{\text{C}}\right) = -3.15\times10^{-5}\text{N} \end{aligned} \tag{1.58}$$

The magnitude of the force

$$|\vec{F}| = \sqrt{F_x^2+F_y^2} = \sqrt{(20\times10^{-6}N)^2+(3.15\times10^{-5}\text{N})^2} = 3.73\times10^{-5}\text{N} \tag{1.59}$$

Now, we could ask ourselves, what would it be the direction of the force? Since the electric charge is positive, then the force will be have the same direction. If the electric charge were negative, then it would move to the opposite direction.

1.5 Electric Field of continuum charge distributions

So far we have calculated the electric field for a point electric charge and for the contribution of several of them. Now we will see how to calculate the electric field of any object that can have length, are or volume.

So to start, suppose in general, that you have certain object as shown in the figure 1.12. We will divide the object in very small little chunks. So the total electric field at point P would be

$$\vec{E} = \sum_{i=1}^{N} k\frac{q_i}{r_i^2}\hat{r}_i \tag{1.60}$$

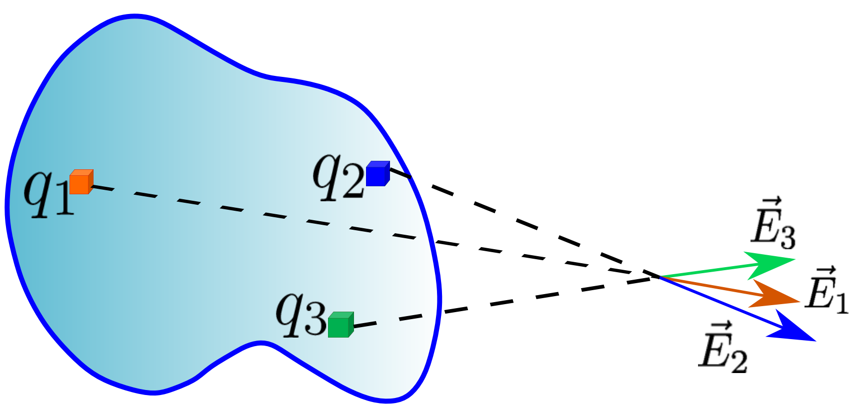


Figure 1.12

However, if we make now the little chunks to be extremely small such that they become infinitesimal, then we can say that the electric charge that each of them enclose is also infinitesimal . Therefore, last equation becomes

$$\vec{E} = \lim_{q_i\to0}\sum_{i=1}^{N} k\frac{q_i}{r_i^2}\hat{r}_i = \int k\frac{dq}{r^2}\hat{r} \tag{1.61}$$

where we have made that every single q_i is now an infinitesimal electric charge dq. The vector $\hat{r}$ is the unitary vector at which the total electric field will point to and from the integration by itself is not easy to know which direction it will be . So , what we will do is to calculate the electric field magnitude given by

$$\boxed{|\vec{E}| = \int k\frac{dq}{r^2}} \tag{1.62}$$

and the direction will be given depending of each case. We will do several exercises so this becomes clear. Now, in order to integrate equation 1.62 we will define the following quantities.

$$\begin{aligned} \lambda = \frac{dq}{dl} &: \text{Linear charge density. Units } \left[\frac{C}{m}\right] \\ \sigma = \frac{dq}{dA} &: \text{Area charge density. Units } \left[\frac{C}{m^2}\right] \\ \rho = \frac{dq}{dV} &: \text{Volume charge density. Units } \left[\frac{C}{m^3}\right] \end{aligned} \tag{1.63}$$

The previous quantities tell us how much electric charge is inside in a length, area or volume respectively. So, depending the problem we want to solve, we will use the different definitions of charge density. Now, be careful that neither λ, σ nor ρ should be constant. It could happen that an object its charge density varies from point to point. However, for the particular case when the charge densities are constant we have that

$$\begin{aligned} \int_0^{Q_{tot}} dq = \lambda \int_0^{L_{tot}} dl &\Longrightarrow Q_{tot} = \lambda L_{tot} \\ \int_0^{Q_{tot}} dq = \sigma \int_0^{A_{tot}} dA &\Longrightarrow Q_{tot} = \sigma A_{tot} \\ \int_0^{Q_{tot}} dq = \rho \int_0^{V_{tot}} dV &\Longrightarrow Q_{tot} = \rho V_{tot} \end{aligned} \tag{1.64}$$

where we labelled as L_{tot} , A_{tot} and V_{tot} as the total length, area and volume respectively. Actually, if you think about it the previous results are logical. If you mulptiply the density times the total length , area or volume , it should give you the total electric charge of an object. As mentioned before, this is not always true, since the charge density could not be constant.

Example 4 : Electric field of a rod

A very thin rod of length L has a positive charge Q distributed uniformly. Calculate the electric field at point P (see figure 1.13), which is located a distance a from one end of the rod.

Solution: We start by making dividing our rod in little chunks of infinitesimal electric charge. Since, the electric charge is uniformly distributed, the linear electric charge density

λ is constant, i.e. any little piece of the rod that we pick will have exactly the same electric charge. From 1.63 we then obtain that

$$dq = \lambda dl \tag{1.65}$$

Now, we integrate (sum infinitesimally) the contribution of electric fields of every single little chunk (see 1.13). All the little pieces with electric charge contributes at the origin to the left. Notice that for this particular case, there is only x component contribution of the electric field. However, remember that the electric field is a vector at point P. So, we need to calculate the integral.

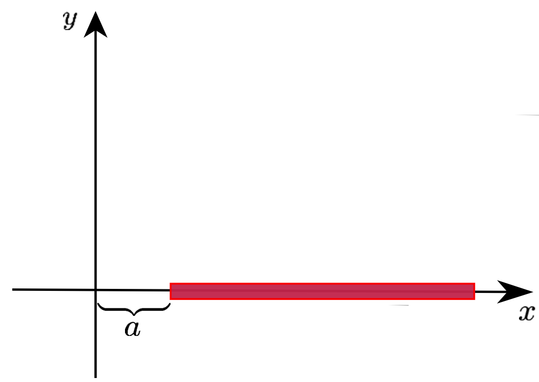


Figure 1.13

$$|\vec{E}| = \int k\frac{dq}{r^2} = k\int \frac{\lambda dl}{r^2} \tag{1.66}$$

Now, recall that r is the distance from the charges to the point of interest (where you are calculating the electric field). So, we can call $r = x$, since the rod lies in the x axis and every single little chunk of charge will have a distance x from the origin. Furthermore, we can call $dl = dx$, since a little length of the rod is a little length in the x axis. Also, we have to delimit where we are integrating the contributions to the total electric field. Since the rod starts a distance a from the origin and its length is L, we therefore have that

$$|\vec{E}| = k\lambda\int_a^{a+L}\frac{dx}{x^2} = -k\lambda\frac{1}{x}\bigg|_a^{a+L} = k\lambda\Big(\frac{1}{a} - \frac{1}{a+L}\Big) = k\lambda\Big(\frac{(a+L)-a}{a(a+L)}\Big) = k\lambda\Big(\frac{L}{a(a+L)}\Big) \tag{1.67}$$

Finally, recall from equation 1.63 that $Q_{tot} = \lambda L_{tot}$, therefore

$$|\vec{E}| = \frac{kQ}{a(a+L)} \tag{1.68}$$

since L is the total length of the rod and Q is the total charge of the object. Finally, to determine the direction, since the electric charge of the rod is positive, we know that electric field lines must be outwards to the rod. Therefore,

$$\vec{E} = -\frac{kQ}{a(a+L)}\hat{i} \tag{1.69}$$

Example 5 : Electric field of a rod in a symmetrical point

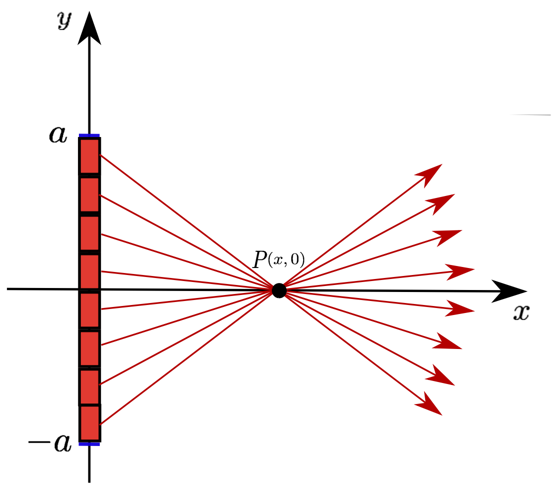


Figure 1.14

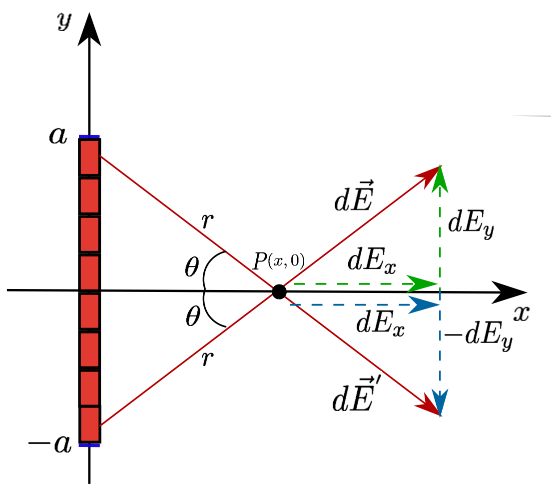


Figure 1.15

The positive charge Q is distributed uniformly along a line of length 2a . It is located along the y axis between $y = -a$ and $y = a$. Find the electric field at point P along the x axis a distance x from the origin.

Solution: Let´s see in figure 1.14 how the electric field vectors would contribute at point $P(x, 0)$. The little chunks in the figure are infinitesimal (of course for drawing purposes they are quite big) and all of them create a differential vector $d\vec{E}$ at P . In principle , we would need to sum the contributions in the x and y component of each vector. However, notice what happens at figure 1.15. The contribution for the little chunk from the top is exactly the same as the one at the very bottom with opposite direction. So, notice how the electric field contributions in the y component cancel out. While the top little chunk will create an electric field vector going to $-y$ with magnitude dE_y , the bottom one will create an electric field vector going $+y$ with exactly the same magnitude. Therefore, all the contributions in the y component will be cancelled out at the end, because every single chunk at the top will have another chunk at the bottom that will cancel its y component. So, the component that is left is just the x component. As shown in the figure 1.15, every dEx goes to the $+x$ direction. So, we have to sum infinitesimally (integrate) all the contributions to the x component. So, we calculate

$$|\vec{E}| = E_x = \int dE_x = \int dE \cos\theta \tag{1.70}$$

where θ is the angle between the x axis and each of the electric field contributions. The electric field magnitude is given by

$$|\vec{E}| = k\lambda \int \frac{dy}{r^2} \cos\theta \tag{1.71}$$

where we used $dq = \lambda dl = \lambda dy$ because a little length dl in the rod can be thought also as a small length dy since the rod lies in the y axis. Now, we can easily see from the figure 1.15 that $\cos\theta$ can be calculated as

$$\cos\theta = \frac{x}{r} \tag{1.72}$$

Therefore, plugging in back to equation 1.71, we obtain

$$|\vec{E}| = k\lambda \int \frac{x dy}{r^3} \tag{1.73}$$

Now,using once again figure 1.15 the distance from the charges to $P(x, 0)$ is $r = (y^2 + x^2)^{1/2}$. Also, notice that x is constant because the point $P(x, 0)$ is already fixed. Therefore,

$$|\vec{E}| = k\lambda x \int \frac{dy}{(y^2 + x^2)^{3/2}} \tag{1.74}$$

Finally, to integrate we make the following change of variables

$$\tan u = \frac{y}{x} \Longrightarrow y = x \tan u, \frac{dy}{du} = x \sec^2 u \Longrightarrow dy = x \sec^2 u du \tag{1.75}$$

Therefore, focusing in our integral, it becomes

$$\int \frac{dy}{(y^2 + x^2)^{3/2}} = \int \frac{x \sec^2 u du}{((x \tan u)^2 + x^2)^{3/2}} = \int \frac{\sec^2 u du}{x^2 (\sec u)^3} = \frac{1}{x^2} \int \frac{du}{\sec u} = \frac{1}{x^2} \int \cos u du = \frac{\sin u}{x^2} \tag{1.76}$$

where from the second step to the third we used the trigonometric property $\tan^2 u + 1 = \sec^2 u$. Finally, we use the trigonometric property

$$\sin\left(\arctan \frac{y}{x}\right) = \frac{y}{x\sqrt{1 + \left(\frac{y}{x}\right)^2}} = \frac{y}{\sqrt{x^2 + y^2}} \tag{1.77}$$

Therefore, including the integration limits , going from $-a$ up to a since it is the range in y where the rod lies, we have the following

$$|\vec{E}| = k\lambda x \left(\frac{y}{x^2 (x^2 + y^2)^{1/2}}\right)_{-a}^{a} = k\lambda \left(\frac{2a}{x(x^2 + a^2)^{1/2}}\right) \tag{1.78}$$

Finally, notice that the total length of the rod is $L_{tot} = 2a$, and recalling that$\lambda L_{tot} = Q_{tot}$, we obtain that the electric field magnitude is

$$|\vec{E}| = k \frac{Q}{x(x^2 + a^2)^{1/2}} \tag{1.79}$$

where Q is the total charge of the object. The direction is to the x direction, therefore the electric field vector is

$$\vec{E} = k \frac{Q}{x(x^2 + a^2)^{1/2}} \hat{i} \tag{1.80}$$

Example 6 : Electric field of a rod, the general case

As you can notice, in the previous two cases of the electric field of a rod, only one component was calculated. However, in general this should not hold. Recall that the electric field. So, we show how should be calculated the electric field for a random point where there is no symmetry that cancels out one component. For the two dimensional case we have

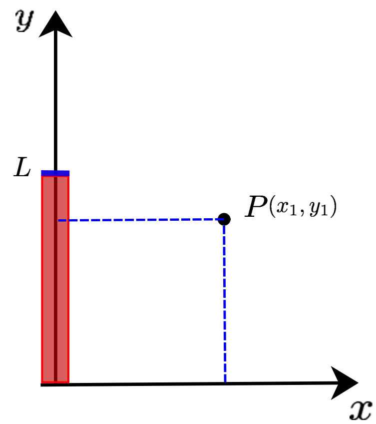


Figure 1.16

$$\begin{aligned} E_{\mathrm{x}} &= k\int \frac{dq}{r^2}\cos\theta \\ E_{\mathrm{y}} &= k\int \frac{dq}{r^2}\sin\theta \end{aligned} \tag{1.81}$$

So, using $\cos\theta = \frac{x_1}{r}$, $\sin\theta = \frac{y}{r}$ and $r = \sqrt{x^2 + (y - y_1)^2}$, we have the integrals

$$\begin{aligned} E_{\mathrm{x}} &= k\lambda x_1 \int_0^L \frac{dy}{(x_1^2 + (y - y_1)^2)^{3/2}} \\ E_{\mathrm{y}} &= k\lambda \int_0^L \frac{(y - y_1)dy}{(x_1^2 + (y - y_1)^2)^{3/2}} \end{aligned} \tag{1.82}$$

Example 7: Electric Field of infinite rod.

Calculate the electric field of an infinite rod at a point $P(x, 0)$

Solution: We could think of solving the following integral

$$|\vec{E}| = k\lambda \int_{-\infty}^{\infty} \frac{xdy}{(x^2 + y^2)^{\frac{3}{2}}} = k\lambda \int_{-\infty}^{\infty} \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \frac{dy}{(x^2 + y^2)} \tag{1.83}$$

where we would be solving the exact same case, but making the integral limits going to infinity (we separate the integral in such a way to make easier to visualize the substitution that we will make). However, instead of integrating from $-\infty$ up to ∞ let´s try another approach. See figure 1.17, and notice that every single little chunk of the rod contributes with a differential electric field at point P. As the angle between the x axis and the electric field line becomes bigger and bigger, we approximate to those chunks at infinity. So, the idea is to integrate the contribution of all the little chunks with electric charge from $\frac{-\pi}{2}$ up to $\frac{-\pi}{2}$ because the chunks that lie at infinity, their electric field lines make an angle almost of $-\frac{\pi}{2}$ respect to the x axis and those at minus infinity make an

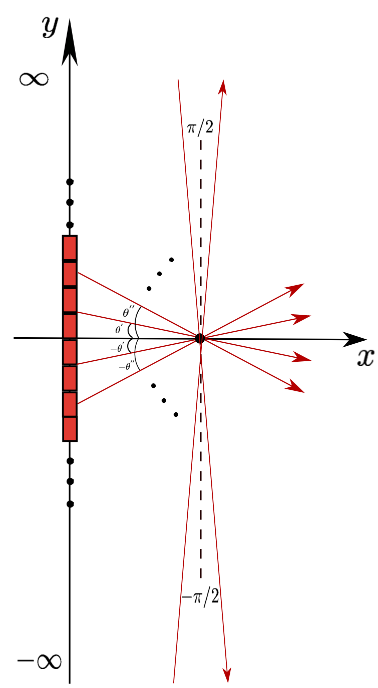


Figure 1.17: Electric Field Contribution of an infinite rod.

angle almost of $\frac{\pi}{2}$ respect to the x axis. So, we need to leave the parameters in terms of the angle between the x axis and the electric field lines. So, defining θ as the angle between any electric field line and the x axis , we have that

$$\tan\theta = \frac{y}{x} \Longrightarrow y = x\tan\theta \Longrightarrow dy = x\sec^2\theta d\theta \tag{1.84}$$

Therefore,

$$dy = \frac{x}{\cos^2\theta}d\theta \tag{1.85}$$

Now, notice that

$$\cos\theta = \frac{x}{(x^2+y^2)^{1/2}} \Longrightarrow \cos^2\theta = \frac{x^2}{x^2+y^2}$$

Therefore substituting $\cos^2\theta$ in equation 1.85, we obtain

$$dy = \frac{x^2+y^2}{x}d\theta \tag{1.86}$$

So, using $\cos\theta$ and dy in equation 1.83 and changing the integral limits to $-\pi/2$ and $\pi/2$, we obtain

$$|\vec{E}| = \frac{k\lambda}{x}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = \frac{k\lambda}{x}\sin\theta\Bigg|_{-\pi/2}^{\pi/2} = \frac{k\lambda}{x}[1-(-1)] = \frac{2k\lambda}{x} = \frac{\lambda}{2\pi\epsilon_0 x} \tag{1.87}$$

where we used the definition of $k = \frac{1}{4\pi\epsilon_0}$. Therefore, the electric field magnitude of an infinite wire is

$$|\vec{E}| = \frac{\lambda}{2\pi\epsilon_0 x} \tag{1.88}$$

Since, the electric charge is positive, all the vector contributions go to $+x$. Therefore,

$$E = \frac{\lambda}{2\pi\epsilon_0 x}\hat{i} \tag{1.89}$$

Example 8: Electric Field Four Rods

Electric charge is distributed uniformly along each side of a square. Two adjacent sides have positive electric charge 20μC (each one). The other two sides have negative electric charge –20μC (each one).

- What are the components E_x and E_y of the resultant electric field at the middle point of the square? Each side of the square length is 20cm.

Solution: Now that we have the formula for the electric field magnitude produced by a charge line at a middle point (equation 1.80).

$$|\vec{E}| = \frac{kQ}{x(x^2+a^2)^{\frac{1}{2}}} \tag{1.90}$$

we can use it for each of the sides of the square. But, careful! you have to consider the directions of the electric fields. Never forget the fact that the electric field at any point is a vector. Remembering that positive charges have outwards electric fields lines, and negative charges have inwards electric field lines; we have that the directions of the electric field vectors of each rod are as shown in figure 1.18. The electric field vector created by the left bar is shown with the orange vector, while the electric field vector created by the right bar is shown with the red vector. Opposite to what we could think in a first instance, the vectors do not cancel out, they sum with $+x$ direction. Similar happens to the y component sum of the electric fields. The electric field direction of the top bar and the bottom bar point to the same $-y$ direction.

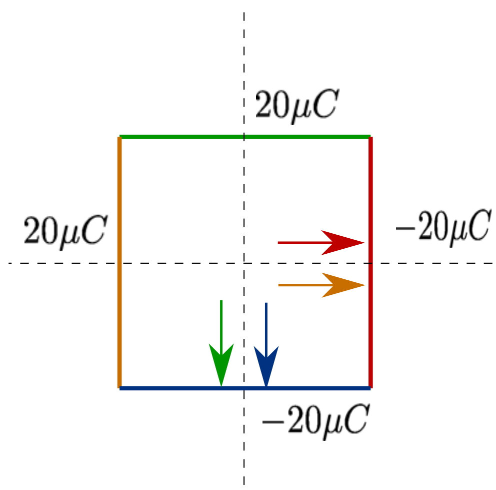


Figure 1.18

Just before we continue with the sum of the electric field components, recall that a in our derivation (Example 6 *Electric field of a rod in a symmetrical point*) is the distance from the bar central point and one of the ends of the rod. Furthermore, x represents the distance from the bar central point to the middle point $P(x, 0)$. In this case, all the distances from the bars central point to the middle point of the square are exactly the same. So, in figure 1.18 you can notice that actually in formula 1.90, $x = a$. Therefore, writing the electric field sum in x, as the contribution of the left and right bar

$$\sum E_x = |E_{\text{leftbar}}| + |E_{\text{rightbar}}| = k\frac{|Q|}{a(a^2+a^2)^{\frac{1}{2}}} + k\frac{|Q|}{a(a^2+a^2)^{\frac{1}{2}}} = k\frac{2|Q|}{a(2a^2)^{\frac{1}{2}}} \tag{1.91}$$

Therefore, plugging in the values:

$$\sum E_x = 9 \cdot 10^9 Nm^2/C^2 \frac{2 \cdot 20 \times 10^{-6} C}{10 \times 10^{-2} m [2 \cdot (10 \times 10^{-2})^2]^{\frac{1}{2}}} = 25,455,844.1\text{N/C} \tag{1.92}$$

For the y component , we have

$$\sum E_y = -|E_{\text{topbar}}| - |E_{\text{bottombar}}| = -k \frac{|Q|}{a(a^2+a^2)^{\frac{1}{2}}} - k \frac{|Q|}{a(a^2+a^2)^{\frac{1}{2}}} \tag{1.93}$$

where we included "−" signs because the electric fields point to the $-y$ direction. So, we have that the sum of the electric fields in the y components is

$$\sum E_y = -\frac{2k|Q|}{a(2a^2)^{\frac{1}{2}}} \tag{1.94}$$

So, notice that the y component is the same magnitude of the x component with a "−" sign. Therefore, we have that the electric field is

$$\vec{E} = 25,455,844.1\text{N/C}\ \hat{i} - 25,455,844.1\text{N/C}\ \hat{j} \tag{1.95}$$

Example 9: Electric Field of a Ring

A ring with total charge $Q = 1.5\mu$C is uniformly distributed. The ring has radius $a = 1.5$m. Find the electric field at point P that is in the axis of the ring a distance $x = 1$m of its center

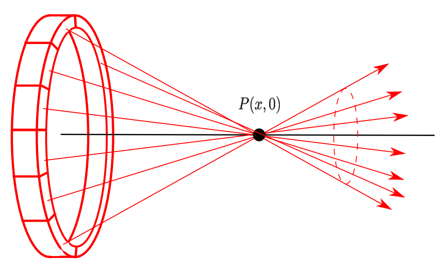


Figure 1.19: Electric Field Contribution of all the little chunks of a ring.

Solution:

The way to tackle this problem is to split once again the object in little chunks with charge, and integrate all the contributions. Also, it will be useful to notice symmetry. Notice, as shown in figure 1.20, that all "y" contributions from the all the small chunks will be cancelled out. The only component that survives is the x component. Therefore, we are interested in calculating

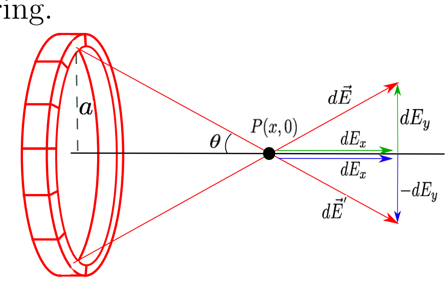


Figure 1.20: Electric Field Contribution of all the little chunks of a ring.

$$|\vec{E}| = \int dE_x = \int dE \cos\theta = |E_x| = k \int \frac{dq}{r^2} cos\theta \tag{1.96}$$

where from figure 1.20, we can notice that we can use $\cos\theta = \frac{x}{r}$, where r is the distance from each of the little small chunks. So, we obtain

$$|\vec{E}| = E_x = k\lambda \int \frac{dl}{r^3} x \tag{1.97}$$

Notice that $r = (x^2 + a^2)^{1/2}$ is a constant, since x is a distance already fixed and a is the radius of the ring. Also, notice that the integration limits must be from 0 up to $2\pi a$ since we want that the differential of length grow from a single point (length zero) up to the total length of the ring, which is the perimeter. So,

$$|\vec{E}| = \frac{k\lambda x}{(x^2 + a^2)^{3/2}} \int_0^{2\pi a} dl = \frac{k\lambda x}{(x^2 + a^2)^{3/2}} [2\pi a] \tag{1.98}$$

Now, recalling that $Q_{tot} = \lambda L_{tot}$, we have that $Q = \lambda 2\pi a$. Therefore,

$$|\vec{E}| = \frac{kQx}{(x^2 + a^2)^{3/2}} \tag{1.99}$$

Finally, plugging in values

$$|\vec{E}| = \frac{(9x10^9 \frac{\mathrm{N}m^2}{\mathrm{C}^2})(1.5x10^{-6}\mathrm{C})(1\mathrm{m})}{[(1m)^2 + (1.5m)^2]^{\frac{3}{2}}} = 2304.13\frac{\mathrm{N}}{\mathrm{C}} \tag{1.100}$$

Example 10: Electric Field of a Disc

Find the electric field of a disc of radius R with uniform positive charge density σ, at a point along its center a distance x . ***Solution:***

The idea to solve this problem is to assume that we can make the disc in infinite number of rings of differential electric charge and sum (integrate) the contribution of all of them. We assume that we can make rings with differential width da and integrate the contribution of all of them at point P. So, we need the result obtained in equation 1.99, with the only difference that this time each ring has a differential charge dq instead of a finite charge Q. Summarizing, we can think of it as:

$$|\vec{E}|_{disc} = \int dE_{Rings} \tag{1.101}$$

i.e. the electric field of the disc will be the sum of all the contributions of every single ring that composes the disc. So, by using equation 1.99, we have

$$|\vec{E}|_{disc} = \int_0^R \frac{kdqx}{(x^2 + a^2)^{\frac{3}{2}}} \tag{1.102}$$

where R is *the radius of the disc* and a is *the radius of the rings* (see figure 1.21). Notice that we used dq instead of Q. The reason is that every ring has a differential charge. Also, notice that we defined the integral from 0 to R. The reason is that we will make that the rings grow, starting from a point $a = 0$ up to the last ring of radius R. Now, in this case, since the disc has a surface, so we use $dq = \sigma dA$.

Now, each ring area, we can think about it as the area between two circles (see figure 1.21, the shaded area). To obtain the differential area of each ring with differential width, we calculate

$$dA = \pi\left((a + da)^2 - a^2\right) \tag{1.103}$$

i.e. the area of the outer circle that conforms the ring with differential width minus the area of the inner circle that conforms the ring. So,

$$dA = \pi\left(a^2 + 2ada + da^2 - a^2\right) \tag{1.104}$$

The term $da^2 \to 0$ because da is infinitesimal. Therefore, we can rid of that term. Therefore,

$$dA = 2\pi ada \tag{1.105}$$

Hence, we have that equation 1.102 becomes

$$|\vec{E}|_{disc} = \int_0^R \frac{kx(\sigma 2\pi ada)}{(x^2 + a^2)^{\frac{3}{2}}} \tag{1.106}$$

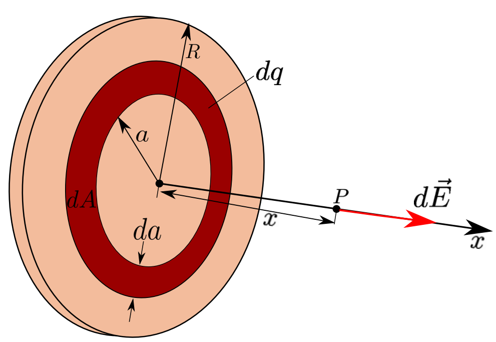


Figure 1.21

Using the following change of variable:

$$u = x^2 + a^2 \Longrightarrow du = 2ada \tag{1.107}$$

So, the integral becomes

$$|\vec{E}| = k\sigma x\pi \int_{x^2}^{x^2+R^2} \frac{du}{u^{\frac{3}{2}}} = k\sigma x\pi \frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}\Bigg|_{x^2}^{x^2+R^2} = -2k\sigma x\pi\left[\frac{1}{\sqrt{x^2 + R^2}} - \frac{1}{x}\right] = 2k\sigma\pi\left[1 - \frac{x}{\sqrt{x^2 + R^2}}\right] \tag{1.108}$$

Therefore the electric field of a positive charged disc is

$$\vec{E} = 2k\sigma\pi\left[1 - \frac{x}{\sqrt{x^2 + R^2}}\right]\hat{i} \tag{1.109}$$

Example 11: Electric Field of an infinite plane

Find the electric field of an infinite uniformly charged plane.
Solution: If you let the radius of a disc becomes infinitely large, then you can not notice any more the edge of the disc, and it becomes the same as infinite plane. So, we have that the electric field of the disc when we let $R \to \infty$

$$|\vec{E}| = \lim_{R\to\infty}\left(2\pi k\sigma\left[1 - \frac{x}{\sqrt{x^2+R^2}}\right]\right) = 2\pi k\sigma = 2\pi\left(\frac{1}{4\pi\epsilon_0}\right)\sigma = \frac{\sigma}{2\epsilon_0} \tag{1.110}$$

where the second term vanished, since $\frac{x}{\sqrt{x^2+R^2}} \to 0$. Also,we used the definition of k constant. Therefore, the magnitude of the electric field of an infinite plane is

$$|\vec{E}| = \frac{\sigma}{2\epsilon_0} \tag{1.111}$$

Example 12: Parabolic Path of Charged Particle in a uniform electric field

An electron moves with a speed $v_o = 3\times 10^5$m/s. It then enters to an uniform electric field $|\vec{E}| = 1\times 10^2 \frac{N}{C}$ which is perpendicular to $\vec{v}_o$ as shown in the figure. Find the position x when the electron hits the plate if its initial position is at the middle point between the plates. The separation of plates is $d = 5$cm (ignore gravity) ***Solution:*** We start by using the only information that we have, the electron feels a force. Therefore,

$$|\vec{F}| = q|\vec{E}| \tag{1.112}$$

Now, this force direction is to $-y$ because the electron is negative and it will be attracted to the positive plate (See Figure 1.22). Also, notice that there are no forces in the x direction so from second Newton's Law we can say that the acceleration in the x component is zero$a_x = 0$. Therefore, the speed of the electron in the x component remains exactly the same. However, the y component velocity changes as

$$m\frac{dv_y}{dt} = q|\vec{E}| \tag{1.113}$$

where we just used second Newtons law in 1.112 and that also the fact that $a_y = \frac{dv_y}{dt}$. So, by separating variables and integrating we obtain.

$$\int_{v_{0y}}^{v_y} m dv_y = q|\vec{E}|\int_0^t dt \tag{1.114}$$

where the limits of integration follow because the particle starts from certain initial velocity v_{0y} up to an arbitrary velocity v_y, when time was 0 up to certain time t respectively.

After integration, we obtain

$$v_y - v_{0y} = \frac{q|\vec{E}|}{m}t \tag{1.115}$$

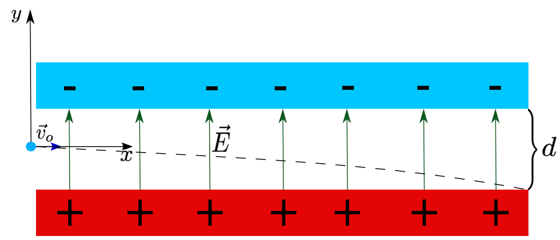


Figure 1.22

However, notice that the velocity in the y component at the very beginning is zero. The exercise says that the incoming electron path is horizontal, so no initial velocity in y exists. So, we are simply left with

$$v_y = \frac{q|\vec{E}|}{m}t \tag{1.116}$$

Now recall that $v_y = \frac{dy}{dt}$, so

$$\frac{dy}{dt} = \frac{q|\vec{E}|}{m}t \Longrightarrow \int_0^y dy = \frac{q|\vec{E}|}{m}\int_0^t tdt \tag{1.117}$$

where the limits of integration follow since the electron starts at time 0 with zero y component, and after an arbitrary time t, the particle has certain position y. After integration, we obtain

$$y = \frac{q|\vec{E}|}{2m}t^2 \tag{1.118}$$

Now, since there are no forces in x, there is no acceleration a_x, therefore there is not change in the velocity in the x component, so we have that:

$$v_x = \frac{x}{t} \Longrightarrow t = \frac{x}{v_x} \tag{1.119}$$

Therefore by plugging t in equation 1.119 into equation 1.118, we obtain

$$y = \frac{q|\vec{E}|}{2m}\left(\frac{x}{v_x}\right)^2 \tag{1.120}$$

Finally, isolating x,

$$x = \pm v_{0x}\sqrt{\frac{2my}{q|\vec{E}|}} \tag{1.121}$$

Since the electron moves to +x, we are left with the positive solution. Plugging in values

$$x = (1000m/s)\sqrt{\frac{2\cdot(9.1\cdot 10^{-31}kg)(-0.5\cdot 10^{-2})}{(-1.6\cdot 10^{-19}C)(1\cdot 10^5N/C)}} \tag{1.122}$$

where notice we included the minus sign of the electron charge, and the negative sign in the position because the particle goes downwards. So, the final answer is

$$x = 0.0159m \quad \text{or written in centimeters} \quad x = 1.59\text{cm} \tag{1.123}$$

1.6 Applications

There are several applications of electrostatics. However, we will get deeper understanding in one of them and make some calculations using what we have learned so far.

Example 13: Electrostatic Precipitator

The electrostatic precipitator is a beautiful and smart application using electric fields and charges. The purpose of this machine is to clean air from pollution particulates. The general idea of how this works is simple. As it can bee seen in figure 1.23 a flow of air with pollution particulates passes trough rods with electric charge. These rods since they are electrically charged they generate an electric field. If the electric field magnitude in every single point between them is high enough ($|\vec{E}| > 6000\frac{N}{C}$) a physical phenomena called as corona discharge takes place. The corona discharge is when the air is submitted under a high electric field such that it behaves as a conductor and the electrons from its molecules get free to move. The electric field in such situations are so high that the electrons from the air molecules are pulled off. So, before the air with pollution particulates arrive to the rods, there is a cloud of electrons already there. Therefore, when the particulates crosses the cloud of electrons, the electrons get stocked to the particulates of pollution. So, they are now electrically charged! Finally, when the air with the pollution particulates now crosses between the plates which are also electrically charged, the particulates feel an electric force due to the electric field generated by the plates! However, the air that is flowing has neutral charge, so the molecules of the air won't feel a force! Therefore, the outgoing air after the plates is clean of pollution particulates or at least with much less particulates, because the plates have trapped them with their electric fields!

Now, let's design an electrostatic precipitator.

1. If the rods of the filter are 50cm long (height of plates also), and we need to create an electric field at least of $E = 3.0 \times 10^6$N/C to make an electrical breakdown. We use a power supply such that makes the air to travel with speed $|\vec{v}| = 60m/s$. What charge magnitude the rods need to have to create an electric field of $E = 3.0 \times 10^6$N/C at the middle point between them? Distance between rods $d = 0.5$cm. Take into two rods.

2. The plates have the same magnitude of electric charge as the rods. Also, we know that they have the same height as the length of the rods. What minimum width they need to have to trap the particles (so that they do not escape from the plates). We can assume that the pollute particulates get an electric charge $q = -1 \times 10^{-9}$C and have mass 0.8×10^{-6}kg. Consider that the distance between plates $d = 0.5$cm is

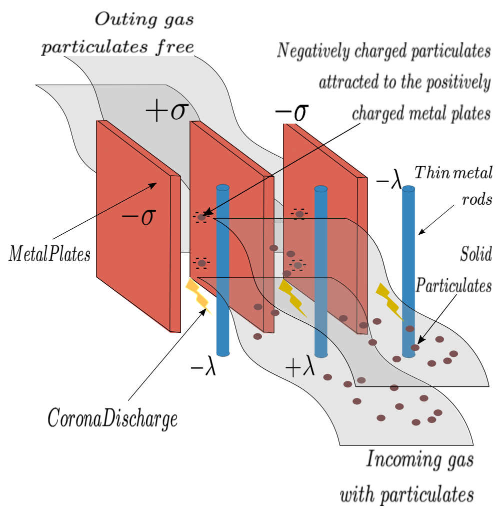


Figure 1.23

much less than the with and height of the plates. Also consider the plates extremely thin so they can be modelled as planes.

Solution:
In order to know the electric field generated by two rods between them, we use the formula we have found of the electric field for an electrically charged rod in equation 1.80. So, the electric field magnitude between the rods due to both rods is

$$|\vec{E}| = \frac{2kQ}{x\sqrt{x^2+a^2}} \tag{1.124}$$

So, we isolate the electric charge of each rod

$$Q = \frac{|\vec{E}|x\sqrt{x^2+a^2}}{2k} \tag{1.125}$$

Therefore, in order to create an electric field of magnitude $|\vec{E}| = 3\times10^6\frac{N}{C}$ at a symmetrical point between both rods , the electric charge of a single rod must be

$$Q = \frac{\left(3\times10^6\frac{\mathrm{N}}{\mathrm{C}}\right)\left(0.25\times10^{-2}\mathrm{m}\sqrt{(0.25{\times}10^{-2}\mathrm{m})^2+(25\times10^{-2}\mathrm{m})^2}\right)}{2\cdot 9\times10^9\frac{\mathrm{Nm}^2}{\mathrm{C}^2}} \approx 1.041\times10^{-7}\mathrm{C} \tag{1.126}$$

where we used half of the distance between the rods $x = 0.25\times10^{-2}$m, and recalling that a is the half of the length of the rod, so $a = 25\times10^{-2}$m.

Now that we know the electric charge of the rods, let's proceed to calculate the minimum width of the plates so that no pollution particulate escapes. When the pollution particulates travel between the plates, we can visualize them independently as in figure

1.22. However, in order that no particulate escapes, we need to place our particulate in figure 1.22 at the very top. Because any particulate at the top (in the negatively charged plate) will reach a maximum distance in x. Any other particulate, will travel less distance because they hit towards the positively charged plate. So, if we calculate what is the distance travelled by an electric charge from the very top , we will get the minimum width required of the plates so that no pollution particulate escapes. A particulate when hits the plate is at $y = -d$ (the particulate travels downwards a distance $-d$ in the y axis when hits the plate, where d is the distance between the plates), placing our reference frame at the initial position of the electric charge when gets between the plates. So, using equation 1.120, we have that

$$-d = \frac{q|\vec{E}|}{2mv_x^2}x^2 \tag{1.127}$$

Now, we do not know exactly the value of the electric field between the plates (the electric field in last equation is the electric field between the plates do not confuse it with the electric field generated by the rods). However, we can use the electric field formula for an infinite plate

$$|\vec{E}| = 2\left(\frac{\sigma}{2\epsilon_0}\right) = \frac{\sigma}{\epsilon_0} \tag{1.128}$$

where the factor of 2 comes from the fact that we have two plates generating an electric field between them. Also, probably you say "Those plates by no means are infinite, Why would I use the formula of an infinite plate?" Well, indeed they are not infinite. However, the distance between the plates is so small compared to the size of the plates, and the particulates so tiny in comparison to the plates, that the approximation is not bad at all. Now, we do not know what is the value of charge density σ. However we can use the definition of sigma, and express the electric field between the plates as

$$|\vec{E}| = \left(\frac{Q}{A}\right)\frac{1}{\epsilon_0} = \frac{Q}{xh\epsilon_0} \tag{1.129}$$

where $A = xh$ is the area of the plates, x the width and h the height of the plates. Hence, equation 1.127 becomes

$$d = -\frac{q}{2mv_x^2}\left(\frac{Q}{xh\epsilon_0}\right)x^2 \tag{1.130}$$

Therefore isolating x from the last equation, we obtain

$$x = -\frac{2mv_x^2 d\epsilon_0 h}{qQ} \tag{1.131}$$

where is important to remark that q is the electric charge that the pollution particulates have due to the electrons stocked on them, and Q is the electric charge of the plates. In this particular case, the exercise mentions explicitly that the electric charge of the rods and the plates is exactly the same. So, plugging in the values we have

$$x = -\frac{2\left(0.8\times10^{-6}\mathrm{kg}\right)\left(60m/s\right)^2\left(0.5\times10^{-2}\mathrm{m}\right)\left(8.85\times10^{-12}\frac{\mathrm{C}^2}{\mathrm{Nm}^2}\right)\left(50\times10^{-2}\mathrm{m}\right)}{\left(-1\times10^{-9}\mathrm{C}\right)\left(1.041\times10^{-7}\mathrm{C}\right)} \approx 1.22\mathrm{m} \tag{1.132}$$

Chapter 2

Gauss Law

We begin this chapter by defining electric flux, and this quantity will lead us to Gauss Law when we calculate electric flux trough closed surfaces (any surface that encloses a volume). As it turns out, Gauss Law is a powerful tool for electroestatics, but also for magnetostatics and graviatation. For electroestatics, Gauss Law permits us to know the magnitude of an electric field given that we know the electric charge that originates it or to know the magnitude of the charge that is creating an electric field. As we will study along this chapter, the Gauss Law is analytically powerful when we choose certain geometries as our enclosing surface.

2.1 Electric Flux

We define a quantity that is proportional to the number of electric field lines that crosses a surface. So, we have the following definition

Definition 2.1.1 *The electric flux through a rectangular surface is given by*

$$\boxed{\Phi = \vec{E} \cdot \vec{A}} \tag{2.1}$$

where $\vec{A}$ is a perpendicular vector to the surface, and its magnitude is the area of the surface.

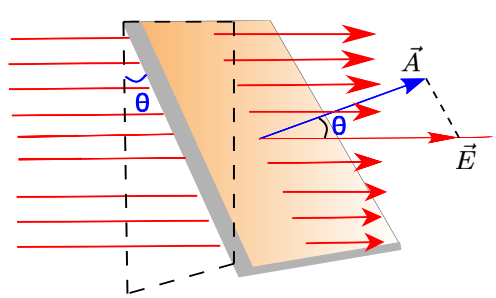


Figure 2.1

Notice that the electric flux is zero when the Electric field lines are perpendicular to the area vector, because no electric field lines are crossing it. Now, probably you think *How could there be a quantity that is related to the number of electric field lines, if those electric field lines are drawn by someone?* i.e. *How can there be something quantitative from something that depends totally to the person who draws the electric field lines?*. Well, during the diagrams shown in the previous chapter, we have exploited a property about electric field lines. *The number of electric field lines per unit area perpendicular to each electric field line (density of electric field lines) is* **proportional to the magnitude of the electric field**. This rule will always hold if you make a diagram such that it is consistent with the electric field that a charge produces.

And, well indeed you can draw the number of electric field lines that you wish (you could be so motivated to even draw 10000000 field lines or more up to infinity). However, the number of electric field lines are always proportional to the magnitude of the electric field in the region were they are drawn. For example, notice how this is true for a single electric charge $+q$. If the electric charge $+q$ is surrounded by a sphere of radius r as shown in figure 2.2 (a surface that will be tangent to every single electric field line) then the density of number of electric field lines is

$$D_0 = \frac{N}{4\pi r_0^2} \tag{2.2}$$

where N is the number of electric field lines (8 for the diagram in figure 2.2) while , the electric field magnitude at any point of the surface of the sphere is

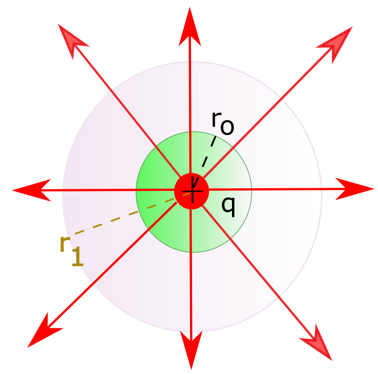


Figure 2.2: Electric charge enclosed by two spheres that are perpendicular to each electric field line.

$$|\vec{E}_0| = k\frac{Q}{r_0^2} \tag{2.3}$$

For another sphere of radius $r_1 > r_0$. The density is

$$D_1 = \frac{N}{4\pi r_1^2} \tag{2.4}$$

and , the electric field magnitude at each point in the second surface is

$$|\vec{E}_1| = k\frac{Q}{r_1^2} \tag{2.5}$$

so $D_0 > D_1$ and also of course $|\vec{E}_0| > |\vec{E}_1|$. So, as we can notice the density varies as $\frac{1}{r^2}$, as the electric field magnitude and the density is smaller at greater r as also of course as the electric field. So, indeed we notice that

$$|\vec{E}| \propto D \tag{2.6}$$

Or the same to say the following,

$$|\vec{E}| \propto \frac{N}{A_\perp} \tag{2.7}$$

where N is the number of electric field lines and A is the magnitude of the area. **Notice**, the key word is ***proportionality***. We do not know exactly what constant of proportionality we should be using to obtain an equality in the last equation. That actually depends on each drawing. Now, if we want the perpendicular unit area, we need $A = A_\perp \cos\theta$. Therefore,

$$N \propto \Phi \tag{2.8}$$

In general, the electric field lines in any diagram are qualitative, and they are deceptive. However, the electric flux definition is not, and it is a quantity that will be extremely useful to obtain quantitative results. We must always have present that the electric field lines that we draw is just a representation of the electric field, and it is the electric field lines that start from the physics and not the way around, i.e. from the picture we cannot deduce all the physics, however from the physics we deduce how the electric field lines should be represented.

Finally, there is an ambiguity in the direction of the vector $\vec{A}$, because we could have chosen it with opposite direction. The electric flux through a rectangular surface is *ill-defined* because there is no way to uniquely determine the direction of vector $\vec{A}$. However, when we choose a closed surface, the ambiguity vanishes as we will see in the next section.

Example 1: Simplest Case of Electric Flux

A side square 0.2m is oriented with its unit normal vector $\hat{n}$ as shown in the figure 2.3. If the electric field $\vec{E}$ of magnitude 4.3×10^3N/C has angle of 70° with respect to the plane as shown in the figure 2.3

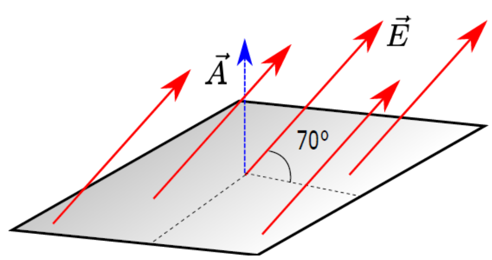


Figure 2.3

- What is the electric flux through the square?
- What would be the electric flux if the square is rotated in such a way that $\vec{A}$ is perpendicular to $\vec{E}$

Solution:

Writting down the electric flux equation for a rectangular surface (equation 2.1), we have

$$\Phi = |\vec{E}||\vec{A}| \cos\theta_{EA} \tag{2.9}$$

where the label EA in the θ angle is written down so that you remember which angle you are actually looking for. This should be obvious from the definition of the dot product. However, we write it down because many students forget what angle they should be using. Now, the area vector magnitude is just the area of the surface, therefore

$$A = (0.2\text{m})^2 = 0.04\ m^2 \tag{2.10}$$

while

$$\cos 20^\circ \approx 0.94 \tag{2.11}$$

Therefore,

$$\Phi = (4.3 \cdot 10^3\ N/C) \cdot (0.94) \cdot (0.04\ m^2) = 163.58 \frac{\text{N}}{\text{C}} \text{m}^2 \tag{2.12}$$

Finally, if we rotate the square such that the electric field in is perpendicular to the electric field , it must be zero.

$$\Phi = |\vec{E}||\vec{A}| \cos 90^\circ = 0 \tag{2.13}$$

because $\cos 90^\circ = 0$. Always that a surface is perpendicular to the electric field lines, then the electric flux is zero.

2.2 Gauss Law

We have defined so far a quite simple case for the electric flux through a rectangular surface. However, what if the surface is not perfectly rectangular? We will split all the surface in small rectangles (or at least try by approximating).

Then, if we want to find what is the flux through a general surface, we need to sum all the contributions of the electric fluxes through each little rectangle

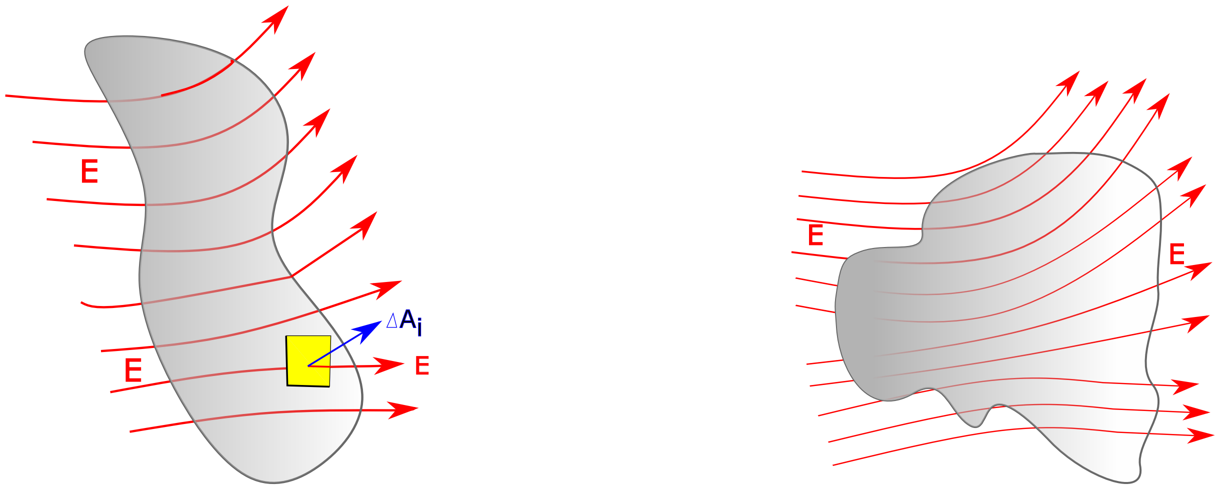
$$\Phi_{tot} \approx \sum_i \Phi_i = \sum_i E_i \cdot \Delta A_i \tag{2.14}$$

If we now let the rectangles be so small, that their area become infinitesimal, we have

$$\Phi_{tot} = \lim_{A_i \to 0} \sum_i E_i \cdot \Delta\vec{A}_i = \int \vec{E} \cdot d\vec{A} \tag{2.15}$$

where we have lost the approximation and have obtained an equality, because the rectangles now are infinitesimally small. Finally, if the surface is closed, i.e. it encloses a volume, then we use the closed integral symbol $\oint$ to denote that the total flux is over a closed surface.

$$\boxed{\Phi = \oint \vec{E} \cdot d\vec{A}} \tag{2.16}$$



(a) (b)

Figure 2.4

And, what is the direction of the perpendicular differential vectors $d\vec{A}$? **By convention the differential vectors $d\vec{A}$ will always point out to the volume that the surface encloses**. So, any electric field line that points towards the volume or "go in" to the volume, then its contribution to the electric flux is negative (because $\vec{E}\cdot d\vec{A} = |\vec{E}|d\vec{A}\cos\theta < 0$, since $\theta > 90°$). Furthermore, any electric field line that points outwards the volume or "go out" from the volume, then its contribution to the electric flux is positive (because $\vec{E} \cdot d\vec{A} = |\vec{E}|d\vec{A}cos\theta > 0$, since $\theta < 90°$) Therefore, from that we have the following remarks

- If the total electric flux in a closed surface is **positive** , then the **number of electric field lines that go out is greater than the number of electric field lines that go in**
- If the total electric flux in a closed surface is **negative**, then the **number of electric field lines that go in is greater than the number of electric field lines that go out**
- If the total electric flux in a closed surface is **zero**, then **the number of electric field lines that go in is equal to the number of electric field lines that go out**

Now, let's see what happens if we enclose a charged particle with a sphere as shown in figure 2.5. We place the particle at the center of the sphere and we calculate the electric flux through the sphere.

So, by using the equation 2.16 we have

$$\Phi = \oint k\frac{q}{r^2}\hat{r} \cdot d\vec{A} \tag{2.17}$$

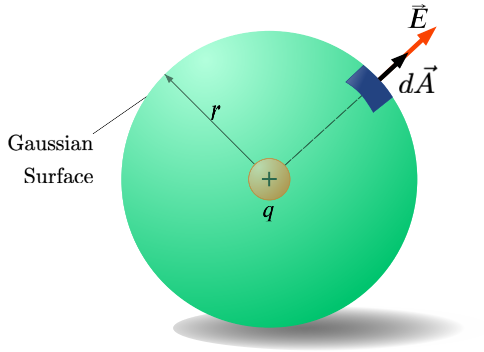


Figure 2.5

where we substituted the electric field of a point charge. Now, every single vector $\hat{r}$ will be parallel to the vectors $d\vec{A}$. The reason is that every single vector $\hat{r}$ is radial to the particle, and $d\vec{A}$ is perpendicular to the surface of the sphere. Therefore, the direction of both vectors si exactly the same (as shown in the figure 2.5). So, the dot product in the latter equation becomes $\hat{r} \cdot d\vec{A} = |\hat{r}||d\vec{A}| \cos 0 = dA$, since the vector $\hat{r}$ is unitary and the $\cos 0 = 1$. Therefore we have

$$\Phi = \oint k\frac{q}{r^2}dA \tag{2.18}$$

Notice now, that for every single differential of area $d\vec{A}$, the electric field is constant because at each differential of area the radius has not changed. In other words, the electric field of a point electric charge is dependant of the radius as $\frac{1}{r^2}$, therefore at each differential of area the electric field does not change. Therefore, for each differential of area dA the electric field is constant! Therefore, $k\frac{q}{r^2}$ is constant and we are left with

$$\Phi = k\frac{q}{r^2} \oint dA = k\frac{q}{r^2} \oint dA = k\frac{q}{r^2} A_{sphere} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}4\pi r^2 \tag{2.19}$$

where we substituted the definition of the constant k and the area of a sphere

Therefore,

$$\boxed{\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}} \tag{2.20}$$

The last equation is very special, and it holds not just for a sphere, it holds for any closed $surface$. We call it as Gauss Law. It is a very powerful tool to get electric fields from any geometry. In this course we will study how to obtain the electric field from very symmetric geometries, or from geometries that lead to easy solutions using Gauss Law. In general, the equation 2.20 can be used for any geometry, however the analytical solution could get nasty, and we would perform the integration by numerical approximation.

Example 2: Electric Flux Through 5 random surfaces

The following figure shows three point charges $q_1 > 0$, $q_2 < 0$ and $q_3 > 0$. Find the electric flux through each of the closed surfaces S_1, S_2 and S_3.

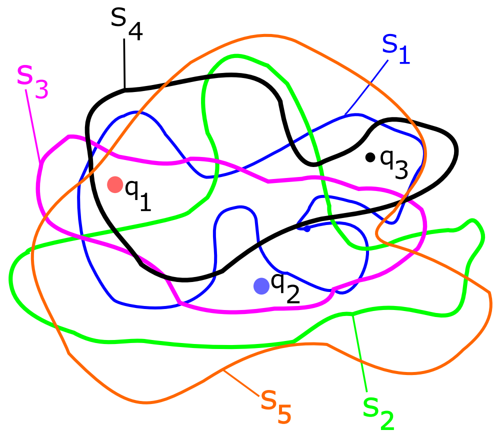


Figure 2.6

Solution: We use the Gauss Law in equation 2.20 to know the electric flux. In order to obtain the electric flux, we just need to know what charges are enclosed by the different surfaces. For the surface S_1 the only charges inside are q_1 and q_3. So the electric flux through S_1 is

$$\Phi_{s_1} = \frac{q_1 + q_3}{\epsilon_0} \tag{2.21}$$

While for the surface S_2, the only charge inside the volume that S_2 encloses is q_2. Therefore, the electric flux is

$$\Phi_{s_2} = -\frac{q_2}{\epsilon_0} \tag{2.22}$$

where explicitly the sign has been written. In the electric flux calculation is important to include the signs of the charges.

For the surface S_3, there are two charges enclosed, q_1 and q_2. Therefore, the electric flux is

$$\Phi_{s_3} = \frac{q_1 - q_2}{\epsilon_0} \tag{2.23}$$

once again, including the negative sign of the electric q_2. Finally, for the surfaces S_4 and S_5, we have that

$$\Phi_{s_4} = \frac{q_1 + q_3}{\epsilon_0} \quad , \quad \Phi_{s_5} = \frac{q_1 - q_2 + q_3}{\epsilon_0} \tag{2.24}$$

Example 3: Electric Flux trough a cube

The cube of the figure 2.7a has sides of length $L = 10$cm. The electric field that passes thorugh it is uniform. The electric field mangitude is $\vec{E} = 4.510^3$N/C and is parallel to the yz plane with an angle of 40° measured from the $+z$ axis to the $+y$ axis.

- What is the electric flux through surface S_3 and S_1?
- What is the total electric flux through the cube?

Solution:

To solve this problem is important to recall that the area vector is perpendicular to the surface and by convention if the surface encloses a volume, the area vectors or the differential area vectors will always point outwards to the surface as shown in figure 2.7. Before starting, let's calculate the area of each face of the cube.

$$A = L^2 = (0.1\text{m})^2 = 0.01\text{m}^2 \tag{2.25}$$

Now, let's use Gauss Law to calculate the total electric flux through the cube

$$\Phi_{cube} = \oint \vec{E} \cdot d\vec{A} = \int_{S_1} \vec{E} \cdot d\vec{A} + \int_{S_2} \vec{E} \cdot d\vec{A} + \int_{S_3} \vec{E} \cdot d\vec{A} + \int_{S_4} \vec{E} \cdot d\vec{A} + \int_{S_5} \vec{E} \cdot d\vec{A} + \int_{S_6} \vec{E} \cdot d\vec{A} \tag{2.26}$$

where we splitted the closed integral into the integration over each of the areas of the cube. In other words, the total electric flux is equal to the electric flux over each of the faces of the cube.

$$\Phi_{cube} = \sum_{i=1}^{N} \Phi_i \tag{2.27}$$

Therefore, to find the electric flux through S_1 and S_3 we need to calculate

$$\Phi_1 = \int_{S_1} \vec{E} \cdot d\vec{A} \qquad , \Phi_3 = \int_{S_3} \vec{E} \cdot d\vec{A} \tag{2.28}$$

Showing explicitly the integration over S_1

$$\Phi_1 = \int_{S_1} \vec{E} \cdot d\vec{A} = \int |\vec{E}| dA \cos\theta = |\vec{E}| \cos\theta \int_{S_1} dA = |\vec{E}| A_{S_1} \cos\theta \tag{2.29}$$

where the the electric field and the $\cos\theta$ go out from the integral because they are constant and the area of the surface S_1 was written as A_{S_1}. Therefore,in this particular exercise, the integration will simply be

$$\Phi_1 = |\vec{E}| A_{S_1} \cos\theta_{EA_{S1}} \qquad , \Phi_3 = |\vec{E}| A_{S_3} \cos\theta_{EA_{S3}} \tag{2.30}$$

because the electric field and the angle are constant, so they get out from the integral and from the integration we just obtain the area of the faces of the cube. This will hold for any the faces of the cube, so we just have to concern about the angles between the electric field and the area vector of each face of the cube, i.e.

$$\Phi_i = |\vec{E}| AS_i \cos\theta_{EA_{Si}} \tag{2.31}$$

where we call all the areas as A because the faces of the cube are all equal; and the label AS_iin the angle θ is just to remember you that the angle in the electric flux is the angle between the electric field and the area vector. As mentioned before this must be obvious from the definition of dot product! So, an apologize in advance for any of you that notice this immediately, however many students make mistakes in this particular step and better to repeat several times what angle must be in there. Now, in figure 2.7 are shown the directions of the area vectors of each face of the cube. So, showing the calculation of all the electric fluxes through each face of the cube, we obtain (using figure 2.7)

$$\Phi_1 = EA\cos 130° = -28.92544244\,(LEFT \quad SIDE) \tag{2.32}$$

$$\Phi_2 = EA\cos 40° = 34.47199994\,(TOP) \tag{2.33}$$

$$\Phi_3 = EA\cos 50° = 28.92544244\,(RIGHT) \tag{2.34}$$

$$\Phi_4 = EA\cos 140° = -34.47199994\,(BOTTOM) \tag{2.35}$$

Notice that for the back and front sides of the cube, the area vectors will be perpendicular to the electric field because the electric field vector lies in the zy plane. Even though there is an angle between the electric field and the z axis, the electric field does not have a x component. Therefore it is completely perpendicular to the front and back area vectors that lie on the $+x$ and $-x$ axis respectively as shown in figures 2.7c and 2.7d. Also, if you are not familiar with the notation $\odot$ and $\otimes$, their meaning is that the vector points outwards to the page and that the vector points inwards to the page respectively. This notation will be widely used in the figures along the book.

$$\Phi_5 = 0\,(FRONT \quad SIDE) \tag{2.36}$$

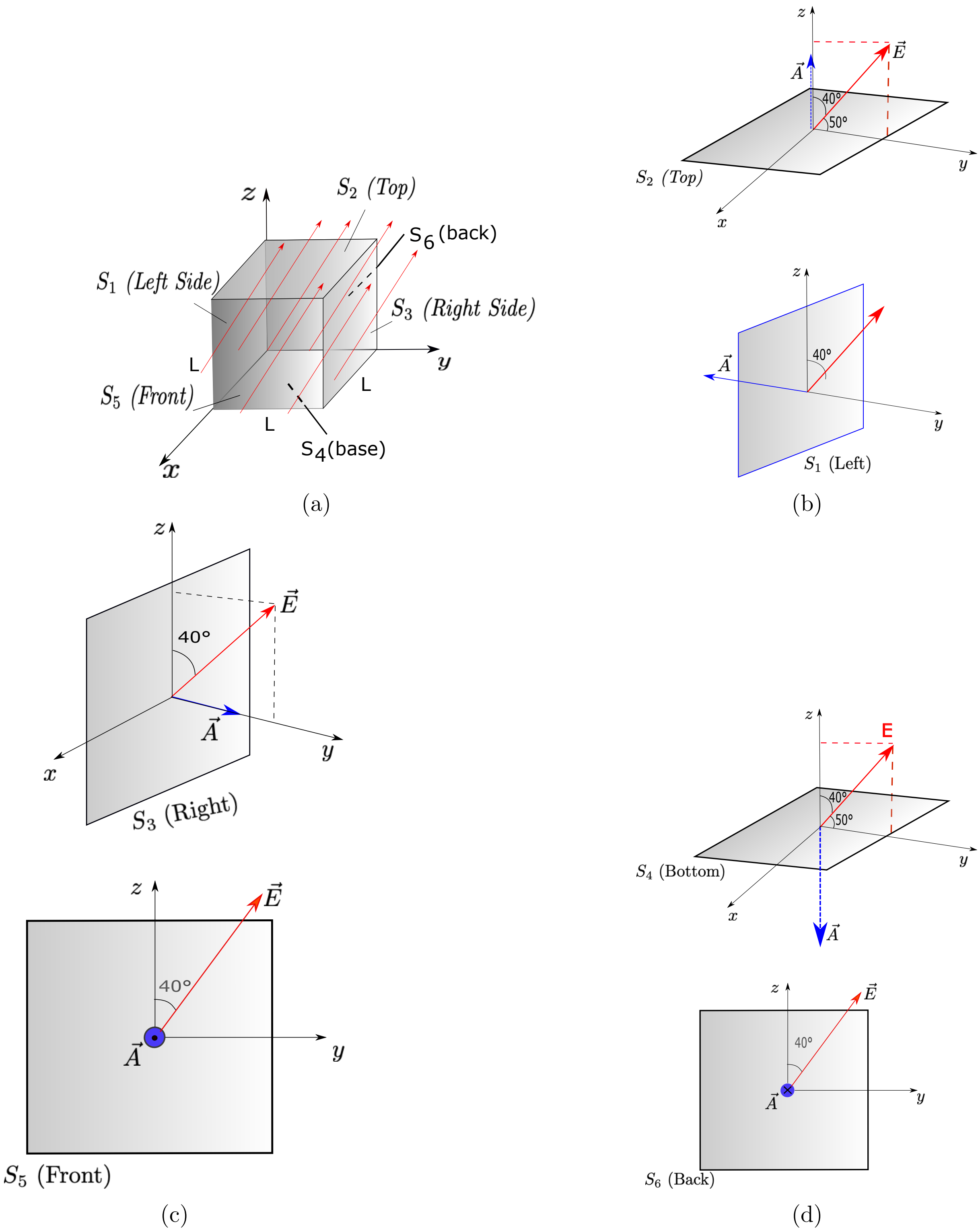
$$\Phi_6 = 0\,(BACK \quad SIDE) \tag{2.37}$$

Hence, notice that the total electric flux is

$$\Phi_{net} = \sum_{i=1}^{6} \Phi_i = 0 \tag{2.38}$$

as it must be! Using Gauss Law in equation 2.20, we see that since there is no electric charge enclosed in the cube, the electric flux must be zero!

Also, the electric flux tells us about the number of electric field lines that goes in and out of the volume enclosed by the surface. Given that the number of the field lines that go in are equal to the field lines that go out, the total electric flux must be zero. Actually it was exactly the electric flux going into the cube in the left side of the cube and the electric flux in the right side of the cube that cancelled out, and the top with the bottom respectively. The electric field lines that go in contributed with opposite sign to the electric flux to the sides where the electric field lines go out.



z
S_2 (Top)
S_6(back)
S_1 (Left Side)
S_3 (Right Side)
L
y
S_5 (Front)
L
L
S_4(base)
x
(a)
z
$\vec{E}$
$\vec{A}$
40°
50°
S_2 (Top)
y
x
z
40°
$\vec{A}$
y
S_1 (Left)
(b)
z
$\vec{E}$
40°
$\vec{A}$
y
x
S_3 (Right)
z
$\vec{E}$
40°
$\vec{A}$
y
S_5 (Front)
(c)
z
E
40°
50°
S_4 (Bottom)
y
x
$\vec{A}$
z
$\vec{E}$
40°
$\vec{A}$
y
S_6 (Back)
(d)

Figure 2.7

Example 4: Electric Flux trough one face of cube given a point charge at middle point

A point charge $q = 5\text{nC}$ is in the center of a side cube $L = 10\text{cm}$. Find the electric flux through one of its faces. ***Solution:***

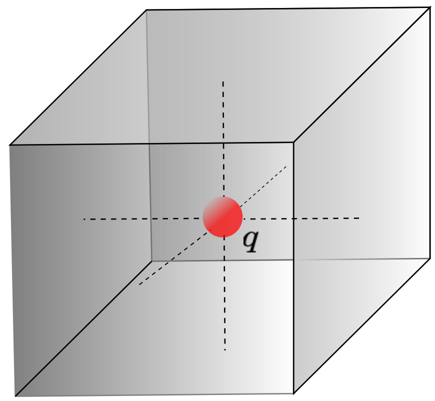


Figure 2.8

By Gauss Law, we know that the electric flux through all the faces of the cube is given by

$$\Phi_{cube} = \frac{q_{enc}}{\epsilon_0} = \frac{5 \cdot 10^{-9}\text{C}}{\epsilon_0} \tag{2.39}$$

And, since the particle is at the center of the cube, the electric flux will be exactly the same through any of the faces. Therefore, the electric flux through just one of the six faces of the cube is given by

$$\Phi_{oneface} = \frac{\Phi_{cube}}{6} = \frac{5 \cdot 10^{-9}\text{C}}{6\epsilon_0} \tag{2.40}$$

Example 5: Gauss Law applied, electric field of an infinite rod

A positive electric charge is uniformly distributed along a very thin and infinite wire. The electric charge per unit length in the wire is λ. Find the electric field at a point at a radial distance r from the wire.

Solution:

One of the most important aspects of solving problems with Gauss Law is to choose a geometry of the Gaussian Surface that makes simple our calculation. How to know if the geometry is such that it makes easy our calculation? Find a geometry such that the electric field is constant at every point and the angle between every single $d\vec{A}$ and $\vec{E}$ is constant . Why we want the electric field and the angle to be constant at every point? The answer is easy, we want from Gauss Law to get out from the integral the electric field

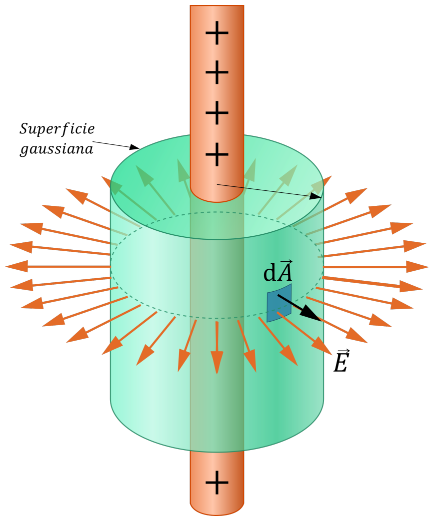


Figure 2.9

as follows

$$\oint \vec{E} \cdot \vec{A} = \oint |\vec{E}| dA \cos\theta_{EdA} = |\vec{E}| \cos\theta_{EdA} \oint dA = |\vec{E}| A_{surface} \cos\theta_{EdA} \tag{2.41}$$

In general, this can only happen if the electric field $\vec{E}$ is constant at every single differential of area $d\vec{A}$. If it is not the case, we have to integrate the dot product. In general, this will not always happen, and sometimes we will have to bravely integrate the dot product in the surface integral. Also, it is desirable that the angle between the electric field and each differential of area is just $\theta_{EdA} = 0$, so that our Gauss Law turns out to be the simple result

$$\Phi = |\vec{E}| A_{surface} = \frac{q_{enc}}{\epsilon_0} \tag{2.42}$$

For example , for the infinite rod, we already know from our discussion in the previous chapter that the electric field lines will be radially outwards as shown in figure 2.9. So for example, choosing an sphere as our Gaussian surface, it would not be the best option because the electric field magnitude will not be constant over the surface of the sphere. Also, the angle is not constant between the electric field vectors and the area vector would not be constant. However, if we choose a cylinder as shown in figure 2.9 every single electric field vector is the same at every single differential of area at the wall surface as shown in the figure 2.9, so it is constant. Also, the angle between the electric field vectors and the differential of areas are the same, just zero! So, the cylinder is the right choice for our Gaussian surface to solve the problem.

So, we start then with our so loved Gauss Law

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \tag{2.43}$$

The closed integral in the left hand side of the last equation can be splitted as follows.

$$\int_{top} \vec{E} \cdot d\vec{A} + \int_{bottom} \vec{E} \cdot d\vec{A} + \int_{wall} \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \tag{2.44}$$

i.e. we integrate over all the surfaces that constitute the cylinder. However, notice that the bottom and top integrals vanish since the electric field and the differential of area are perpendicular, so $E \cdot d\vec{A} = 0$. Therefore, we are left with

$$\int_{wall} \vec{E} \cdot d\vec{A} = |E| \int_{wall} dA = |E| A_{wall} = \frac{q_{enc}}{\epsilon_0} \tag{2.45}$$

where from the first step to the second we used $\vec{E} \cdot \vec{A} = |E| A_{wall} \cos 0 = |E| A_{wall}$ because the electric field lines and $d\vec{A}$ are parallel. Finally, the area of the wall of the cylinder is the same as a rectangle with width $2\pi r$ and height l, so $A_{wall} = 2\pi r l$. Therefore, last equation becomes

$$|E| 2\pi r l = \frac{q_{enc}}{\epsilon_0} \Longrightarrow |E| = \frac{q_{enc}}{2\pi r l \epsilon_0} \tag{2.46}$$

Finally, notice that charge density $\lambda = \frac{q_{enc}}{l}$, so the electric field is given by

$$|E| = \frac{\lambda}{2\pi r \epsilon_0} \tag{2.47}$$

Notice how easy it was to solve this problem by just choosing the right Gaussian surface in comparison to the way we solved this problem in chapter one.

Example 6: Gauss Law applied, enclosing charge in rods and particles

An infinitely long line charge having a uniform charge per unit length $\lambda = 1.5 \frac{\mu\text{C}}{\text{m}}$ lies a distance d from point O as shown in the figure. Determine the total electric flux through the surface of a sphere of radius $R = 2$cm, centered at O resulting from this line charge and the electric charges $+q$ which charge is 1.2μC

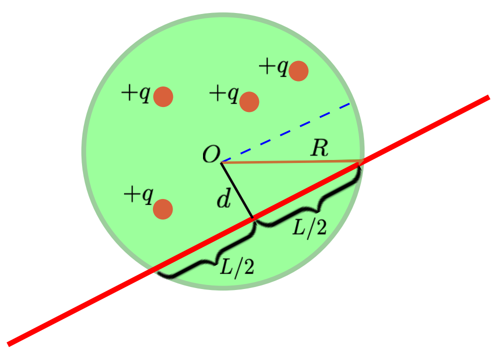


Figure 2.10

Solution:

We have to consider all the electric charge enclosed by the sphere. So, we can write the electric flux through the sphere

$$\Phi = \frac{q_{enc}}{\epsilon_0} = \frac{4q + q_{enc-rod}}{\epsilon_0} \tag{2.48}$$

where $q_{enc-rod}$ is the rod enclosed charge by the sphere. So, we can use the fact that $q_{enc-rod} = \lambda L$ to obtain the rod enclosed charge. Therefore, the question now is what is "L". We just use trigonometry.

$$\left(\frac{L}{2}\right)^2 + d^2 = R^2 \qquad \Longrightarrow \left(\frac{L}{2}\right)^2 = R^2 - d^2 \tag{2.49}$$

$$L = 2\sqrt{R^2 - d^2} \qquad \Longrightarrow \tag{2.50}$$

Therefore, the rod enclosed charge is

$$q_{enc-rod} = \lambda L = 2\lambda\sqrt{R^2 - d^2} \tag{2.51}$$

Therefore, the electric flux is

$$\Phi = \frac{q_{enc}}{\epsilon_0} = \frac{4 \cdot 1.2 \times 10^{-6}C + 2 \cdot (1.5 \cdot 10^{-6}C/m)\sqrt{(2 \cdot 10^{-2}m)^2 - (1.5 \cdot 10^{-2}m)^2}}{8.85 \cdot 10^{-12}\frac{\mathrm{C}^2}{\mathrm{Nm}^2}} \tag{2.52}$$

$$\Phi = 546857.20\frac{N}{C}m \tag{2.53}$$

Example 7: Gauss Law applied, electric field of an infinite plane

Find the electric field due to an infinite plane of uniformly distributed positive charge with surface charge density σ.

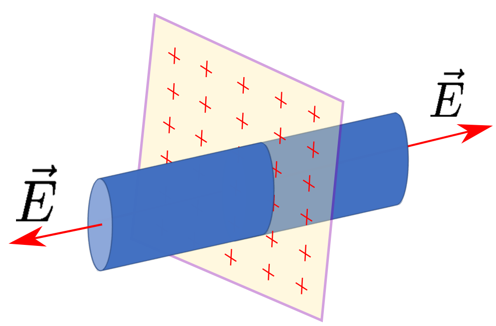


Figure 2.11

Solution:
Once again we need a smart choice of our Gaussian surface. If we recall, from the previous chapter all the electric field lines of an infinite plane are as shown in the figure. In the previous chapter we analyzed a disc and then found the electric field of an infinite plane by

sending the radius of the disc up to infinity. And when we analyzed the disc we concluded that there were no electric field contributions other than radially since they cancel out by symmetry. So, using a cylinder once again will do the job. We place the cylinder as shown in figure 2.11. The cylinder will have to enclose electric charge of the plane, if it is not the case the right hand side from equation 2.20 will be zero and no electric field we will be able to obtain . So the cylinder top and bottom must be on opposite sides of the plane as shown in the figure 2.11. Therefore, writing down Gauss Law for the cylinder

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \int_{top} \vec{E} \cdot d\vec{A} + \int_{wall} \vec{E} \cdot d\vec{A} + \int_{bottom} \vec{E} \cdot d\vec{A} \tag{2.54}$$

where we wrote the closed integral over all the surfaces of the cylinder.The integral over surface *wall* is zero, because there is no electric field vector crossing through that surface. Now,

$$\int_{top} \vec{E} \cdot d\hat{A} = \int_{top} |\vec{E}| dA \cos 0 = |\vec{E}| \int_{top} dA = |\vec{E}| A_{top} \tag{2.55}$$

And,

$$\int_{bottom} \vec{E} \cdot d\hat{A} = \int_{bottom} |\vec{E}| dA \cos 0 = |\vec{E}| \int_{bottom} dA = |\vec{E}| A_{bottom} \tag{2.56}$$

Also, we have that

$$A_{top} = A_{bottom} \qquad \text{so calling them just as } A \Longrightarrow \tag{2.57}$$

Therefore

$$\Phi = 2|\vec{E}|A = \frac{q_{enc}}{\epsilon_0} \Longrightarrow |\vec{E}| = \frac{q_{enc}}{2A\epsilon_0} = \frac{\sigma}{2\epsilon_0} \tag{2.58}$$

where we used the fact that $\sigma = \frac{q_{enc}}{A}$

2.3 Conductors and Insulators

In reality, there are many kinds of materials with different properties, however for purposes of this course we will divide all materials in two general sets, *conductors* and *insulators*.

Definition 2.3.1 *A conductor is any material where the electric charges can move easily from one region to another in the material.*

Definition 2.3.2 *An insulator is any material that does not allow the electric charge to move easily through it. When these materials are submitted to a threshold magnitude of electric field, they behave as conductors.*

2.3.1 Properties of Conductors

Experimentally, the time it takes a good conductor to reach equilibrium is on the order of 1×10^{-16}s , which for most purposes can be considered instantaneous.Therefore, when they are isolated (no external electric field) we will consider them as **electrostatic systems**. By taking into account this assumption, we have that conductors have the following properties

1. **Inside an isolated conductor $\vec{E}=0$.**
   If $\vec{E}\neq 0$, there would exist a force that would cause the charges to move (recall that $\vec{F}=q\vec{E}$). But as we mentioned, we are taking into account everything as electrostatic,hence nothing moves.

2. **The net (total) charge within an isolated conductor is zero**
   This property holds since $\vec{E}=0$ inside any isolated conductor, so using the contour of the conductor as a Gaussian Surface $\oint \vec{E}d\vec{A}=0$, therefore by Gauss Law $q_{enc}=0$

3. **If an isolated conductor has electric charge, it must necessarily reside on the surface.**
   Inside any isolated conductor $\vec{E}=0$, so the total electric charge must be 0, otherwise there would be an electric field. However, it could happen that certain conductors have extra positive charge or extra negative charge, however since these extra charges are unable to be within the conductor (otherwise $E\neq 0$), the only place where they can reside is on the surface.

4. **If an isolated conductor has charge on the surface, then the electric field just outside (infinitesimally near to the surface) the conductor is**

$$\vec{E}=\frac{\sigma}{\epsilon_0}\hat{n} \tag{2.59}$$

   where $\hat{n}$ is a perpendicular vector to the surface of the conductor. This property is demonstrated in the following exercise

Example 8: Electric Field just outside of a conductor

Suppose a conductor is in electrostatic equilibrium. What is the electric field just outside the conductor?

Solution:

Once again, we pick a cylinder as our Gaussian surface. However, this time we place it so that one circular surface its infinitesimally separated from the surface of the conductor, and the other circular surface is infinitesimally inside the conductor. Before continuing, analyze figure 2.12, the electric field produced by the conductor must be necessarily pointing outwards because remember that there is no electric field inside the conductor. Also, there is no electric field pointing to the sides perpendicular to the vector $\hat{n}$ since all charges are static, otherwise they could move (recall that $\vec{F}=q\vec{E}$,so if there is an electric

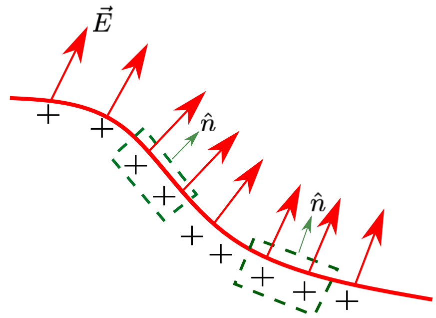


Figure 2.12

field to the sides, all those charged particles would move). So, writing down Gauss Law for this case

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \int_{top} \vec{E} \cdot d\vec{A} + \underbrace{\int_{wall} \vec{E} \cdot d\vec{A}}_{=0} + \underbrace{\int_{bottom} \vec{E} \cdot d\vec{A}}_{=0} \tag{2.60}$$

where the last two integrals will vanish since there is no electric field passing through those surfaces($\vec{E} = 0$). Therefore, we are left with

$$\int_{top} \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \tag{2.61}$$

Notice that every single differential vector $d\vec{A}$ in our Gaussian surface have the vector direction of $\hat{n}$, so the $d\vec{A}$ are parallel to the electric field (see figure 2.12). This is true because we are calculating the electric field so near to the surface of the conductor, that the electric field lines are very good approximated to be perfectly perpendicular to the surface. So, we have

$$\int_{top} |\vec{E}| \cdot dA \cos 0 = \frac{q_{enc}}{\epsilon_0} \tag{2.62}$$

Now, the electric field that passes through the top surface can also be approximated as constant, since the electric field lines are approximated as perfectly perpendicular to the surface of the conductor and since every single electric field line is separated at approximately the same distance from the surface, all electric field have approximately the same magnitude. Therefore,

$$|\vec{E}| \int_{s_1} dA = |\vec{E}| A_{s_1} = \frac{q_{enc}}{\epsilon_0} \qquad \Longrightarrow |\vec{E}| = \frac{q_{enc}}{A_{s_1}\epsilon_0} = \frac{\sigma}{\epsilon_0} \tag{2.63}$$

where we used $\sigma = \frac{q_{enc}}{A_{s_1}}$. The direction of the electric field is perpendicular to the surface, since every single electric field line is pointing outwards to the surface of the conductor. So

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n} \tag{2.64}$$

Example 9: Electric Field of a Spherical Insulator

Positive charge Q is uniformly distributed over the entire volume of a spherical insulator of radius R. Find the magnitude of the electric field at any point a distance r from the center of the sphere.

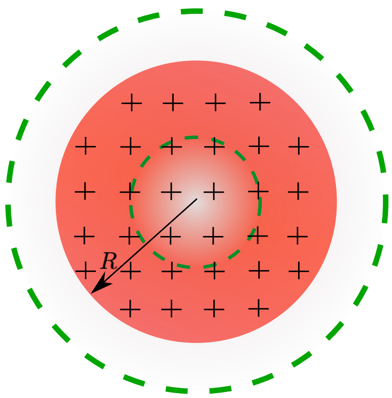


Figure 2.13

Solution:

We split the exercise in two cases: when r $\leq$ R and r $>$ R.
Let´s start with r $>$ R.

We use a spherical Gaussian Surface with radius r (the green sphere out the sphere shown in figure 2.13). Writing down Gauss Law we have

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \Longrightarrow \Phi = |\vec{E}| \int dA = |\vec{E}| A_{sphere} = \frac{q_{enc}}{\epsilon_0} \qquad \Longrightarrow |\vec{E}| = \frac{q_{enc}}{A_{sphere}\epsilon_0} \tag{2.65}$$

where we have used the fact that the electric field is the same at every point at a distance r from the center of the spherical insulator. So, just plugging in the area of a sphere we obtain

$$|\vec{E}| = \frac{q_{enc}}{4\pi r^2 \epsilon_0} = k\frac{Q}{r^2} \qquad r > R \tag{2.66}$$

where we used the fact that the enclosed charge is the total charge of the sphere ($q_{enc} = Q$).

Now for $r < R$, we will use once again a sphere that is inside the real spherical insulator. So, we have

$$\Phi = \oint E \cdot d\vec{A} = EA_{sphere} = \frac{q_{enc}}{\epsilon_0} \tag{2.67}$$

where we used the fact that the electric field is exactly the same (so constant) at every single point a distance r from the center of the spherical insulator. However, this time the enclosed charge is not all the electric charge of the sphere ($q_{enc} \neq Q$). This time the

Gaussian Surface is enclosing just some fraction of the total charge. In order to calculate the electric charge that is enclosed by the sphere of radius r we calculate

$$q_{enc} = \int_0^r \rho dV_{sphere} = \rho \int_0^r 4\pi r^2 dr = \rho \frac{4\pi}{3} r^3 \tag{2.68}$$

where ρ can get out of the integral because the electric charge is uniformly distributed, so the electric charge density is constant (every single little piece of differential volume contains exactly the same electric charge). Notice actually that in this very specific case the enclosed charge is just $q_{enc} = \rho \cdot V_{sphere}$. However, as we will see in next section this is not always the case. Also, we used the differential of volume of a sphere. Therefore, equation 2.67 becomes

$$E 4\pi r^2 = \frac{\rho 4\pi r^3}{3\epsilon_0} \qquad \Longrightarrow |\vec{E}| = \frac{\rho r}{3\epsilon_0} \tag{2.69}$$

If we do not know ρ we can use:

$$\rho \cdot Vol_{conductor} = \rho \frac{4}{3}\pi R^3 = Q \qquad \Longrightarrow \rho = \frac{3Q}{4\pi R^3} \tag{2.70}$$

because the electric charge is uniformly distributed.Just careful that this time the volume used is the volume of the insulator sphere, not the Gaussian Surface. Therefore, plugging in the charge density ρ in equation 2.69 we obtain

$$|\vec{E}| = \left(\frac{3Q}{4\pi R^3}\right) \frac{r}{3\epsilon_0} = k\frac{Q}{R^3} r \tag{2.71}$$

So, we have found that the electric field for a spherical insulator behaves as

$$|\vec{E}| = \begin{cases} k\frac{Q}{r^2} & \text{for } r > R \\ k\frac{Q}{R^3} r & \text{for } r \leq R \end{cases} \tag{2.72}$$

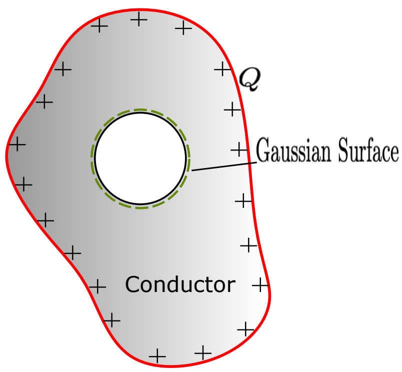
Notice that the electric field outside an spherical insulator behaves exactly as a point particle.

2.3.2 Charge Induction in Conductors

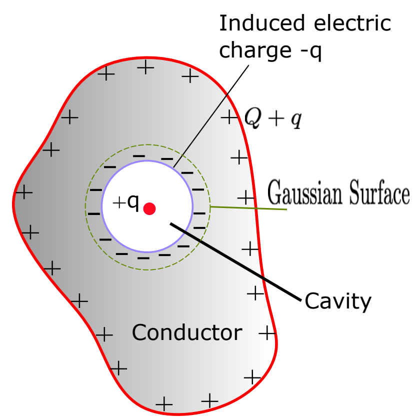
Four properties have been mentioned for conductors in electrostatic equilibrium. Let's mention one more property in this section. Suppose that a conductor has cavities. If the conductor has electric charge Q, could some of the electric charge Q reside in the wall of the cavity? Gauss Law, will give us the answer. Let's create a Gaussian surface that is extremely near to the surface of the wall of the cavity as shown in figure 2.14a. The Gaussian Surface is infinitesimally near to the surface of the cavity as shown in the figure 2.14a. So, writing down the possible electric flux through the Gaussian surface

$$\Phi = \oint \underbrace{\vec{E}}_{=0} \cdot d\vec{A} = 0 \tag{2.73}$$

must be zero, because the electric field must be zero since the Gaussian Surface is inside the conductor. The Gaussian surface is infinitesimally near to the surface of the cavity, but it is inside the conductor, so the electric field is zero. Therefore, by using Gauss law, the enclosed charge by the Gaussian surface is zero! So the answer is no! If a conductor with electric charge Q has empty cavities, there is no way that some of that electric charge can reside on the surface of the inner walls of the cavities. So the only place where there can be electric charge as we already know, is on the outer surface. However, what happens if there is an electric charge inside the cavity as shown in figure 2.14b? Now, we cannot say that the enclosed charge by the Gaussian surface is zero because we have explicitly placed the electric charge inside the cavity and the Gaussian surface.



(a)



(b)

Figure 2.14

However, the electric field inside the conductor must be zero! So what happens? We say that a charge is *induced* in the wall of the cavity so that the enclosed charge is zero and there is no contradiction in Gauss Law. If the electric charge that is enclosed is zero, then

$$q_{enc} = q + q_{wall} = 0 \Longrightarrow q_{wall} = -q \tag{2.74}$$

So, the induced charge is the same as the electric charge inside the cavity with opposite sign. However, that is not all what takes place. Think about this. The electric charge with opposite sign that now is on the surface of the inner cavity had to come from somewhere! Indeed, it came from the material of the conductor. Either positive or negative charges from the conductor had to move to the inner surface so that it became electrically charged. However, if the electric charge came from the material of the conductor, then its electric charge is not neutral anymore. Either it became positively charged or negatively charged depending on what kind of charges moved from the material to the inner surface. However, everything must remain on equilibrium! And having a net electric charge in the conductor does not guarantee us that. Therefore, what happens is that certain electric charge moves from inside the conductor to the outer surface. However, this electric charge will now be of the same sign of the electric charge as the one inside the cavity. So, at the end the material inside the conductor gets once again neutral, since the same amount of $-q$

electric charge that moved to the inner surface will move to the outer surface but with opposite sign $+q$. So, for any conductor we have the following property

If an isolated conductor has cavities, then it cannot have electric charge in the inner surface of the cavities. However, if a conductor has electric charges inside a cavity, then the wall or surface of the inner cavities obtain electric charge of the same magnitude but opposite sign of the charges inside the cavity; while the outside surface obtain extra charge of the same sign of the charges inside the cavity

Example 10: Sphere inside an spherical shell (charge Induction)

A solid insulator sphere of radius r_a has a total (net) positive charge Q uniformly distributed throughout its volume. A spherical conductive shell, with inner radius r_b and outer radius r_c, is concentric with the solid sphere and has a net charge of $-5Q$. Another conductor sphere is concentric to the two mentioned objects, with total charge $9Q$. With the application of Gauss law, find the electric field in the regions I,II, III,IV,V and VI in the figure and the electric charge in the walls of the spherical conductive shells. The whole system is in electrostatic equilibrium.

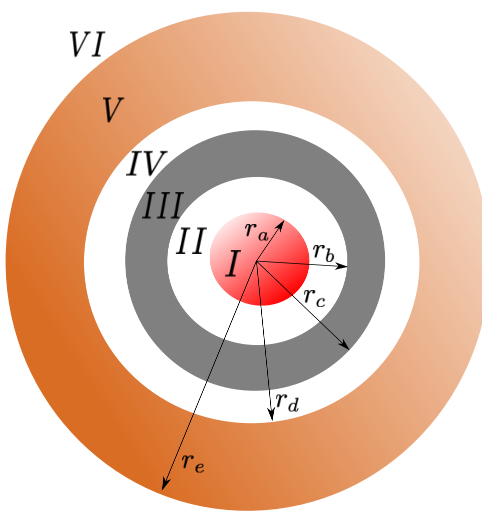


Figure 2.15

Solution:

We first analyze the behavior of the electric field.
In the region I we are inside an spherical insulator, therefore we use the result of the exercise 9, equation 2.71. So,

$$\vec{E} = k\frac{Qr}{R^3}\hat{r} \qquad \text{for } r < r_a \tag{2.75}$$

For the region II, we can use once again the result of the previous exercise for a spherical insulator but when we calculate the electric field outside of it (equation 2.66)

$$\vec{E} = k\frac{Q}{r^2}\hat{r} \qquad \text{for } r_a \leq r \leq r_b \tag{2.76}$$

For the region III, since we are inside a conductor, we know that the electric field must be zero.

$$|\vec{E}| = 0\frac{N}{C}m \qquad \text{for } r_b < r < r_c \tag{2.77}$$

For region IV, we shall use Gauss law. We make a Gaussian surface surrounding the shell of region III . So,

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} = -\frac{4Q}{\epsilon_0} \tag{2.78}$$

where we have that the enclosed charge is

$$q_{enc} = Q - 5Q = -4Q \tag{2.79}$$

Since everything is spherical symmetric we have that the electric field will be constant for all $d\vec{A}$. Given that the enclosed electric charge is negative, then the electric flux will be negative, so every electric field line will have exactly the opposite direction of each differential of area. Therefore $\vec{E} \cdot d\vec{A} = |\vec{E}d\vec{A}\cos\theta_{EdA} = |\vec{E}d\vec{A}\cos(-180°) = -|\vec{E}|d\vec{A}$. Hence, equation 2.78 becomes

$$-|\vec{E}|A_{sphere} = \frac{-4Q}{\epsilon_0} \Longrightarrow |\vec{E}| = \frac{4Q}{A_{sphere}\epsilon_0} = \frac{4Q}{4\pi\epsilon_0 r^2} = k\frac{4Q}{r^2} \tag{2.80}$$

So,

$$\vec{E} = -k\frac{4Q}{r^2}\hat{r} \qquad \text{for } r_c \leq r \leq r_d \tag{2.81}$$

For the region V, since we are inside a conductor, we know that the electric field must be zero.

$$|\vec{E}| = 0\frac{N}{C}m \qquad \text{for } r_d < r < r_e \tag{2.82}$$

For the region VI, we shall use once again Gauss law. We make a Gaussian surface surrounding all objects. For this final case the enclosed charge is

$$q_{enc} = Q - 5Q + 9Q = 5Q \tag{2.83}$$

The electric flux will be positive, because the total electric charge enclosed is positive, therefore the field lines will be outwards. Due to the spherical symmetry, we will have $\oint \vec{E} \cdot d\vec{A} = \vec{E}|A_{sphere}$. Therefore, we have

$$|E|A_{sphere} = \frac{5Q}{\epsilon_0} \Longrightarrow |\vec{E}| = \frac{5Q}{A_{sphere}\epsilon_0} = \frac{5Q}{4\pi\epsilon_0 r^2} = k\frac{5Q}{r^2} \tag{2.84}$$

Therefore,

$$\vec{E} = k\frac{5Q}{r^2}\hat{r} \qquad \text{for } r > r_e \tag{2.85}$$

Now, for the electric charges at the walls we must use what we have learned about charge induction.

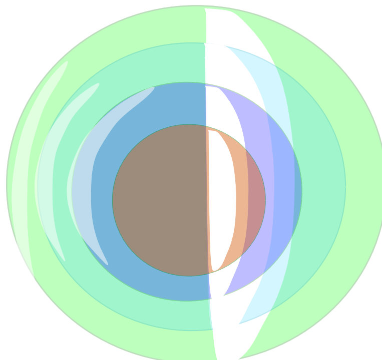
For $r = r_a$ we actually do not know the electric charge. We know that in the whole volume of the sphere the charge is Q, but exactly at the surface we do not know. The red sphere is an insulator, so the electric charge shouldn't be exactly at the surface when $r = r_a$.
For $r = r_b$ by charge induction, the electric charge has to be the same magnitude but opposite sign of the charge inside the cavity, so the electric charge is $-Q$. For $r = r_c$, the new electric charge is the total electric charge of the conductor plus an induced electric charge. This induced electric charge has the same magnitude and sign as the electric charge inside the cavity. So the electric charge is $-5Q + Q = -4Q$

For $r = r_d$ by charge induction, the electric charge has to be the same magnitude but opposite sign of the charge of everything that is inside the cavity of the shell with inner radius r_d, so the electric charge is $4Q$. For $r = r_e$, in analogy to what happened with the first conductor, the electric charge is $9Q - 4Q = 5Q$

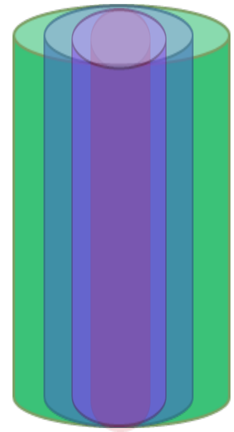
2.4 Symmetrical Non-Uniform Charge Distributions

So far we have assumed that electric charge is uniformly distributed in the objects that we have analyzed. However, this could be highly idealized. In many situations, we could probably wait enough to let the charges to distribute uniformly all along the object. However, in many real cases charges will tend to be more likely in certain regions of the object. In cases where the distribution of charge is not uniform nor symmetrical, the integral $\oint \vec{E} \cdot d\vec{A}$ in Gauss Law could get highly difficult to solve and unpractical analytically to use this approach. However, even though the distribution is not uniform, if there is symmetry we can use easily Gauss Law. We mention two particular cases, spherical symmetry and cylindrical symmetry.

If we place the object to analyze at the origin, whenever its charge density depends only on the distance from the origin, we say is spherically symmetric, i.e. any function of charge density only dependant of r as $\rho(r)$ is spherically symmetric. We say so, because suppose you take a vector with certain length $r = r_0$. If such vector direction changes with any angle in the direction of the unitary vectors $\hat{\theta}$ or $\hat{\phi}$, the charge with such distance $r = r_0$ is exactly the same. So, visually we can think of these kind of charge distributions as spherical shells, where every spherical shell has certain different electric charge (this is shown in figure 2.16a). In any spherical symmetric charge distribution, the electric field will be radially directed because the electric charge and field are invariant under rotation. So, is not surprise that for these cases we use as our Gaussian surface an sphere.



(a) Spherical symmetric charge distribution. Every spherical shell layer has different electric charge.



(b) Cylindrical symmetric charge distribution. Every cylindrical layer has different electric charge.

Figure 2.16

Now, if the object to analyze has certain axis to which the charge density only depends on the radial distance from the axis to any point, then we say it has cylindrical symmetry. For such charge distributions we can think as the object made of cylindrical layers with different charge each of them (this is shown in figure 2.16b). As expected, for such distributions, we will use a Gaussian surface in the shape of a cylinder.

Example 11: Electric Field of a charge distribution with cylindrical symmetry

An insulator cylinder of infinite length and radius R has volume charge density as

$$\rho(r) = \begin{cases} \rho_0 \left(a - \frac{r}{R}\right) & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases} \tag{2.86}$$

where $\rho_0 = 1 \cdot 10^{-9} \frac{C}{m^3}$, $a = 2$ and radius R.

- Find the electric field as function of r inside the cylinder
- Find the electric field as function of r outside the cylinder.

Solution:

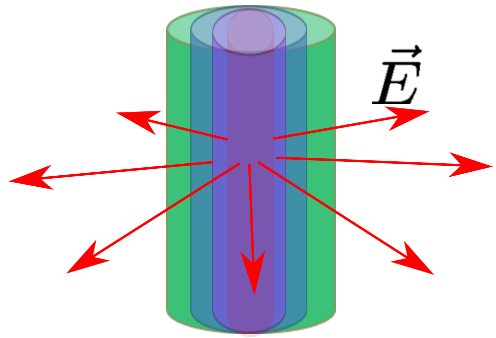


Figure 2.17

We proceed to calculate Gauss Law, by using a cylinder as our Gaussian surface. The cylinder will have certain length l and radius r. So, given that we have cylindrical symmetry, $\vec{E}$ and $d\vec{A}$ are always parallel. Therefore, we have that

$$\oint \vec{E} \cdot d\vec{A} = |\vec{E}|A_{cylinder} = |\vec{E}|2\pi rl = \frac{q_{enc}}{\epsilon_0} \tag{2.87}$$

where we just used the side area of the cylinder, given that there are no electric field lines crossing the top and bottom surfaces of the cylinder. So, the equality

$$|\vec{E}|2\pi rl = \frac{q_{enc}}{\epsilon_0} \tag{2.88}$$

holds no matter if we take points inside or outside the cylinder ($r < R$ or $r > R$). We have just used the fact that the electric field lines have cylindrical symmetry. We are concerned now about how much electric charge we are enclosing. If we are inside the cylinder ($r < R$), then we have that

$$q_{enc} = \int_0^r \rho dV = \int_0^r \rho_0 \left(a - \frac{r}{R}\right)(2\pi rldr) = \rho_0 2\pi l \int_0^r \left(ar - \frac{r^2}{R}\right) dr \tag{2.89}$$

where we used the differential of volume of a cylinder $dV = 2\pi rldr$. So, evaluating the integral we have that

$$q_{enc} = \rho_0 2\pi l \left[\frac{ar^2}{2} - \frac{r^3}{3R}\right]_0^r = \rho_0 2\pi l \left[r^2 - \frac{r^3}{3R}\right] \tag{2.90}$$

Hence, substituting the enclosed charge by our Gaussian cylinder of radius $r < R$ in equation 2.88, we obtain

$$|\vec{E}| = \frac{\rho_0 2\pi l}{2\pi rl\epsilon_0}\left[r^2 - \frac{r^3}{3R}\right] = \frac{\rho_0}{\epsilon_0}\left[r - \frac{r^2}{3R}\right] \tag{2.91}$$

Now, if our Gaussian cylinder radius is $r \geq R$, we have enclosed all the electric charge in the cylinder. Hence,

$$Q_{tot} = \rho_0 2\pi lR^2 \left[1 - \frac{1}{3}\right] = \frac{4}{3}\rho\pi lR^2 \tag{2.92}$$

where we used the radius of the cylinder in equation 2.90. So, for either outside the cylinder or at the radius of the cylinder, we have enclosed all possible electric charge of the cylinder. Hence, using the total charge of the cylinder, we have that equation 2.88 becomes

$$|\vec{E}| = \frac{2\rho_0 R^2}{3\epsilon_0 r} \tag{2.93}$$

So, summarizing we have obtained

$$|\vec{E}| = \begin{cases} \frac{\rho_0}{\epsilon_0}\left[r - \frac{r^2}{3R}\right] & \text{for } r < R \\ \frac{2\rho_0 R^2}{3\epsilon_0 r} & \text{for } r \geq R \end{cases} \tag{2.94}$$

Example 12: Electric Field of a charge distribution with spherical symmetry

A non-uniform distribution charge, is spherically symetric. Its charge distribution is given by

$$\rho(r) = \begin{cases} \rho_0(1 - 4r/3R) & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases} \tag{2.95}$$

where ρ_0 is a positive constant. Find the electric field produced by the spherical symmetric charge distribution when $r \leq R$ (inside the charge distribution) and when $r > R$ (outside the charge distribution) . ***Solution:***

We will use Gauss Law, and take advantage of the spherical symmetry. So, we use as Gaussian surface an sphere of radius r

$$\int \vec{E} \cdot d\vec{A} = |\vec{E}| A_{sphere} = |\vec{E}|\left(4\pi r^2\right) = \frac{q_{enc}}{\epsilon_0} \tag{2.96}$$

where the last equality will hold not matter if the Gaussian surface is inside or outside of the charge distribution. Now, if $r < R$, when the Gaussian sphere is inside of the spherical charge distribution we have that the enclosed charge is

$$q_{enc} = \int_0^r \rho dV = \int_0^r \rho_0 \left(1 - \frac{4r}{3R}\right)[4\pi r^2 dr] = \rho_0 4\pi \int_0^r \left(r^2 - \frac{4r^3}{3R}\right) dr = \rho_0 4\pi \left(\frac{r^3}{3} - \frac{r^4}{3R}\right) \tag{2.97}$$

where we used a differential volume of a sphere $dV = 4\pi r^2 dr$. Therefore, inside the spherical charge distribution, the electric field is

$$|\vec{E}| = \frac{\rho_0}{\epsilon_0}\left(\frac{r}{3} - \frac{r^2}{3R}\right) \tag{2.98}$$

where we substituted the enclosed charge (equation 2.97) in equation 2.96. Now, notice that when we enclose all the possible electric charge by letting our Gaussian sphere be of radius $r = R$, we have that

$$q_{enc} = 0\ C \tag{2.99}$$

by substituting $r = R$ in equation 2.97. So, in total the sphere is neutrally charged! And this shouldn't surprise, this can happen in any real situation. For example, if you enclose certain volume of your body with a Gaussian surface, for sure there is probably certain electric charge in there. We are made of electrons and protons of our atoms, so probably in certain times some electric charges are in certain regions of our body. However, if now you use a Gaussian sphere that encloses your whole body, now you are neutrally charged! This happens with the sphere we are analyzing, if we enclose certain volume inside the sphere, there is electric charge and therefore electric field, however once we are at the surface of the sphere and outside of it no electric field is produced because all electric charge is neutralized due to the layers of different electric charge of the sphere.

$$|\vec{E}| = 0\ C \tag{2.100}$$

Therefore, we have that the electric field generated by the spherical charge distribution is

$$|\vec{E}| = \begin{cases} \frac{\rho_0}{\epsilon_0}\left(\frac{r}{3} - \frac{r^2}{3R}\right) & \text{for } r < R \\ 0 & \text{for } r \geq R \end{cases} \tag{2.101}$$

Example 13: Hydrogen Atom

From high school (probably before) we were taught that the hydrogen atom is composed by a proton and an electron. And many times it is taught that hydrogen atom looks like in figure 2.18a, an electron that is cycling around the proton. However, this picture is not right at all. Something much more interesting takes place. Quantum mechanics tells us that at very tiny scales, we can not tell with certainty what is the position and velocity of an electron . We can only know with certain probability where the electron is. The ground state of the hydrogen atom is the lowest energy state of the hydrogen atom. In such case, quantum mechanics states that the probability density (the probability to find the electron in an spherical shell between a distance r and $r + dr$) is given by

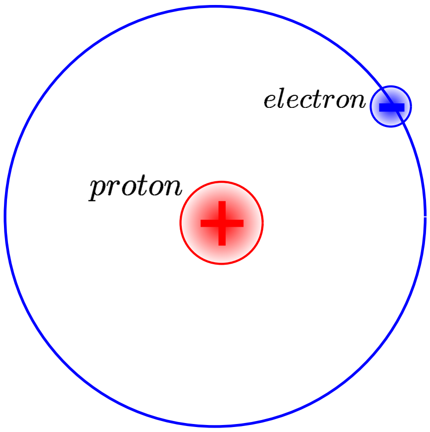
$$|\Psi(r)|^2 = \frac{1}{\pi a_0^3} e^{-r/a_0} \tag{2.102}$$

So, the electron probability density rises a charge density. We can think as a cloud around the proton of certain probability of finding the electron there as shown in figure 2.18b. So, this cloud gives a spherical symmetric non uniform charge density distribution

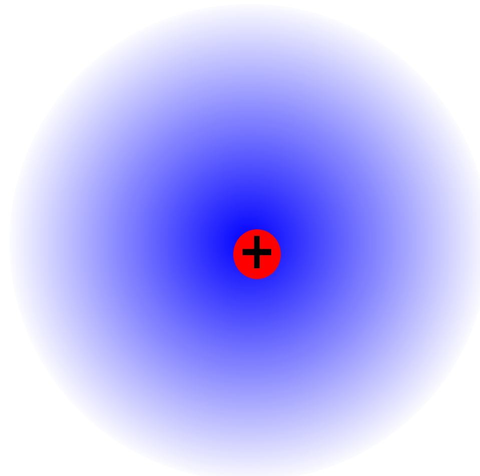
$$\rho(r) = -\frac{Q}{\pi a_0^3} e^{-2r/a_0} \tag{2.103}$$

where $-Q$ is the charge of the electron.

- Find the enclosed charge by an sphere of radius r centered at the origin
- Find the electric field generated by the hydrogen atom in its ground state



(a) Classical picture of an hydrogen atom. This picture is far from being right



(b) Modern picture of the ground state of the hydrogen atom. There is a cloud of probability around the proton of finding the electron. Such cloud rises an spherical non uniform charge distribution.

Figure 2.18

$$q_{enc} = \int_0^r \rho(r)dV + Q \tag{2.104}$$

where Q is the charge of the proton. So, focusing first in the integral in the last equation

$$\int_0^r \rho(r)dV = -\int_0^r \frac{Q}{\pi a_0^3} e^{-2r/a_0}\left(4\pi r^2 dr\right) \tag{2.105}$$

where we have included the minus sign of the electron, and used the fact that the differential of volume for a sphere is $4\pi r^2 dr$. Therefore we want to solve the following integral

$$\int_0^r \rho(r)dV = -\int_0^r \frac{Q}{a_0^3} e^{-2r/a_0} 4r^2 dr \tag{2.106}$$

By making the use of the following variable

$$u = \frac{2r}{a_0} \quad \text{and} \quad du = \frac{2dr}{a_0} \quad \text{, we would have that} \quad \frac{u^2 du}{2} = \frac{4r^2 dr}{a_0^3} \tag{2.107}$$

so, we can express the integral as

$$-\frac{Q}{2}\int_0^{2r/a_0} u^2 e^{-u} du \tag{2.108}$$

We proceed to integrate by parts, so we obtain

$$-\frac{Q}{2}\int_0^{2r/a_0} u^2 e^{-u} du = -\frac{Q}{2}\left[-u^2 e^{-u}\Big|_0^{2r/a_0} + 2\int_0^{2r/a_0} e^{-u} u du\right] \tag{2.109}$$

The integral in the last term, is solved by integrating by parts once again

$$\int_0^{2r/a_0} e^{-u} u du = -ue^{-u}\Big|_0^{2r/a_0} - \int_0^{2r/a_0} e^{-u} du = -ue^{-u}\Big|_0^{2r/a_0} - e^{-u}\Big|_0^{2r/a_0} \tag{2.110}$$

Therefore, equation 2.108 becomes

$$-\frac{Q}{2}\int_0^{2r/a_0} u^2 e^{-u} du = -\frac{Q}{2}\Big[-u^2e^{-u} + 2\Big(-ue^{-u} - e^{-u}\Big)\Big]_0^{2r/a_0} \tag{2.111}$$

Evaluating the limits of integration, we are left with

$$-\frac{Q}{2}\int_0^{2r/a_0} u^2 e^{-u} du = \frac{Q}{2}e^{-2r/a_0}\Big((2r/a_0)^2 + 2(2r/a_0) + 2\Big) - Q \tag{2.112}$$

So, finally plugging the last equation in 2.104, we have that the total enclosed charge is

$$q_{enc} = \frac{Q}{2}e^{-2r/a_0}\Big((2r/a_0)^2 + 2(2r/a_0) + 2\Big) \tag{2.113}$$

Therefore, we have an enclosed charge different to zero as expected from the classical point of view. Notice also, that as we should expect, if $r \to 0$, we are left with just the proton electric charge. Finally to obtain the electric field produced by the hydrogen atom we use Gauss Law.

As mentioned before, we use an sphere to enclose both charges. As, previously, when we have a spherical symmetrical distribution of electric charge, by using an sphere as our Gaussian surface, we will obtain from the left hand side of Gauss Law $|\vec{E}|A_{sphere} = |\vec{E}|4\pi r^2$. Therefore,

$$|\vec{E}|4\pi r^2 = \frac{Q}{2\epsilon_0}e^{-2r/a_0}\Big((2r/a_0)^2 + 2(2r/a_0) + 2\Big)$$

So, the magnitude of the electric field generated by an hydrogen atom is given by

$$|\vec{E}| = k\frac{Q}{2r^2}e^{-2r/a_0}\Big((2r/a_0)^2 + 2(2r/a_0) + 2\Big) \tag{2.114}$$

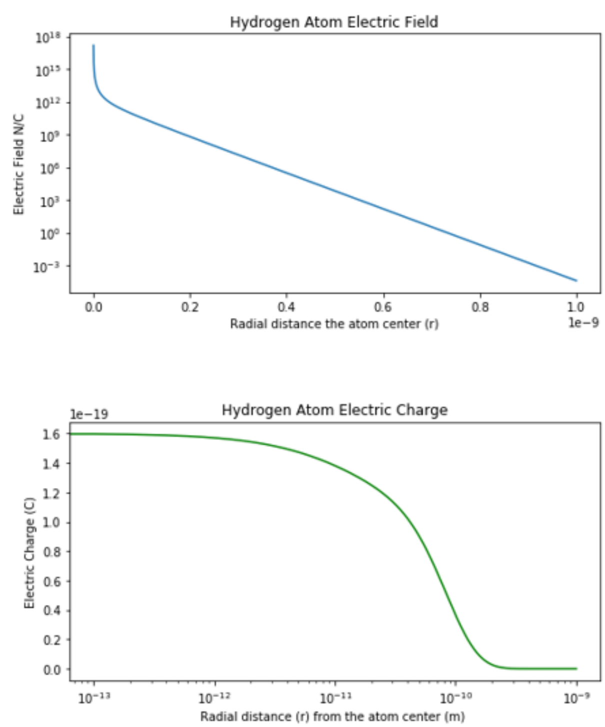


Figure 2.19

Chapter 3

Electric Potential

So far we have discussed about electric fields and how they are the responsible for electric force. However, we have not mentioned a crucial quantity in physics, *energy*. How is the energy related to the electric fields? Is there a *potential energy* associated to an electric field? Indeed, there is. In this chapter we will discuss about the potential energy associated to an electric field, and the electric potential which is highly related to energy. Also, we will mention a new variable, called *voltage* which is much more common to hear about it in industry and daily applications as house electrical devices. As it will turn out, dealing with electric potential is much easier than electric fields. The reason is simple, the electric potential is a scalar, so we do not have to worry about the components of the electric field.

3.1 Electric Potential Difference(Voltage)

First of all, we should ask ourselves, when a charged particle moves from a point A in space to a point B. What is making the *work* so that the charge moves? We could see the analogous to the gravitational case, when we let an apple fall, what makes it move is the *gravitational field* and it is the one exerting a force on the apple and doing a *work* to move it. So similarly, the electric field does the work when a charged particle moves from a point A to a point B. Now, suppose an electric field moves a particle and we calculate the work done by the electric field, so we will have the following integral

$$W = \int_A^B \vec{F} \cdot d\vec{l} \tag{3.1}$$

where $d\vec{l}$ is a differential vector that is tangent to every single point in the path that the charges follows when moves. As it turns out,it can be shown that the electric force is a *conservative force.* And, given that the electric force is *conservative*, it can be written as

$$\vec{F} = -q\nabla V \tag{3.2}$$

and

$$\vec{E} = -\nabla V \tag{3.3}$$

Therefore, we can write the work done by the electric field as

$$W = \int_A^B \vec{F} \cdot d\vec{l} = q \int_A^B \vec{E} \cdot d\vec{l} = -q \int_A^B \nabla V \cdot d\vec{r} = -q(V_B - V_A) = -q\Delta V \qquad (3.4)$$

The function V is called electric potential and it is *scalar* with units $\frac{J}{C}$.However, the electric potential by itself is not a physical quantity, it is the difference of electric potential that is physical. Therefore,

$$\boxed{\Delta V = -\frac{W}{q}} \qquad (3.5)$$

This last quantity ΔV is called *voltage* and its units are $V \equiv \frac{J}{C}$ called *volts*. Why is it that the electric potential is not physical? Because it depends on the reference frame we have chosen to make our measurements. However, the *potential difference* is physical eliminating the redundancy of the reference frame we have chosen. We will discuss this progressively, and at the end of the chapter hopefully it will be much clearer why. Now, recalling that the work done by a member of the system (our system now is the electric field and the particle which is moving) we are analyzing.

$$W_{int} = -\Delta U \qquad (3.6)$$

where ΔU is the change in the potential energy; so we can combine the last equation 3.6 with 3.5 to obtain

$$\boxed{\Delta U = q\Delta V} \qquad (3.7)$$

The last equation tells us something important about the potential energy. Whenever there is drastic change of potential, or there is a huge voltage($\Delta V \gg 0$), there will be a drastic change in potential energy. Recall from mechanics, that whenever we have conservative forces the conservation of energy is written as

$$\Delta E = \Delta U + \Delta K = 0 \qquad (3.8)$$

Therefore,

$$\Delta K = -\Delta U \qquad (3.9)$$

so if there is a drastic change in the potential energy given a big voltage ($\Delta V \gg 0$), charges will obtain kinetic energy. Therefore, they will start flow rapidly . That is the reason why high voltages are dangerous.

Now, we could ask ourselves, does the potential difference or the voltage depends on the particle that moves? Well, the answer is no! From equation 3.4 if we equal the third term with the last one we have

$$q \int_B^A \vec{E} \cdot d\vec{l} = -q\Delta V \Longrightarrow \Delta V = -\int \vec{E} \cdot d\vec{l} \qquad (3.10)$$

Look how actually the last equation is independent of the electric charge that is moving. The electric field in equation 3.10 is the one produced by the charge that influences the one that is moving. Probably you think, but I see in equation 3.7 a direct relation of the potential difference (voltage) with the electric charge that moves, what is going on? As we will see, ΔU is dependent of the electric charge that moves, so at the end they cancel out. Finally, if the electric field is constant we are left with

$$\int_0^d \vec{E}\cdot d\vec{l} = \vec{E}\cdot\left(\int_0^d d\vec{l}\right) \tag{3.11}$$

where we asssumed that the length of the end points of our interest to measure the voltage starts from zero up top to a certain distance d. Therefore, for constant electric fields we have

$$\boxed{\Delta V = -\vec{E}\cdot\vec{d}} \tag{3.12}$$

Example 1: Electric Potential Difference of Point Charges

Determine the electric potential difference in arbitrary points in space r_i and r_f generated by a point electric charge Q.

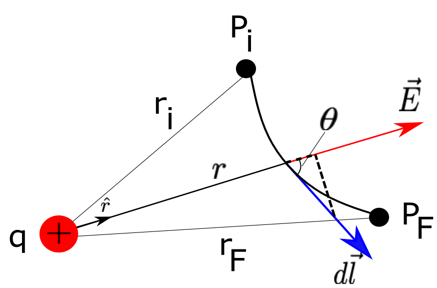


Figure 3.1

Solution:

Let's take an arbitrary electric charge Q and place it as shown in the figure. So by using equation 3.10, we have

$$\Delta V = -\int \vec{E}\cdot d\vec{l} = -\int \frac{kQ}{r^2}\hat{r}\cdot d\vec{l} = -\int \frac{kQ}{r^2} dl\cos\theta \tag{3.13}$$

where we substituted the electric field produced by a point electric charge and used $|\hat{r}| = 1$ because it is a unitary vector. Now, see (see figure) and notice that $\cos\theta = \frac{dr}{dl} \implies dr = dl\cos\theta$. Therefore, incluiding the limits of integration in the last equation and solving the integral we have

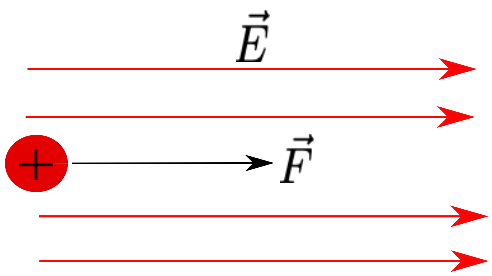
$$\Delta V = -\int_{r_i}^{r_f} \frac{kQ}{r^2} dr = kQ\frac{1}{r}\bigg|_{r_i}^{r_f} = kQ\left(\frac{1}{r_f} - \frac{1}{r_i}\right) \tag{3.14}$$

So this result holds for any punctual electric charge Q. As it will be used quite often , we explicitly show the result.

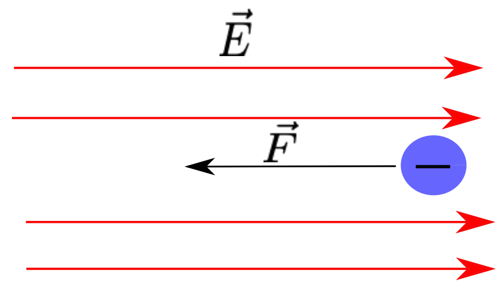
$$\boxed{\Delta V = kQ\left(\frac{1}{r_f} - \frac{1}{r_i}\right)} \tag{3.15}$$

Example 2: Meaning of the sign of the voltage

A proton is released from rest in uniform electric field, as result the proton starts to move. a) Does the electric potential increase or decrease towards the points where the proton moves? b) Does the potential energy of the proton increase or decrease? c) What would be the answers in a) and b) if it is an electron instead?



(a)



(b)

Figure 3.2

Solution:

We will use equation3.12 to determine whether the potential difference increases or decreases when a proton moves in a electric field. Therefore, we need to determine first of all the direction of the vector $\vec{d}$, which in principle is always arbitrary, you can select two different points and determine such as your $\vec{d}$ vector. However, we will use this vector $\vec{d}$ as the vector of the displacement of the proton. So, to dertmine the direction of the vector $\vec{d}$ let's start with the electric force equation

$$\vec{F} = q\vec{E} \tag{3.16}$$

From this equation we can obtain first what would it be the direction of the proton moving. Since the electric charge of the proton is positive, then the force and electric field have the same direction. Therefore, the proton moves to the same direction of the electric field. Therefore, the distance vector $\vec{d}$ will be pointing to the same direction as the electric field.

Using equation 3.12 for a constant electric field,

$$\Delta V = -\vec{E} \cdot \vec{d} = -|\vec{E}||\vec{d}| \cos 0 = -|\vec{E}||\vec{d}| \tag{3.17}$$

where the angle $\theta = 0$ since the electric field and the displacement vector $\vec{d}$ have exactly the same direction . Therefore,

$$\Delta V < 0 \tag{3.18}$$

Hence, the electric potential decreases towards the direction of the proton movement. Now, by using equation 3.7, we have that

$$\Delta U = q\Delta V < 0 \tag{3.19}$$

the potential energy decreases as expected because the kinetic energy increases.

For the case when we have an electron, we have that

$$\vec{F} = -|q_e|\vec{E} \tag{3.20}$$

where we make explicit that the electric charge of the electron is negative. So, the electron moves opposite to the direction of the electric field. Therefore,

$$\Delta V = \vec{E} \cdot \vec{d} = -|\vec{E}||\vec{d}| \cos 180 = |\vec{E}||\vec{d}| \tag{3.21}$$

Hence,

$$\Delta V > 0 \tag{3.22}$$

Therefore, the electric potential increases towards to the direction of the movement of the electron. Now, for the potential energy we must obtain that it decreases also, since the electron is moving and it should lose potential energy and gain kinetic energy. So, lets see if this holds. Starting once again with equation 3.12 , we have

$$\Delta U = -|q_e|\Delta V < 0 \tag{3.23}$$

Since,$\Delta V > 0$ So the potential energy decreases as it should!

As you can notice, we actually never needed to use the exact values of the electric charge of the proton neither of the electron. It was more a quantitave analysis. So, in general, we can say the following

Negative electric charges will move to the direction where the voltage is positive (electric potential increases), and the positive charges will move towards the direction where the voltage is negative (electric potential decreases). Also, the sign of ΔV, tells us in which direction the electric field points.

Example 3: Proton moving in a constant electric field

Two metal plates are placed front one to each other 0.1cm distance. The plates generate an electric field of magnitude $|\vec{E}| = 1\frac{\text{N}}{\text{C}}$ as shown in the figure. If a proton is released from rest from the positive plate. (proton mass 1.6×10^{-27}kg). What would be the speed of the when it hits the negative plate?

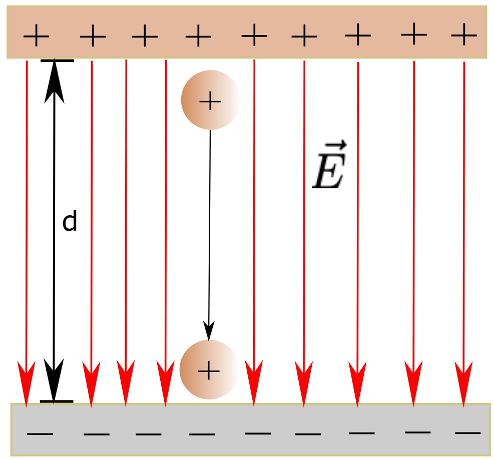


Figure 3.3

Solution:

Since we want to know the speed, somehow we need to relate it with voltage. From equation 3.7 we know that there is a relation between voltage and potential energy.So , we start with the equation of conservation of energy when there are only conservative forces,

$$\Delta U + \Delta K = 0 \Rightarrow \Delta U = -\Delta K \tag{3.24}$$

Using equation 3.7

$$q\Delta V = -\Delta K \tag{3.25}$$

Substituting the potential difference for a constant electric field (equation 3.12) and the definition of kinetic energy difference

$$-q\vec{E} \cdot \vec{d} = -(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2) \tag{3.26}$$

where v_i stands for initial speed and v_f for final speed (do not confuse with electric potential). Now, since the exercise explicitly mentions that the proton is released from rest then $v_i = 0$. Therefore, by isolating v_f in the last equation we have

$$v_f = \sqrt{\frac{2q|\vec{E}|d}{m}} \tag{3.27}$$

where we used the fact that the dot product will be positive, since we know that the proton moves to the same direction as the electric field so the angle between $\vec{E}$ and $\vec{d}$ is 0. Substituing values we have

$$v_f = \sqrt{\frac{2(1.6 \cdot 10^{-19}C)(1N/C)(0.1 \cdot 10^{-2}m)}{1.67 \cdot 10^{-27}kg}} = 437.74\frac{\text{m}}{\text{s}} \tag{3.28}$$

Example 4: Calculating speed of 3 charges in a triangular configuration

Three equal point charges of 1.2μC and 3×10^{-8}kg of mass are placed in the vertices of an equilateral triangle of length 0.50m. If the charges are released from rest, What is the speed of the charges after the triangle has duplicated each side length?

Solution:

From the figure, the electric charges in the middle triangle are in the initial configuration, while in the triangle of sides r_f are at the final configuration after they have moved. Once again, as in the previous exercise we will use conservation of energy to solve this problem. In this case, we have to consider the total energy of the system. So we have to compute

$$\Delta U_T = \Delta U_{12} + \Delta U_{13} + \Delta U_{23} \tag{3.29}$$

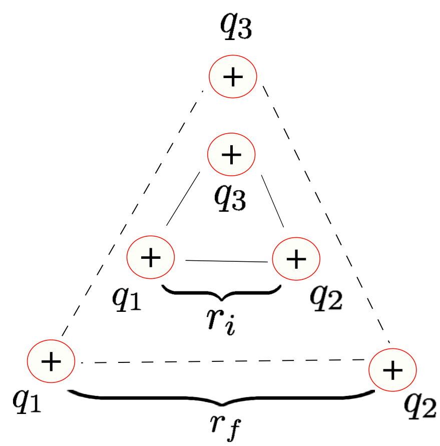


Figure 3.4

where we are calculating the difference of potential energy between the pairs. However, since they are in a equilateral triangle, they will be separeted the same distance between them. Also, $q_1 = q_2 = q_3$. Therefore,

$$\Delta U_T = 3\Delta U_{12} = 3\Delta U_{13} = 3\Delta U_{23} \tag{3.30}$$

So, calling $q = q_1 = q_2 = q_3$. We have

$$\Delta U_T = 3q\Delta V = 3q\left(kq\left[\frac{1}{r_f} - \frac{1}{r_i}\right]\right) = 3kq^2\left[\frac{1}{r_f} - \frac{1}{r_i}\right] \tag{3.31}$$

where we just substituted equation 3.15, the electric potential difference generated by a punctual electric charge. So, now using conservation of energy

$$\Delta U_T + \Delta K_T = 0 \Longrightarrow 3kq^2\left[\frac{1}{r_f} - \frac{1}{r_i}\right] + \frac{3}{2}mv_f^2 = 0 \tag{3.32}$$

where we did not include $\frac{3}{2}v_i^2$ because the charges start from rest. Notice, how the 3 included in the potential energy difference and the 3 in the kinetic energy will be cancelled out. This is due to the symmetry of the exercise, but this is not necessarily true always. For instance, particles could have started in any random configuration and with aleatory initial velocities. Finally, isolating v_f, we obain

$$v_f = \sqrt{\frac{-2kq^2\left[\frac{1}{r_f} - \frac{1}{r_i}\right]}{m}} \tag{3.33}$$

Now, since the triangle duplicated its sides length, we have that

$$r_f = 2r_i \qquad \Rightarrow \frac{1}{r_f} - \frac{1}{r_i} = \frac{1}{2r_i} - \frac{1}{r_i} = -\frac{1}{2r_i} \tag{3.34}$$

Therefore,

$$v_f = \sqrt{\frac{kq^2}{mr_i}} \tag{3.35}$$

Finally, plugging in the corresponding values we obtain

$$v_f = \sqrt{\frac{(9\cdot 10^9 \frac{Nm^2}{C^2})(1.20\cdot 10^{-6}C)^2}{(3\cdot 10^{-8}kg)(0.5m)}} = 929.51 \quad m/s \tag{3.36}$$

3.2 Electric Potential

We have defined the difference of potential, and we have called it as voltage. Now, it is convenient to define a reference frame to calculate only the *potential*. In electrostatics, we will define a reference potential such that at infinity it is zero

$$V_\infty = 0 \tag{3.37}$$

Therefore, we can calculate the potential of one electric charge, by using equation 3.15

$$\Delta V = V - V_\infty = k\frac{q}{r} \tag{3.38}$$

where $\frac{1}{r_i}$ was depreciated since $1/r \to 0$ when $r \to \infty$. So,

$$V = k\frac{q}{r} \tag{3.39}$$

where r is the distance from the electric charge to the point where we are calculating the potential. Now, I want to emphasize that the potential by itself is not something physical. It is the **potential difference** that is measurable. However, even though that it is measurable the difference of the potential, it is useful to have a definition of potential by itself. Let me make the analogy with potential energy in mechanics. When you calculate the potential energy of a ball that is about to fall, you use certain reference frame. For example, you probably say that at the *floor* the potential energy is zero. However, probably a guy at the second floor in a building says that the potential energy is zero at the second floor. So who is correct or incorrect? Well, no one is incorrect. The thing is that to start to define the potential energy the guy in the building and you used certain reference frame. However, when you calculate the ***change or difference of potential energy*** you could now use energy conservation law and deduce some physics as the velocity at which the ball hits the first floor. So, something similar is with the potential, we can use a reference frame to establish the electric potential, and then do some physics when we calculate the potential difference. Now, is important to remember to include the electric charge sign in equation 3.39. Finally, if we have N particles and we are interested to obtain the potential at certain point in space, we just need to sum the contribution of all particles

$$V = \sum_{i=1}^{N} k\frac{q_i}{r_i} \tag{3.40}$$

Example 5: Electric potential simplest case calculation

Two point electric charges $q_1 = 5\mu C$ and $q_2 = -3\mu C$ are placed in the xy plane at $(0,0)$ and $(0,4)$ respectively. Find the electric potential at the point $P(-1,3)$m .
Solution:
We need to calculate

$$V = k\frac{q_1}{r_1} + k\frac{q_2}{r_2} \tag{3.41}$$

From the figure we can easily construct traingles to find distances r_1 and r_2 . which are the distances from the electric charge q_1 and q_2 to point P respectively. So,

$$r_2^2 = 1^2 + 1^2 \quad \Rightarrow r_2 = \sqrt{2}\text{m} \tag{3.42}$$

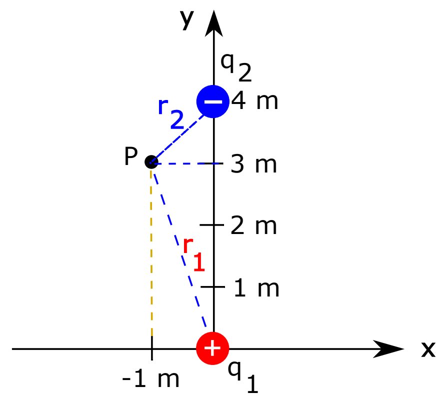


Figure 3.5

while

$$r_1 = \sqrt{1^2 + 3^2} = \sqrt{10}\text{m} \tag{3.43}$$

Therefore, plugging values in equation 3.41

$$V = 9 \cdot 10^9 \frac{Nm^2}{C^2}\left(\frac{5 \cdot 10^{-6}C}{\sqrt{10}m} - \frac{3 \cdot 10^{-6}C}{\sqrt{2}m}\right) = -4861.6\text{V} \tag{3.44}$$

3.3 Electric Potential Energy

Suppose, we have an electric charge q_1 at certain random point in space. Now, electric charge q_1 will create an electric field, and therefore if an external agent were to move an electric charge q_2 from infinity up to a random point P in space, the work done by the external agent would be

$$W = \Delta U = (U_P - U_\infty) = q_2(V_P - V_\infty) = qV_P \tag{3.45}$$

where V_P means the potential at point P. The second potential term vanished because as we have mentioned our reference potential is such that $V_\infty = 0$. However, if the potential is zero at infinity, also must be the potential energy in the last equation. Therefore we are left with

$$U_P = q_2 V_P \tag{3.46}$$

But potential $V_P = k\frac{q_1}{r_P}$, where r_P just means the distance from electric charge q_1 to point P. Therefore, we have that the potential energy of the configuration of two electric charges is given by

$$\boxed{U = k\frac{q_1 q_2}{r}} \tag{3.47}$$

where the labels P were removed since we have chosen a random point and the last equation is a general relationship. Now. what if we bring a third electric charge from infinite? Well, this electric charge has to be moved against the electric field produced by q_1 and q_2. So, we have

$$U = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_2 q_3}{r_{23}} \tag{3.48}$$

where the distances between the charges have been labelled as r_{12} (distance between charges q_1 and q_2), r_{13} (distance between q_1 and q_3),r_{23} (distance between q_2 and q_3) . What about bringing now a fourth electric charge from infinity ? Now that electric charge moves against the electric field created by q_1, q_2 and q_3. Therefore, we have

$$U = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_1 q_4}{r_{14}} + k\frac{q_2 q_3}{r_{23}} + k\frac{q_2 q_4}{r_{24}} + k\frac{q_3 q_4}{r_{34}} \tag{3.49}$$

And we could continue up to the number of electric charges that we are interested to bring from infinity. It is matter to write down the pairs without repeating. So, we can write a general formula for the electric potential energy for N particles as

$$\boxed{U = k\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{q_i q_j}{r_{ij}}} \tag{3.50}$$

Example 6: Potential Energy of 3 charges in triangular configuration

Three point charges are located in a equilateral triangle as shown in the figure. Its length are 35cm and $Q = 5\mu$C
(a) What is the potential energy of the system?

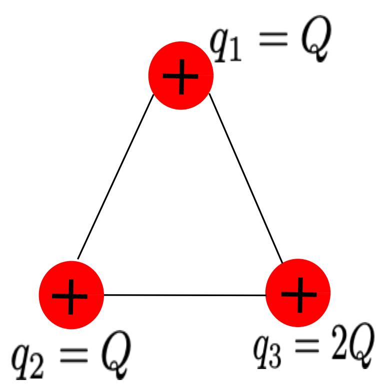


Figure 3.6

Solution:

To calculate the electric potential energy remember to take pairs without repeating. So,

$$U = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_2 q_3}{r_{23}} \tag{3.51}$$

Since,

$$r_{12} = r_{13} = r_{23} \tag{3.52}$$

and

$$q_1 = q_2 = Q \qquad q_3 = 2Q \tag{3.53}$$

We have that:

$$U = k\frac{Q^2}{r}[1+2+2] = 5k\frac{Q^2}{r} \tag{3.54}$$

where all distances were named as r and the electric charges in 3.51 were substituted. Plugging in values, we obtain

$$U = 5 \cdot \left(9 \cdot 10^9 \frac{Nm^2}{C^2}\right) \frac{(5 \cdot 10^{-6}C)^2}{0.35m} = 3.215\text{J} \tag{3.55}$$

3.4 Electric Potential and Potential Energy of continuous distributions of electric charge

In order to find the electric potential generated by continuous distributions of charge at a point P, we have two options:

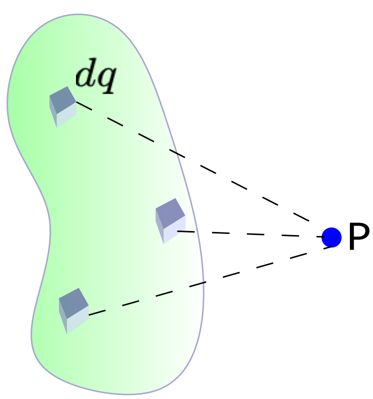


Figure 3.7

- We can split the object in little chunks and consider them as infinitesimal charges as shown in figure 3.7. Every single infinitesimal charge contributes a differential potential

$$dV = k\frac{dq}{r} \tag{3.56}$$

Then we sum infinitesimally all the contributions of the infinitesimal potentials at P, so

$$\boxed{V = \int k\frac{dq}{r}} \tag{3.57}$$

- If we know the electric field generated by an object, then we use the definition of the electric potential difference

$$\Delta V = V_f - V_i = -\int_{P_i}^{P_f} \vec{E}\cdot d\vec{l} \tag{3.58}$$

Taking our reference potential $V = 0$ at $P_i = \infty$, we have that

$$\boxed{V = \int_{\infty}^{P} \vec{E}\cdot d\vec{l}} \tag{3.59}$$

Now, suppose you want to calculate the potential energy of a continuous distribution. Let's start with the potential energy we have found for a set of N particles (equation 3.50)

$$U = k\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{q_iq_j}{r_{ij}} \tag{3.60}$$

The last equation can also be written as

$$U = \frac{k}{2}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\frac{q_iq_j}{r_{ij}} \tag{3.61}$$

where the factor of $\frac{1}{2}$ arises because now you are double counting the pairs. The only pair forbidden is the electric charge q_i with itself. Now, q_i can get out of the second sum and express the last equation as

$$U = \frac{1}{2}\sum_{i=1}^{N} q_i\left(\sum_{j\neq i}^{N} k\frac{q_j}{r_{ij}}\right) \tag{3.62}$$

where notice that in the last equation the expression inside the parenthesis is the potential of all particles different of q_i at the position of electric charge q_i! (See equation 3.40) Hence, last equation can be written as

$$U = \frac{1}{2}\sum_{i=1}^{N} q_iV(r_i) \tag{3.63}$$

So far we have just written the equation that we already used for the potential energy of N electric charges in another format. However, starting from last equation we can make the following, we split the object with continuous charge distribution in extremely small infinitesimal chunks with electric charge dq and integrate the complete contribution

$$U = \frac{1}{2}\int dqV = \int \rho V dVol \tag{3.64}$$

where we used $dq = \rho dVol$. Now, all along this book, we have been using V as potential. In many texts is common to also use ϕ. For this small derivation I will write ϕ for potential so that volume V is not confused with potential, and in order to avoid being writing $dVol$. So

$$U = \frac{1}{2}\int \rho\phi dV \tag{3.65}$$

where the integration limits must be such that encloses all the electric charge. However, we can make the integral over all space! Because any region in space that does no contain electric charge, $\rho = 0$ and will not contribute to the integral. Now , using Gauss Law in differential form , we can rewrite the last equation as

$$U = \frac{\epsilon_0}{2}\int \left(\nabla\cdot\vec{E}\right)\phi dV = U = \frac{\epsilon_0}{2}\int \left[\nabla\cdot\left(\vec{E}\phi\right) - \vec{E}\cdot\nabla\phi\right] \tag{3.66}$$

where the first term is zero because is a total derivative and we're taking the integral over all of space and $\phi(r) = 0$ as $r \to \infty$, so it vanishes. Now, recalling that $\vec{E} = -\nabla\phi$ (equation 3.3, just remember, for this deduction we are calling the potential as ϕ instead of V, so that it is not confused with differential of volume), then the last equation becomes

$$\boxed{U = \frac{\epsilon_0}{2}\int_{allspace} \vec{E}\cdot\vec{E}dV} \tag{3.67}$$

where

$$\boxed{u = \frac{\epsilon_0}{2}\vec{E}\cdot\vec{E}} \tag{3.68}$$

is the energy density (units $\frac{\mathrm{J}}{\mathrm{m}^3}$).

Example 7: Electric potential of a finite rod

Find the electric potential at point $P = (3, 1)$m due to a charge line placed in the y axis from $y = 0$m to $y = 5$m. The line charge has a charge density given by $\lambda = 4\frac{\mu C}{m}$. Along your calculations you can use the result of the following integral ,

$$\int \frac{du}{\sqrt{u^2 + c^2}} = \ln\left(u + \sqrt{u^2 + c^2}\right) \tag{3.69}$$

Solution:

We can calculate the potential either with equation 3.57 or equation 3.59 However, we do not know the electric field of a finite rod in a point which is not symmetrical. Therefore, we are going to use equation 3.57, splitting the rod in infinitesimal chunks and

integrate the contribution of all of them.
Now, the difficulty arises in defining the distance r from the electric charges on the rod to the point P. However, by using the triangle shown in the figure, we can define the hypotenuse distance as ,

$$r = \sqrt{3^2 + (y-1)^2} \tag{3.70}$$

which will be the distance from the electric charges to the point P. How do we know it will work? Well, first notice that r in the last equation is not **fixed**. So, when we integrate, the value of y will change from the lower limit of integration up to the upper limit of integration. Secondly, even though we used a particular triangle to define the distance r, the distance $y-1$ in the vertical direction is something general that applies to all charges above the base of the triangle, and also the distance 3 in the horizontal direction applies to all charges in the rod above of the triangle base. Probably you think that equation 3.70 does not apply to the charges below the base of the triangle, however that is not the case. We can prove for instance that it applies for the electric charge at the origin. By Pythagoras, the distance d from the electric charge at the origin to the point P, should be

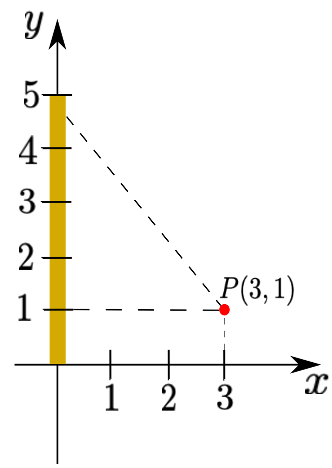


Figure 3.8

$$d = \sqrt{3^2 + 1^2} \tag{3.71}$$

Using our formula of r (equation 3.70) at the origin (x=0,y=0)

$$r = \sqrt{3^2 + (0-1)^2} = \sqrt{3^2 + 1^2} \tag{3.72}$$

which is exactly to d. So, the formula in equation 3.70 applies to any electric charge in the rod. So,

$$V = k\int \frac{dq}{r} = k\int_0^5 \frac{\lambda dy}{\sqrt{3^2 + (y-1)^2}} \tag{3.73}$$

where the limits of integration go from 0 up to 5, because is the range in the y axis that covers all the rod with electric charges. Now, just making a change of variables

$$u = y - 1 \Longrightarrow du = dy \tag{3.74}$$

We can write the integral in equation 3.73 as

$$V = k\lambda \int_{-1}^{4} \frac{du}{\sqrt{3^2 + u^2}} \tag{3.75}$$

where the new limits of integration follows because when $y = 0$ (lower limit in the integration) by equation 3.74 $u = -1$ and when $y = 5$ by once again equation 3.73 $u = 4$. Now, by using the integral result in equation 3.69, we have that the last equation becomes

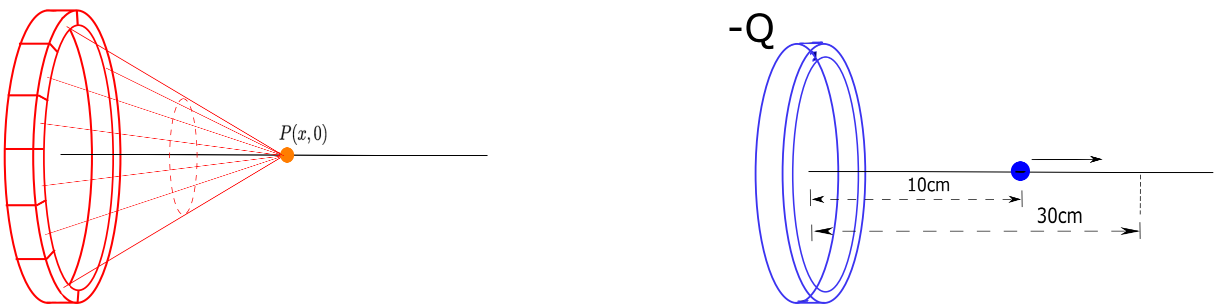
$$V = k\lambda\left[\ln(4+\sqrt{16+9}) - \ln(-1+\sqrt{1+9})\right] = \left(9\cdot 10^9\frac{Nm^2}{c^2}\right)\left(4\cdot 10^{-6}\frac{c}{m}\right)\ln\left[\frac{9}{\sqrt{10}-1}\right] \tag{3.76}$$

Therefore,

$$V = 51,338.24\text{V} \tag{3.77}$$

Example 8: Electric potential of a ring

An electric charge Q is distributed uniformly in a ring of radius a. Find the electric potential at P in the ring axis a distance x from its center. (b) Assume that $Q = -1\text{nC}$ and (its radio) $a = 20\text{cm}$. If an electron is released from rest at $x_i = 10\text{cm}$, What will it be the speed when $x_f = 30\text{cm}$. ***Solution:***



(a) (b)

Figure 3.9

In order to obtain the speed of the electron, we can use conservation of energy

$$\Delta U = -\Delta K \Longrightarrow q_e\Delta V = -\frac{1}{2}m_e v_f^2$$

where the initial velocity cancelled out because the electron starts from rest, and we used m_e as the mass of the electron, q_e as the electric charge of the electron and used equation 3.7 for the potential energy difference. Therefore,

$$v_f = \sqrt{\frac{-2q_e\Delta V}{m_e}} \tag{3.78}$$

All variables from in last equation are known, except ΔV, however we can calculate the potential at the initial point $x = 10\text{cm}$ and the potential at $x = 30\text{cm}$ and take the difference. Firstly we need to know what is the potential of a ring, so

$$V = k\int\frac{dq}{r} = k\int_0^{2\pi a}\frac{\lambda dl}{r} \tag{3.79}$$

where the limits of integration follow because we want to cover the whole ring. The length l will start from a point up to the perimeter of the ring. Now, notice that r is constant,

$$r = \sqrt{x^2 + a^2} \tag{3.80}$$

because x is a fixed distance from the ring to the point of interest, and a is the radius of the ring which does not change. Therefore we can take r out from the integral ,

$$V = \frac{k\lambda}{\sqrt{x^2 + a^2}} \int_0^{2\pi a} dl = \frac{k\lambda}{\sqrt{x^2 + a^2}} 2\pi a = \frac{kQ}{\sqrt{x^2 + a^2}} \tag{3.81}$$

where we used that the total length of the ring is the perimeter $L = 2\pi a$, and $\lambda L = Q$. Hence, equation 3.78 becomes

$$v_f = \sqrt{\frac{-2kQq_e}{m_e}\left[\frac{1}{\sqrt{x_f^2 + a^2}} - \frac{1}{\sqrt{x_i^2 + a^2}}\right]} \tag{3.82}$$

where we used the potential difference at points x_f and x_i. By just plugging in the values

$$v_f = \sqrt{\frac{-2 \cdot (9 \cdot 10^9 \frac{Nm^2}{c^2})(-1.6 \cdot 10^{-19}C)(-1 \cdot 10^{-9}C)}{9.1 \cdot 10^{-31}kg}\left[2.77\frac{1}{\mathrm{m}} - 4.47\frac{1}{\mathrm{m}}\right]} = 2,319,530.07\mathrm{m/s} \tag{3.83}$$

where the electron mass $m_e = 9.1 \times 10^{-31}$kg and electric charge $q_e = 1.6 \times 10^{19}C$ was used and that

$$\frac{1}{\sqrt{(30 \cdot 10^{-2}m)^2 + (20 \cdot 10^{-2}m)^2}} \approx 2.77\frac{1}{\mathrm{m}} \tag{3.84}$$

$$\frac{1}{\sqrt{(10 \cdot 10^{-2}m)^2 + (20 \cdot 10^{-2}m)^2}} \approx 4.47\frac{1}{\mathrm{m}} \tag{3.85}$$

Example 9: Electric potential of a disc

Find the electric potential at a point P along the axis perpendicular to the disc of radius R.

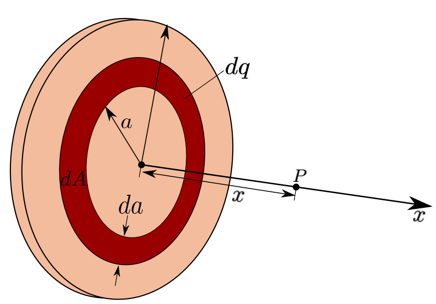


Figure 3.10

Solution:

The way to proceed will be very similar when we found the electric field of a disc. Once we know the potential of a ring, we can assume that the disc is made of many rings, each of them with infinitesimal electric charge. So, we sum infinitesimally (integrate) the contribution of all the rings at point P. In other words,

$$V_{disc} = \int dV_{rings} \tag{3.86}$$

we sum the contribution of all the differential potential created by each ring in the disc. So, the last equation becomes

$$V = k \int \frac{dq}{\sqrt{x^2 + a^2}} \tag{3.87}$$

where we used equation 3.81 for an infinitesimal electric charge in each ring. Now,

$$dq = \sigma dA = \sigma 2\pi a da \tag{3.88}$$

where we used equation 1.105. (If these calculations do not make much more sense to you, review in chapter 1 the exercise where we find the electric field of a disc). Hence,

$$V = k \int_0^R \frac{\sigma 2\pi a da}{\sqrt{x^2 + a^2}} = 2\pi k\sigma \int_0^R \frac{a da}{\sqrt{x^2 + a^2}} \tag{3.89}$$

where the limits of integration are from $a = 0$ up to $a = R$ because we want to make the radius of the rings grow from a point ($a = 0$) up to the radius of the disc($a = R$). Finally, to do the integration, lets just do the following change of variable

$$u = x^2 + a^2 \Longrightarrow du = 2ada \tag{3.90}$$

Therefore, the integral in equation 3.89 now reads as

$$V = \pi k\sigma \int_{x^2}^{x^2+R^2} \frac{du}{u^{1/2}} = \pi k\sigma \int_{x^2}^{x^2+R^2} u^{-1/2} du = 2\pi k\sigma u^{1/2}\Bigg|_{x^2}^{x^2+R^2} = 2\pi k\sigma \left[\sqrt{x^2 + R^2} - x\right] \tag{3.91}$$

where the new limits follow because when $a = 0$ (lower limit in the old variable) by equation 3.90 $u = x^2$, and when $a = R$(lower limit in the old variable) by once again equation 3.90 $u = x^2 + R^2$

Example 10: Electric potential of a conductor sphere

A solid conductor sphere of radius R has total electric charge Q. Find the potential at a distance r from the center of the sphere with :
a)$r > R$,
b)$r = R$,
c)$r < R$

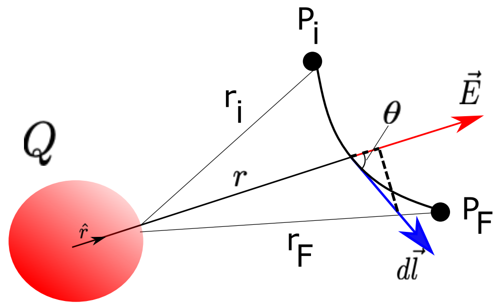


Figure 3.11

Solution:

This time instead of splitting the object in many little pieces with differential electric charge, we will use equation 3.59

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l} \tag{3.92}$$

because we already know what is the electric field of a sphere. Recall that a sphere with electric charge Q behaves as a point electric charge, so its electric field is

$$\vec{E} = k\frac{Q}{r^2}\hat{r} \tag{3.93}$$

Therefore,

$$V = -k\int_{\infty}^{r} \frac{Q}{r^2} dl\cos\theta \tag{3.94}$$

Now, recall $d\vec{l}$ is a tangent vector to the path that joins the end points in the integral for the calculation of the potential. See figure 3.11 and notice that

$$\cos\theta = \frac{dr}{dl} \qquad \Rightarrow dr = dl\cos\theta \tag{3.95}$$

Therefore equation 3.94 becomes,

$$V = -k\int_{\infty}^{r} \frac{Q}{r^2} dr = -kQ\left[-\frac{1}{r}\right]_{\infty}^{r} = \frac{kQ}{r} \qquad \Rightarrow \tag{3.96}$$

$$V = \frac{kQ}{r} \qquad r > R \tag{3.97}$$

So, we have found the potential for any outside point of the sphere.Now, if we make:

$$\lim r \to R \tag{3.98}$$

We have that,

$$\boxed{V = \frac{kQ}{R} \qquad \text{at} \quad r = R} \tag{3.99}$$

Just one more region to go. Taking any arbitrary point inside the sphere and a point at $r = R$, the potential difference between both points

$$\Delta V = V_R - V_{inside} = -\int \vec{E} \cdot d\vec{l} \tag{3.100}$$

However, the electric field inside any conductor is zero, hence

$$V_R - V_{inside} = 0 \quad \Rightarrow V_{inside} = V_R \tag{3.101}$$

So, we have found an important result!

> The electric potential at any point (inside or the surface) of any conductor is exactly the same! We used specifically an sphere, however this holds for any conductor.

Example 11: Electric potential of sharped conductors

Two conductor spheres with radius r_1 and r_2 where $r_2 > r_1$, are connected by a extremely long and thin conductor wire. If the spheres are separated several meters, such that it can be considered that the electric charge of one does not affect the electric charge distribution of the other.

a)What sphere generates more electric field ?

Solution:

The breaking point of this exercise is to notice that

$$V_{r_1} = V_{r_2} \tag{3.102}$$

Figure 3.12

i.e. the electric potential of both spheres is exactly the same! Why? Because both spheres are conductors and they are connected by a conductor. From the last exercise we learned that a conductor is at the same potential at any point of the conductor. Therefore, since all spheres and cable are conductors, we can consider the whole system like just one conductor with the same potential. So,

$$k\frac{q_1}{r_1} = k\frac{q_2}{r_2} \tag{3.103}$$

where we just used the potential of a spherical conductor (equation 3.99). So, by moving around the factors in the last equation we obtain.

$$\frac{r_2}{r_1} = \frac{q_2}{q_1} \tag{3.104}$$

Now the magnitude of the electric fields of the spheres are

$$E_2 = k\frac{q_2}{r_2^2} \qquad E_1 = k\frac{q_1}{r_1^2} \tag{3.105}$$

where E_2 is the electric field magnitude of the sphere of radius r_2 and E_1 the electric field magnitude of the sphere with radius r_1. So taking the ratio of the two electric fields we have,

$$\frac{E_2}{E_1} = \frac{k\frac{q_2}{r_2^2}}{k\frac{q_1}{r_1^2}} = \left(\frac{q_2}{q_1}\right)\left(\frac{r_1}{r_2}\right)^2 \tag{3.106}$$

No using equation 3.104 in the ratio of electric charges,

$$\frac{E_2}{E_1} = \frac{r_2}{r_1}\left(\frac{r_1}{r_2}\right)^2 = \frac{r_1}{r_2} \quad \Longrightarrow E_2 = \frac{r_1}{r_2}E_1 \tag{3.107}$$

Since,

$$r_1 < r_2 \quad \Rightarrow E_2 < E_1 \tag{3.108}$$

A quite incredible conclusion. Not intuitive at all! Conductors with sharped shapes produce bigger electric fields. We used two spheres, however there is a general property of conductors.

Conductors in electrostatic equilibrium, their regions with lower surface curvature radius generate greater electric fields than regions with greater surface radius.

The last property is exploited in the design of lightning rods. A very sharped object will have lower surface curvature, and even making the electric field grow abruptly tending to infinity (if $r \to 0$). So, when the electric charges flowing in a lighting are landing to the Earth, they will be attracted by the high electric fields of the lighting rod, and making safe surrounding regions.

Example 12: Simple exercise, recovering electric field from electric potential

If the electric potential in certain region of space is given by $V = 2x^3y^2 - 3xz + 5y^2z - z^2$. Find the electric field at the point $(0, 1, 1)$m.
Solution:
This exercise is to show the power to be dealing with potential instead of electric fields. If we have the potential function, we can just calculate

$$\vec{E} = -\nabla V \tag{3.109}$$

to obtain the electric field. Once we have the electric field, we can calculate forces, and motion of particles under the force exerted by the electric field. There is a huge richness by having the electric potential.
Now,

$$\vec{E} = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z}\right) \tag{3.110}$$

So,

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z} \tag{3.111}$$

So:

$$E_x = -(6x^2y^2 - 3z)\Big|_{0,1,1} = 3\text{N/C} \tag{3.112}$$

$$E_y = -(4x^3y + 10yz)\Big|_{0,1,1} = -10 \quad N/C \tag{3.113}$$

$$E_z = -(-3x + 5y^2 - 2z)\Big|_{0,1,1} = -3 \quad N/C \tag{3.114}$$

Hence,

$$\vec{E} = (3, -10, -3) \quad N/C \tag{3.115}$$

Part II

Circuit Basics

Chapter 4

Capacitance and Capacitors

We will study in this chapter a widely used component in electric circuits, *the capacitor*. We will start with a rather theoretical definition of a capacitor and we will study from theoretical perspective the properties of a capacitor. Afterwards, we study the calculation of the capacitance of several capacitors connected in different configurations in an electrical circuit. We finish the chapter with the study of dielectrics and how the capacitors are affected when these materials are introduced.

4.1 Capacitors

Definition 4.1.1 *A capacitor is any pair of conductors with electric charges* $+Q$ *and* $-Q$

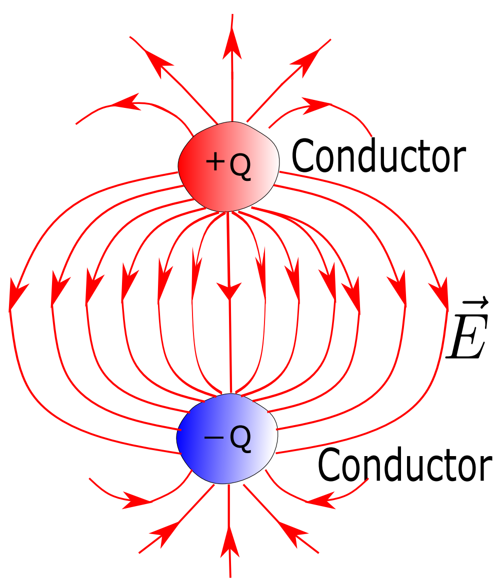


Figure 4.1

Just that!? A complete chapter of a book to something defined so simple? Indeed, my dear reader. However, it turns out that when we land this abstraction to real conductors we can construct electrical circuits with many functions with the use of capacitors. So, by now please be patient and follow me.

A capacitor since is constituted by two conductors, with opposite charges, an electric field is created between them. As we have learned in the last chapter, when there is an electric field, there is potential energy stored in the electric field. Therefore, we can say that capacitors store electric energy in form of electric potential energy given that certain electric field is created between the conductors.

Probably you ask yourself, Why necessarily conductors? Why not also two insulators with charges $+Q$ and $-Q$ can be considered as a capacitor? The theoretical answer is that we want to define two objects such that there is no ambiguity in the calculation of ΔV. Recall from last chapter that a conductor has exactly the same electric potential at the surface and inside the conductor. Therefore, when we calculate the potential difference between the conductors, we can take any point at each conductor and no problem will arise. However, if we use insulators instead, the potential is not exactly the same at any point in the insulator. Therefore, to define the potential difference, it would depend on the locations on each insulator that we choose to measure the voltage. The practical answer is that we want to store electric charge in the capacitors, therefore we need a material in which charges easily move from one point to other. Actually, in circuits what we do is to connect conductor wires (generally copper) to a battery which supplies a potential difference. Those cables also are connected to the capacitor and since the capacitor and the cables are conductors, the electric charges start to move easily through the cables to the capacitors where the charges are stored.

Now, we know from the potential difference created by one electric charge

$$\Delta V = kQ\left(\frac{1}{r_f} - \frac{1}{r_i}\right) \tag{4.1}$$

that the voltage is proportional to the electric charge, no matter the shape of the conductors, the potential difference by superposition principle will always be proportional to the electric charge. Therefore, we can write the potential difference as

$$\Delta V = CQ \tag{4.2}$$

where C is a constant of proportionality. Therefore, if we isolate C and we impose the condition that we calculate the absolute value of the electric charge and the potential difference we obtain

$$\boxed{C = \frac{|Q|}{|\Delta V|}} \tag{4.3}$$

so, the constant C is called as **Capacitance** of the capacitor and its units are $\mathrm{F} \equiv \frac{\mathrm{C}}{\mathrm{V}}$ (Farad).

By definition the capacitance is always positive. Why is it so? Well, we have two electric charges in our capacitor $+Q$ and $-Q$. So, we eliminate the redundancy when we calculate the capacitance if we measure the positive or the negative charge. Also, the absolute value in the voltage eliminates the redundancy if we measure the potential difference starting from the positive to the negative charge or the way around. It is quite important to remark that many of the electromagnetism texts or circuits books does not make explicitly the absolute value in the charge Q and the voltage ΔV. It is assumed and taken as obvious that you already know so. It will become obvious and tedious to be writing repeatedly the absolute value bars. However, for pedagogical reasons we'll keep the absolute value symbol in the first exercises.

To have a better grasp about what capacitance means, let's start with some analogies. The calorific capacity is the amount of energy an object can store or release per unit of mass given that there exist certain change of temperature. The **capacitance** in analogy is the amount of electric charge that a capacitor can store or release given certain voltage (potential difference). The capacity of a milk carton is the amount of milk it can store. The **capacitance** is the capacity of a capacitor to store (also release) electric charge given a electric potential difference (voltage).

Let's start with the calculation of capacitance of some different geometries. As it turns out, the capacitance only depends on the geometry of the conductors that store the electric charge.

Example 1: Capacitance of parallel plates capacitor

A capacitor of parallel plates consists of two large conductive plates placed very close to each other in such a way that the electric field between them can be considered to be uniform. If the area of the plates is A and the separation between them is d, calculate its capacitance.

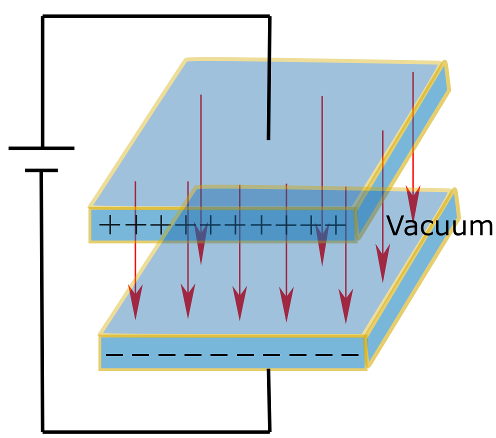


Figure 4.2

Solution:
We need to calculate

$$C = \frac{|Q|}{|\Delta V|} \tag{4.4}$$

So , lets start by calculating ΔV. In general,

$$\Delta V = -\int \vec{E} \cdot d\vec{l} \tag{4.5}$$

However, the electric field between the plates is almost uniform (constant), therefore

$$\Delta V = -\vec{E} \cdot \vec{d} \tag{4.6}$$

However, we are interested in the absolute value of the dot product, therefore

$$|\Delta V| = |\vec{E}||\vec{d}| \tag{4.7}$$

Now, the electric charge in the positive plate can be written as

$$Q = \sigma A \tag{4.8}$$

while the electric field magnitude between the two plates is the sum of the electric field produced by each plate. Therefore,

$$|\vec{E}| = \frac{\sigma}{\epsilon_0} \tag{4.9}$$

where we used the magnitude of the electric field of an infinite plane (equation 1.111) and summed them twice. Therefore, let's use what we have found for ΔV, $|\vec{E}|$ and Q in the calculation of the capacitance, and we obtain

$$C = \frac{\sigma A}{\frac{\sigma}{\epsilon_0} d} \tag{4.10}$$

Therefore, the capacitance for any capacitor with parallel plates is given by

$$\boxed{C = \frac{A\epsilon_0}{d}} \tag{4.11}$$

Notice that the capacitance is only dependant of geometrical factors of the plates. As mentioned before, the capacitance will only depend on the geometry of the capacitor.

Example 2: Capacitance of concentric cylinders capacitor

A long cylindrical conductor of radius r_d and linear density of charge $+\lambda$ is surrounded by a cylindrical conductive shell of internal radius r_0 and linear density of charge $-\lambda$. Calculate the capacitance per unit length for this capacitor, assume that there is vacuum in the space between the cylinders.

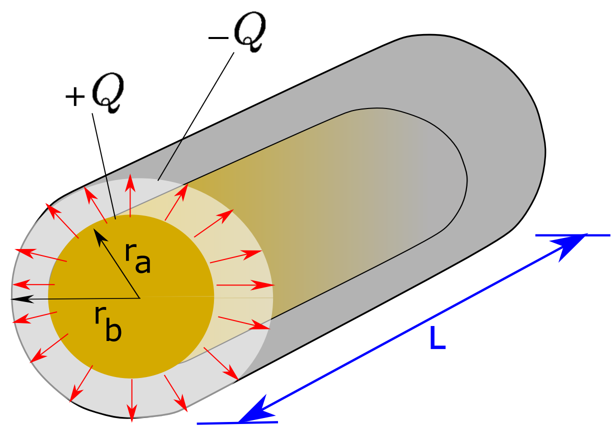


Figure 4.3

Solution:
First of all, if the capacitor is long enough in comparison to $r_B - r_A$, we can ignore the edge effects. Secondly we have to consider just the electric field created inside the capacitor. Since the two objects that constitute the capacitor are conductors, then the only electric field that exists inside is the one created by the inner cylinder. Thirdly, strictly speaking we should consider σ for the charge density in the inner cylinder. However, for simplification we take as λ for the charge density. So, starting by the calculation of the potential difference

$$\Delta V = -\int \vec{E} \cdot d\vec{l} = -\int_{r_A}^{r_B} \frac{\lambda}{2\pi\epsilon_0 r} dr = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_B}{r_A}\right) \tag{4.12}$$

where we used the electric field of a thin infinite rod 1.17, and used as limits of integration the radius of the inner cylinder and the outer cylinder. Hey! Wait a minute! It is not a thin rod, neither it is infinite! Indeed, however we use this electric field for simplification of the calculation. Now, we are interested in the absolute value of the voltage, so

$$|\Delta V| = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_B}{r_A}\right) \tag{4.13}$$

Therefore, the capacitance,

$$C = \frac{|Q|}{|\Delta V|} = \frac{\lambda L}{\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_B}{r_A}\right)} \tag{4.14}$$

where we used that the positive charge is $|Q| = +Q = \lambda L$. Therefore, just cancelling out common factors in numerator and denonminator and rearranging terms in the last equation, we obtain

$$\boxed{C = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{r_B}{r_A}\right)}} \tag{4.15}$$

Notice, the capacitance for the cylindrical capacitor depends only on length, and radius of the inner and outer cylinder. Geometrical properties of the capacitor once again define the capacitance.

Example 3: Capacitance of concentric spherical shells capacitor

Two concentric spherical conductive shells are separated by vacuum. The inner shell has total charge + Q and outer radius r_a, while the outer shell has net charge - Q and inner radius r_b. Find the capacitance of this spherical capacitor.

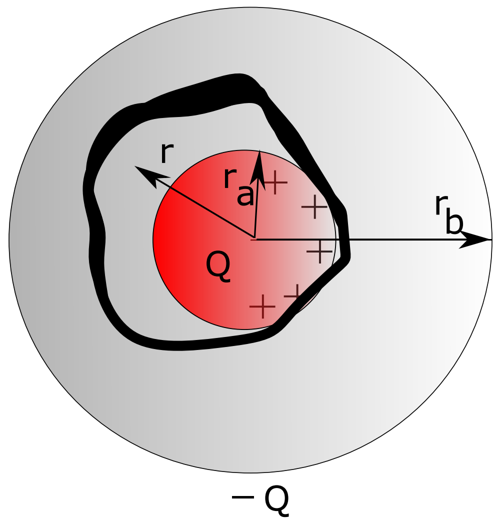


Figure 4.4

Solution: Once again, we must consider just the electric field created by the inner sphere, because the electric field created by the outer sphere points outside of the conductor, therefore we are not interested in that electric field. Just the one between the conductors. Therefore, the potential difference

$$\Delta V = -\int \vec{E} \cdot d\vec{l} = -\int k\frac{Q}{r^2}dr \tag{4.16}$$

where the electric field of a spherical object was substituted (equation 2.66). Now, giving limits of integration the radius of the inner sphere and the outer sphere

$$\Delta V = -kQ\int_{r_A}^{r_B} \frac{dr}{r^2} = kQr^{-1}\Big|_{r_A}^{r_B} = kQ\left[\frac{1}{r_B} - \frac{1}{r_A}\right] = kQ\left[\frac{r_A - r_B}{r_A r_B}\right] \tag{4.17}$$

However, $r_B > r_A$. And we need absolute value of the potential difference. Therefore,

$$|\Delta V| = kQ\left[\frac{r_B - r_A}{r_B r_A}\right] \tag{4.18}$$

where we just interchanged $r_B \leftrightarrow r_A$ and now the last equation is positive. So, finally

$$C = \frac{|Q|}{|\Delta V|} = \frac{Q}{kQ\left[\frac{r_B - r_A}{r_B r_A}\right]} \tag{4.19}$$

By just arranging the last equation, we have that for any spherical capacitor, its capacitance is given by

$$C = \frac{r_B r_A}{k(r_B - r_A)} \tag{4.20}$$

where once again, notice that the capacitance only depends on the geometrical properties of the sphere, nothing else matters to the capacitance.

4.2 Combination of Capacitors

The capacitors are manufactured with determined standard values of capacitance. If for certain application we require a different value of capacitance. How can we obtain it? The answer is that we make combinations of the way we connect the capacitors in a circuit. So we define the following ways to connect the capacitors

Definition 4.2.1 *When two or more capacitors are connected in such a way that their electric charge Q is exactly the same, we say that they are* ***connected in series*** *(See figure 4.5a)*

Definition 4.2.2 *When two or more capacitors are connected in such a way that their voltage ΔV is exactly the same, we say that they are* ***connected in parallel*** *(See figure 4.5b)*

From such definitions, we can deduce how to calculate the equivalent capacitance. Suppose a circuit in parallel as shown in 4.5b. So the electric charge stored in an equivalent capacitor is given by

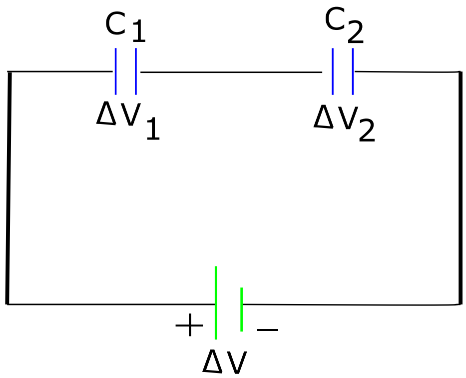
$$Q_{eq} = Q_1 + Q_2 \tag{4.21}$$

Now, from equation 4.3, the electric charge in one capacitor is given by $Q = C\Delta V$. Therefore, the last equation can be written as

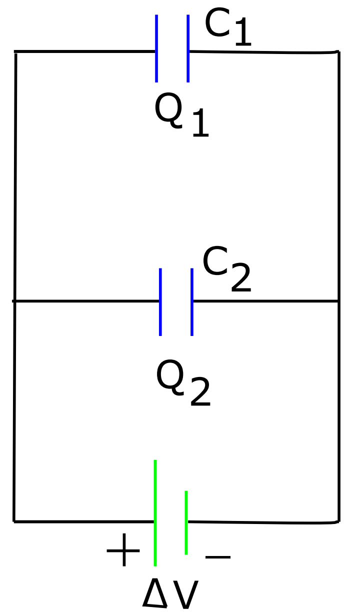
$$Q_{eq} = C_1\Delta V + C_2\Delta V = (C_1 + C_2)\Delta V \tag{4.22}$$

where $C_!$ is the capacitance of the capacitor with electric charge Q_1 and C_2 the capacitance for the capacitor with electric charge Q_2. Also, we used the same voltage for both capacitors because they are in parallel. However, $Q_{eq} = C_{eq}\Delta V$. Therefore,

$$C_{eq} = C_1 + C_2 \tag{4.23}$$



(a)



(b)

Figure 4.5

where the last result applies for any pair of capacitors connected in parallel. If we have in general N capacitors, **connected in parallel the equivalent capacitance is found by**

$$\boxed{C_{eq} = \sum_{i=1}^{N} C_i} \tag{4.24}$$

However, What if they are connected in series? Suppose now, two capacitors connected in series as shown in figure 4.5a. The potential difference is now given by

$$\Delta V_{eq} = \Delta V_1 + \Delta V_2 \tag{4.25}$$

Now, from equation 4.3, isolating voltage in one capacitor, we have that $\Delta V = \frac{Q}{C}$. Therefore, the last equation can be written as

$$\frac{Q_{eq}}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} \tag{4.26}$$

However,we know that

$$Q_{eq} = Q_1 = Q_2 \tag{4.27}$$

because they are connected in series. Therefore, equation 4.26 becomes

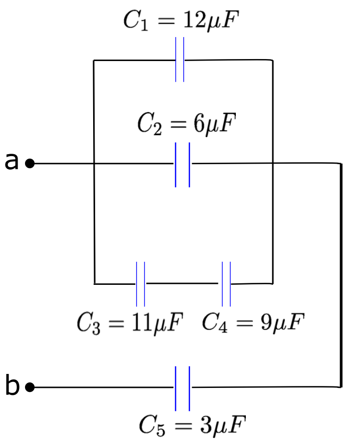
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{4.28}$$

the last result applies for any pair of capacitors connected in series. If we have in general N capacitors, **connected in series the equivalent capacitance is found by**

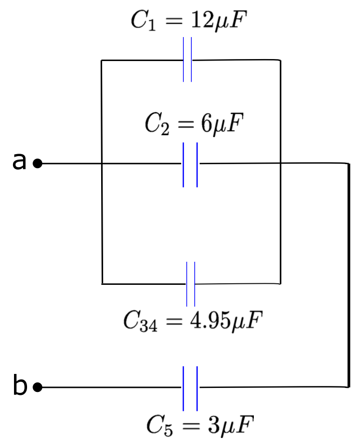
$$\boxed{\frac{1}{C_{eq}} = \sum_{i=1}^{N} \frac{1}{C_i}} \tag{4.29}$$

Example 4: Equivalent capacitance of several capacitors in a circuit

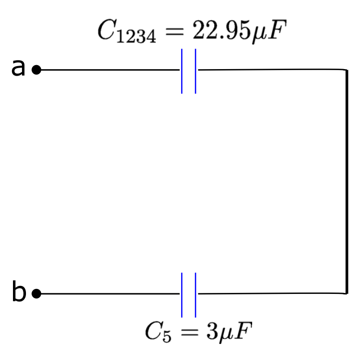
Find the equivalent capacitance of the combination of capacitors shown in figure 4.6a.



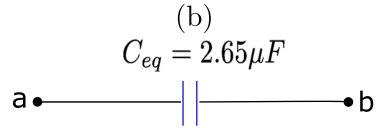
(a)



(b)



(c)



(d)

Figure 4.6

Solution:

Anytime you have a complicated circuit, start by reducing it in *equivalent* capacitors until you have obtained just one *equivalent* capacitor. In this example, we can start firstly noticing that capacitors C_3 and C_4 are connected in series. Therefore, the equivalent capacitance of C_3 and C_4 capacitors is given by

$$\frac{1}{C_{34}} = \frac{1}{C_3} + \frac{1}{C_4} = \frac{C_4 + C_3}{C_3 C_4} \quad \Rightarrow C_{34} = \frac{C_3 C_4}{C_4 + C_3} \tag{4.30}$$

where we just labelled with "34" to the equivalent capacitance of the capacitors C_3 and C_4. Therefore, plugging values we have

$$C_{34} = \frac{(11\times 10^{-6}F)(9\times 10^{-6}F)}{9\times 10^{-6}F + 11\times 10^{-6}F} = C_{34} = 4.95\times 10^{-6}F \tag{4.31}$$

Now, the new circuit with the equivalent capacitor C_{34} looks as the one shown in figure 4.7b. Evidently, capacitors C_1, C_2 and C_{34} are connected in parallel. Therefore, obtaining the equvialent capacitance of the three capacitors,

$$C_{1234} = C_1 + C_2 + C_{34} \tag{4.32}$$

where the capacitance is just the sum because they are in parallel. By just plugging values, we obtain that the equivalent capacitance of the three capacitors is

$$C_{1234} = 12\times 10^{-6}F + 6\times 10^{-6}F + 4.95\times 10^{-6}F = 22.95\times 10^{-6}F \tag{4.33}$$

So, we obtain the circuit shown in figure 4.7c, where we have now the new equivalent capacitor C_{1234}. Finally, the capacitors C_{1234} and C_5 are in series, therefore their equivalent capacitance is

$$\frac{1}{C_{12345}} = \frac{1}{C_{1234}} + \frac{1}{C_5} \quad \Rightarrow \frac{1}{C_{12345}} = \frac{C_5 + C_{1234}}{C_{1234}\cdot C_5} \quad \Rightarrow C_{12345} = \frac{C_{1234}\cdot C_5}{C_5 + C_{1234}} \tag{4.34}$$

By plugging in the values

$$C_{12345} = \frac{(22.95\times 10^{-6}F)(3\times 10^{-6}F)}{(3\times 10^{-6}F) + (2.295\times 10^{-5}F)} = 2.65\times 10^{-6}F \tag{4.35}$$

Since we have covered all the capacitors in the circuit, we have found the equivalent capacitance of all the circuit (See figure 4.7d), therefore

$$\boxed{C_{eq} = 2.65\cdot 10^{-6}F} \tag{4.36}$$

Example 5: Voltage and charge of several capacitors in a circuit

Find the voltage and electric charge of each capacitor in the circuit shown in figure 4.7a, when a potential difference $\Delta V_{ab} = 15V$ is applied in a and b.

Solution:
Let's start to find the equivalent capacitance of the complete circuit. Firstly, we can notice easily from figure 4.7a that C_1 and C_2 are in series. Therefore,

$$C_{12} = \frac{C_1 C_2}{C_1 + C_2} = \frac{(15\times 10^{-6}F)(3\times 10^{-6}F)}{(18\times 10^{-6}F)} = 2.5\times 10^{-6}F \tag{4.37}$$

where we have already plugged in the values . Now, the circuit looks like the one shown in the figure 4.7b. We can make a new equivalent capacitor of C_{12} and C_3 in parallel, therefore

$$C_{123} = C_{12} + C_3 = 2.5\times 10^{-6}F + 6\times 10^{-6}F = 8.5\times 10^{-6}F \tag{4.38}$$

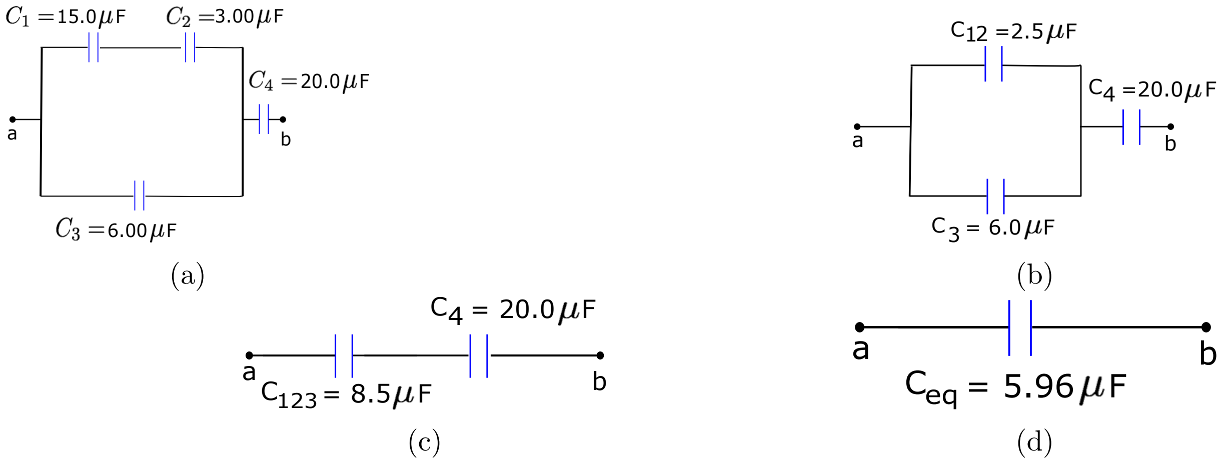


Figure 4.7

Finally, the circuit looks like in figure 4.7c. We have that C_{123} and C_4 are in series, therefore the equivalent capacitance of both capacitors is given by

$$C_{1234} = \frac{C_{123}C_4}{C_{123}+C_4} = \frac{(8.5\times10^{-6}F)(20\times10^{-6}F)}{(28.5\times10^{-6}F)} = 5.96\times10^{-6}F \tag{4.39}$$

Since we have covered all capacitors, we can say that the equivalent capacitance of the circuit is

$$C_{eq} = 5.96\times10^{-6}F \tag{4.40}$$

Now, if we want to know the voltage in each capacitor and the electric charge stored in each capacitor, we need to go backwards, i.e. we need to start from the equivalent capacitor of the whole system shown in figure 4.7d, and start to go steps backwards until we have the complete circuit once again (figure 4.7a). Let's do the exercise and it will become much clearer what we mean with this. Firstly the electric potential difference of the equivalent capacitor is given by

$$\Delta V_{eq} = \frac{Q_{eq}}{C_{eq}} \qquad \text{therefore, } Q_{eq} = C_{eq}\Delta V_{eq} = (5.96\times10^{-6}F)(15V) = 8.94\times10^{-5}C \tag{4.41}$$

where notice that we used $\Delta V_{ab} = 15V$ because the equivalent voltage is equal to the voltage to which the whole system is submitted to. This is much clearer if we see figure 4.7d. The equivalent capacitor of the whole system is directly connected to the voltage supply ΔV_{ab}. Now, when go once step back to the final circuit (figure 4.7c), the capacitors C_{123} and C_4 are in series, therefore they have the same electric charge. Hence,

$$Q_{eq} = Q_{123} = Q_4 \tag{4.42}$$

So, we have obtained actually the first electric charge. The capacitor C_4 has the same electric charge as the equivalent capacitor of the complete circuit. So

$$Q_4 = 8.94\times10^{-5}C \tag{4.43}$$

Now, the potential difference of capacitor C_4, can also be found

$$\Delta V_4 = \frac{Q_4}{C_4} = \frac{8.94 \times 10^{-5} C}{20 \times 10^{-6} F} = 4.47V \tag{4.44}$$

If we go back now one step back (figure 4.7b), we have that capacitors C_{12} and C_3 are in parallel, therefore their voltages are equal to the voltage of the equivalent capacitor, i.e.

$$\Delta V_{123} = \Delta V_{12} = \Delta V_3 \tag{4.45}$$

So, if we know the voltage ΔV_{123} we can obtain the other two. So,

$$\Delta V_{123} = \frac{Q_{123}}{C_{123}} = \frac{8.94 \times 10^{5} C}{8.5 \times 10^{-6} F} = 10.53\text{V} \tag{4.46}$$

Hence, we can obtain now

$$Q_3 = C_3 \Delta V_3 = C_3 \Delta V_{123} = (6 \times 10^{-6} F)(10.53V) = 6.318 \times 10^{-5} C \tag{4.47}$$

while the electric charge in the equivalent capacitor C_{12}

$$Q_{12} = C_{12} \Delta V_{12} = C_{12} \Delta V_{123} = (2.5 \times 10^{-6} F)(10.53V) = 2.63 \times 10^{-5} C \tag{4.48}$$

Probably you feel terribly tedious all the calculations. When you practice many circuits, many of middle steps will become obvious to you and the calculation will become extremely fast. Now, from figure 4.7a, we can see that the capacitors C_1 and C_2 are in series, therefore their electric charge is exactly the same to the equivalent capacitor

$$Q_{12} = Q_1 = Q_2 \tag{4.49}$$

While the voltages

$$\Delta V_1 = \frac{Q_1}{C_1} = \frac{Q_{12}}{C_1} = \frac{2.6325 \cdot 10^{-5} C}{15 \times 10^{-6} F} = 1.755V \tag{4.50}$$

$$\Delta V_2 = \frac{Q_2}{C_2} = \frac{Q_{12}}{C_2} = \frac{2.6325 \cdot 10^{-5} C}{3 \times 10^{-6} F} = 8.775V \tag{4.51}$$

And we are done! Let's summarize in the following table what we obtained

Capacitor	Electric Charge (Q)	Voltage (ΔV)
C_1	2.63×10^{-5}C	1.755V
C_2	2.63×10^{-5}C	8.775V
C_3	6.318×10^{-5}C	10.53V
C_4	8.94×10^{-5}C	4.47V

4.3 Energy stored in Capacitors

We already have an equation for the potential energy for any electric field

$$U = \frac{\epsilon_0}{2}\int_{all\ space} \vec{E}\cdot\vec{E}dV \tag{4.52}$$

however, in order to use the last equation we need to know the electric field of the conductors of the capacitor. Let´s do another approach for capacitors. Recall that $\Delta U = q\Delta V$. And, when an external agent applies a work we have $W = \Delta U = q\Delta V$. So, if we let the potential energy difference be infinitesimal

$$dW = dq\Delta V = \frac{qdq}{C} \tag{4.53}$$

where $\Delta V = \frac{q}{C}$ was used. Therefore, integrating both sides of the last equation

$$W = \int_0^Q \frac{qdq}{C} = \frac{Q^2}{2C} \tag{4.54}$$

Therefore, the energy as work to bring that charge to the capacitor, is stored as potential energy. So,

$$U = \frac{Q^2}{2C} \tag{4.55}$$

Also, you can say that the potential energy U is equal to the work W when we bring the electric charges from infinity. Hey! wait a minute! What if I did not bring the charge from infinity? Remember that potential energy needs a reference frame, our reference frame is such that at infinity the potential energy is zero. So, if you did not bring the charge from infinity, do not worry! What you need to do is calculate the *difference of potential energy*. Now that will make sense, because you calculate the potential energy when the charge is at the capacitor minus the potential energy when the charge was at some other point, i.e.

$$U_{\text{at the capacitor}} - U_{atsomefinitepoint} = (U_{\text{at the capacitor}} - U_\infty) - (U_{atsomefinitepoint} - U_\infty) \tag{4.56}$$

where the potential U_∞ was explicitly written to show you that even if it were not zero, you get rid of it! The reference frame is removed once you take the difference. So, do not worry if in equation 4.55 we take it supposing we bring the charges from infinity .

Finally, we can express the potential energy in different forms, with use of some algebra

$$U = \frac{Q^2}{2C} = \frac{Q^2\Delta V}{2Q} = \frac{Q\Delta V}{2} \tag{4.57}$$

where $C = \frac{Q}{\Delta V}$ was used. Also,

$$U = \frac{Q^2}{2C} = \frac{(C\Delta V)^2}{2C} = \frac{C(\Delta V)^2}{2} \tag{4.58}$$

So, we have the following three formulas to calculate the potential energy for any capacitor with vacuum between its conductors (no matter the shape)

$$\boxed{U = \frac{Q^2}{2C} = \frac{Q\Delta V}{2} = \frac{C(\Delta V)^2}{2}} \tag{4.59}$$

Example 6: Calculating several properties of a cylindrical capacitor

A cylindrical capacitor (vacuum between its conductors) of 15 m length store 3.20×10^{-9}J of energy when the potential difference between the two conductors is 4.0 Volts.
(a) Calculate the magnitude of the electric charge.
(b) Calculate the fraction of the radius of the capacitor $\frac{r_B}{r_A}$
(c) What happens if its length is of 1 m?.
(d) What happens if its length is of 1 cm?

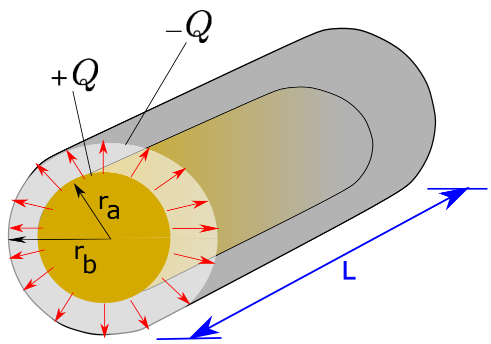


Figure 4.8

Solution:
Easy peasy, let's use the formula we have derived for the electric potential energy of a capacitor,

$$U = \frac{1}{2} Q \Delta V \tag{4.60}$$

Then, let's just isolate Q, substitute values and obtain the electric charge

$$Q = \frac{2U}{\Delta V} = \frac{2 \cdot 3.20 \times 10^{-9}\text{J}}{4.0\text{V}} = 1.6 \times 10^{-9}\text{C} \tag{4.61}$$

Now, for question in b) , we fortunately have in the voltage formula of a cylindrical capacitor the ratio $\frac{r_B}{r_A}$ (equation 4.13) . So, it is just matter to do some algebra

$$\Delta V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_B}{r_A}\right) \Rightarrow \frac{2\pi\epsilon_0 \Delta V}{\lambda} = \ln\left(\frac{r_B}{r_A}\right) \Rightarrow \frac{2\pi\epsilon_0 \Delta V}{Q} L = \ln\left(\frac{r_B}{r_A}\right) \Rightarrow \frac{r_B}{r_A} = e^{\frac{2\pi\epsilon_0 \Delta V}{Q} L} \tag{4.62}$$

So, plugging in the values in the last expression we obtain

$$\frac{r_B}{r_A} = 2.83 \tag{4.63}$$

Gosh! It means that the outer radius must be more than two and half times the radius of the inner cylinder to store a miserable amount of electric charge 1.6×10^{-9}C

$$r_B = 2.83 r_A \tag{4.64}$$

This is not a good design of a capacitor at all! Now, to answer the question in c) , if $L = 1$m the factors in the exponential to calculate the ratio (equation 4.62) become

$$\frac{2\pi\epsilon_0 \Delta V}{Q} L = 0.139 \tag{4.65}$$

Therefore,

$$\frac{r_B}{r_A} = e^{0.139} = 1.149 \tag{4.66}$$

So,

$$r_B = 1.149 r_A \tag{4.67}$$

well, much better than the last case. Even though, having a capacitor of one meter?! Not good at all, unless you want a capacitor with huge capacitance, storing lots of electric charge. However, this capacitor is not the case.

Finally, if now instead $L = 1cm$ the exponent factors become

$$\frac{2\pi\epsilon_0 \Delta V}{Q} L = 1.39 \cdot 10^{-3} \tag{4.68}$$

hence

$$\frac{r_B}{r_A} = e^{1.39 \cdot 10^{-3}} = 1.001 \tag{4.69}$$

So, now

$$r_B = 1.001 r_A \tag{4.70}$$

Now the inner and outer cylinders must be equal to retain the electric charge 1.6×10^{-9}C, however a cylinder of 1cm makes much more sense to be practical.

Example 7: Potential energy stored in a spherical capacitor

Part I) A conductor sphere of radius R has electric charge Q. Calculate the energy in the electric field in function of the distance r from the center of the sphere for

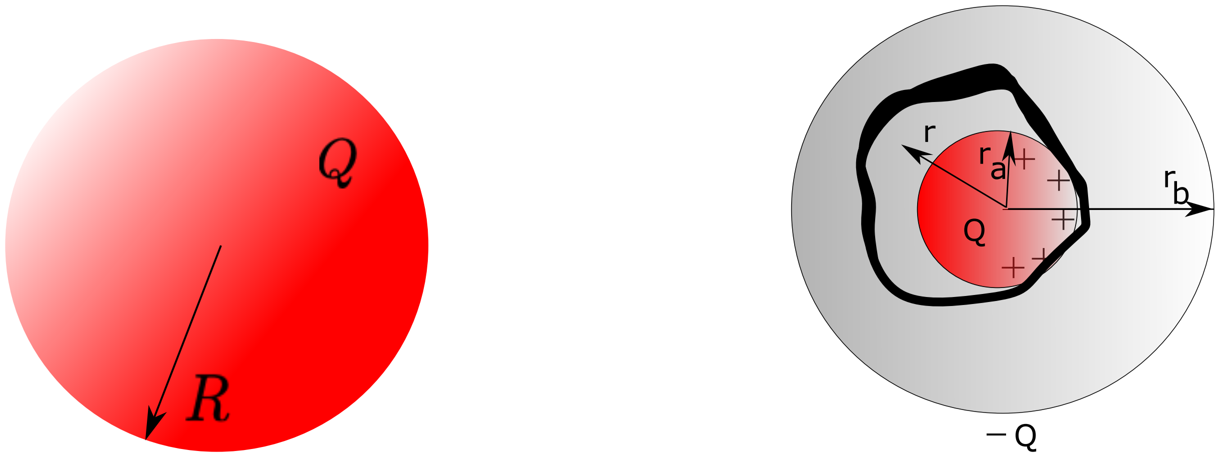
- $r > R$
- $r \leq R$
- Calculate the total energy stored in the electric field of a sphere, if $R = 1$cm, $Q = 1.5$nC a distance $r = 3.5$cm

Part II) Show that by using

$$U = \int \frac{1}{2} \epsilon_0 \vec{E} \cdot \vec{E} dV \tag{4.71}$$

a spherical capacitor leads to the potential energy of a capacitor

$$U = \frac{Q^2}{2C} \tag{4.72}$$



(a) (b)

Figure 4.9

Solution:

For part I, we just have to solve 4.71 inside and outside the conductor. Inside the conductor, when $r < R$, we know that the electric field is zero, therefore equation 4.71 becomes zero. However, outside there is an electric field

$$U = \int_R^r \frac{1}{2}\epsilon_0 \vec{E}\cdot\vec{E}dV = \int_R^r \frac{1}{2}\epsilon_0\left(\frac{kQ}{r^2}\right)^2 4\pi r^2 dr = \frac{1}{2}\epsilon_0 k^2 Q^2 4\pi \int_R^r \frac{dr}{r^2} \tag{4.73}$$

where we used the differential volume of a sphere $dV = 4\pi r^2 dr$. Using that $\frac{1}{k} = 4\pi\epsilon_0$ and solving the integral we have

$$U = \frac{1}{2}kQ^2\left[-\left(\frac{1}{r}\right)\right]_R^r = \frac{1}{2}kQ^2\left[\frac{1}{R} - \frac{1}{r}\right] \tag{4.74}$$

Therefore, we obtained that

$$U = \begin{cases} 0 & \text{for } r \leq R \\ \frac{1}{2}kQ^2\left[\frac{1}{R} - \frac{1}{r}\right] & \text{for } r > R \end{cases} \tag{4.75}$$

For the second question , where the total energy with specific values is asked,

$$U = 0 + \frac{1}{2}kQ^2\left[\frac{1}{R} - \frac{1}{r}\right] \tag{4.76}$$

where the zero was written just to make notice that we have also considered the energy inside the sphere. So, plugging in values, we have that

$$U = \frac{1}{2}\left(9\times10^{-9}\frac{Nm^2}{C^2}(1.5\times10^{-9}C)^2\right)\left[\frac{1}{1\cdot10^{-2}m} - \frac{1}{3.5\cdot10^{-2}m}\right] = 7.23\cdot10^{-7}J \tag{4.77}$$

Finally, for part II what we want to do is corroborate that indeed formula in equation 4.71 will lead us to the same result that we obtained in equation 4.54.It must! We have mentioned before that 4.71 is a general result, so applied to a very specific case should give the same for any capacitor. So, first of all by plugging the electric field of a spherical conductor in equation 4.71, we obtain

$$U = \frac{1}{2}\epsilon_0 \int_{r_A}^{r_B} \left(k\frac{Q}{r^2}\right)^2 4\pi r^2 dr \tag{4.78}$$

So, solving the integration, we have

$$U = \frac{2\pi k^2 Q^2}{\epsilon_0} \int_{r_A}^{r_B} \frac{dr}{r^2} = -\frac{2\pi k^2 Q^2}{\epsilon_0} \frac{1}{r}\bigg|_{r_A}^{r_B} = -\frac{2\pi k^2 Q^2}{\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A}\right) \tag{4.79}$$

Finally let's do some algebra

$$U = 2\pi\epsilon_0 k^2 Q^2 \left(\frac{r_B - r_A}{r_A r_B}\right) = 2\pi\epsilon_0 k Q^2 \left(\frac{k(r_B - r_A)}{r_A r_B}\right) \tag{4.80}$$

However, notice that the second term in last equality is exactly the inverse ($\frac{1}{C}$) of the capacitance of a spherical capacitor (see equation 4.20). Therefore,

$$U = 2\pi\epsilon_0 k Q^2 \frac{1}{C} = 2\pi\epsilon_0 \left(\frac{1}{4\pi\epsilon_0}\right) Q^2 \frac{1}{C} = \frac{Q^2}{2C} \tag{4.81}$$

Nice! We have shown that indeed the potential energy of a spherical capacitor is given by equation 4.72. If you follow a similar procedure for any capacitor, does not matter the shape of the capacitor, always you will obtain equation 4.72.

4.4 Dielectrics

So far we have studied capacitors, assuming there is nothing between the two conductors. However, in reality there is certain material in between (at least there is air, unless somehow you remove it from the medium). If we want the capacitor to work, the material that we introduce must be an **insulator**, other way there would not be a potential difference ΔV. Recall that conductors have exactly the same electric potential everywhere, so if you introduce a conductor, everything would be a conductor and no potential difference would exist. The insulator that we introduce is called **dielectric**. The question now that arises is "How will the capacitance be affected? Will it be greater or smaller?". *Michael Faraday* found **experimentally that when a dielectric is introduced in a capacitor, the capacitance increased proportionally to the capacitance without dielectric as**

$$C = \kappa C_0 \tag{4.82}$$

where C_0 is the capacitance without dielectric or vacuum and κ is called as **dielectric constant** (Do not confuse with $k = 9\times10^9 \frac{\mathrm{Nm}^2}{\mathrm{C}^2}$). The dielectric constant depends on each material and it is always greater than one.

Now, the capacitance always increases in the presence of a dielectric, however there are two possible scenarios

- **If the voltage is maintained fixed** the electric charge increases as

$$\boxed{Q = \kappa Q_0} \tag{4.83}$$

where Q_0 is the electric charge in the capacitor before introducing the dielectric. The reason $Q = \kappa Q_0$ is that $C = \kappa C_0$ and $C_0 = \frac{Q_0}{\Delta V_0}$. Therefore, the only way that $C = \kappa C_0$ is that $Q = \kappa Q_0$ because the voltage does not change $V = V_0$. Now, if the voltage is fixed, then the electric field cannot change!

$$\boxed{\vec{E} = \vec{E}_0} \tag{4.84}$$

where $\vec{E}$ is the electric field with dielectric and $\vec{E}_0$ is the electric field without dielectric. Think once again about the formula $\Delta V = -\vec{E} \cdot \vec{d}$. Since, the the distance between the capacitors do not change, and ΔV is fixed, $\vec{E}$ must be also constant. Finally, the energy

$$U = \kappa U_0 \tag{4.85}$$

where U_0 is the potential energy with not dielectric. Why is this? By doing some algebra

$$U = \frac{Q^2}{2C} = \frac{\kappa^2 Q_0^2}{2\kappa C_0} = \kappa \frac{Q_0^2}{2C_0} = \kappa U_0 \tag{4.86}$$

where once again all the labels 0 means when there was no dielectric (vacuum). Now, probably you would think, with analogy with what we have done so far that the energy density is

$$u = \frac{\epsilon_0}{2} \vec{E} \cdot \vec{E} = \frac{\epsilon_0}{2} \vec{E}_0 \cdot \vec{E}_0 \tag{4.87}$$

However, this is not true! There is a quite deeper reason why the last equation is not true and it is beyond the scope of this course. However, if you are curious and want to know why, I mention it in next section. Meanwhile, just trust me that

$$\boxed{u = \frac{\kappa \epsilon_0}{2} \vec{E} \cdot \vec{E}} \tag{4.88}$$

where $\vec{E}$ is the electric field vector with dielectric!

- **If the electric charge in the capacitor is maintained fixed** the voltage in the capacitor decreases as

$$\boxed{\Delta V = \frac{\Delta V_0}{\kappa}} \tag{4.89}$$

because $C = \kappa \frac{Q}{\Delta V}$, and $Q = Q_0$, therefore the only way this holds is that $\Delta V = \frac{\Delta V_0}{\kappa}$. From last equation we can easily see that when the electric charge is fixed, the magnitude of the electric field must also reduce as

$$\boxed{|\vec{E}| = \frac{|E_0|}{\kappa}} \tag{4.90}$$

because the electric potential difference ΔV is always proportional to the electric field. (Recall for a constant electric field $\Delta V = -\vec{E} \cdot \vec{d}$).
The potential energy in the case of constant electric charge, is given by

$$U = \frac{U_0}{\kappa} \tag{4.91}$$

where U_0 is the potential energy with vacuum (no dielectric). The reason of the last equation can be shown with some algebra,

$$U = \frac{Q^2}{2C} = \frac{Q_0^2}{2\kappa C_0} = \frac{1}{\kappa}\frac{Q_0^2}{2C_0} = \frac{U_0}{\kappa} \tag{4.92}$$

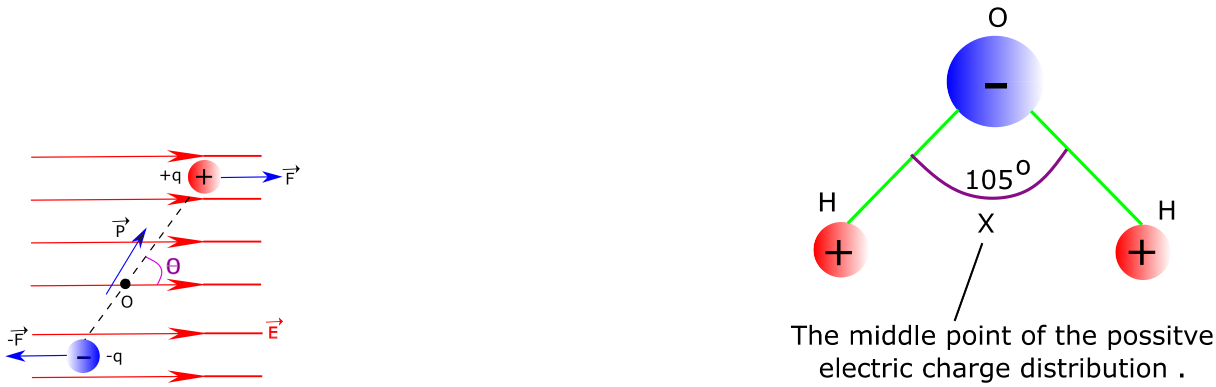
so the potential energy decreases! Finally, the energy density when the electric charge is constant is also given by equation 4.88.

So , let's summarize what we have discussed so far in the following table.

ΔV fixed constant	Q fixed constant
$\vec{E} = \vec{E}_0$	$\vec{E} = \frac{\vec{E}_0}{\kappa}$
$Q = \kappa Q_0$	$Q = Q_0$
$\Delta V = V_0$	$\Delta V = \frac{\Delta V_0}{\kappa}$
$U = \kappa U_0$	$U = \frac{U_0}{\kappa}$
$C = \kappa C_0$	$C = \kappa C_0$

Table 4.1

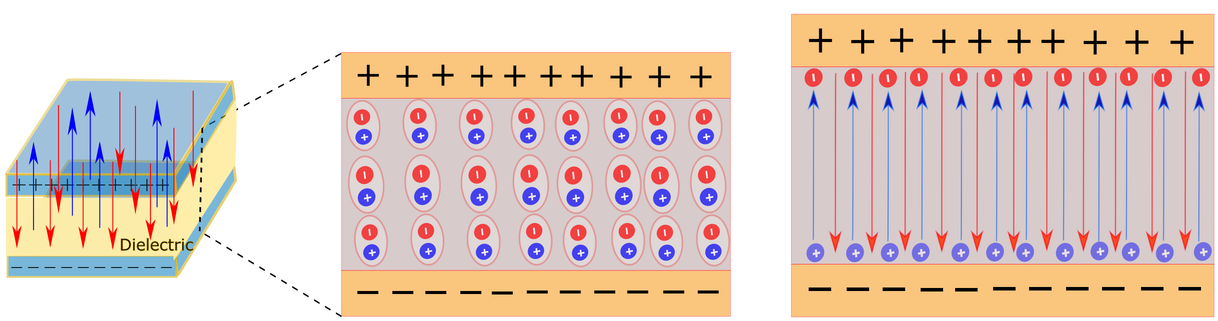
Now, what is the physics behind all these results? We have mentioned how to calculate different variables in presence of a dielectric , and all based in an experimental fact $C = \kappa C_0$. However, why does this happen? To start our discussion think about a dipole as shown in figure 4.10a One of the charges will feel a force to right and one to left. Therefore a torque is produced, and around the point O the two charges will start to move in circular motion until they have reached an equilibrium and stop moving. Well, this phenomena happens as well to molecules. Of course some molecules will tend to rotate more than others, it depends completely in the charge distribution in the molecule or the geometry of the molecule. For example, the water molecule has a charge distribution so that negative electric charge is mostly at opposite region to the positive charge (See figure 4.10b) . Therefore, the water molecule can be modelled as a dipole and when there is an



(a) (b)

Figure 4.10

external electric field, the complete molecule will tend to rotate until it reaches an equilibrium. When there exist a separation between the positive charge and negative charge in a molecule, we say that the molecule is ***polarized***. When the polarization is present in a molecule even when there is no external electric field, we say that the molecule is a ***polar molecule***, oppositely if the polarization in a molecule is not present all the time, even when there is no external electric field we just say is that the molecule is ***not polar molecule***.



(a) (b)

Figure 4.11

Now, suppose a capacitor with conductor plates with charge density $\pm\sigma$, maintains its **electric charge fixed** (electric charge is trapped). If you introduce now a dielectric, the molecules of the dielectric will tend to **polarize**. If the insulator material is made of not polar molecules, the polarization will be partial. However, even if the polarization is partial there will be an interesting effect on the electric charge of the dielectric material. A surface of electric charge will be induced at the surface of the plate with opposite sign (see figure 4.11a). Since there is a new induced electric charge density σ_i at the surface of the plate conductor,there will be a new electric field with opposite direction (see figures

4.11a and figure 4.11b). Therefore, the magnitude of the electric field between the plates

$$|\vec{E}| = |\vec{E}_0| - |\vec{E}_i| \tag{4.93}$$

so, as we see the electric field must go down! In this special case (electric charge constant), we know that $\vec{E} = \frac{\vec{E}_0}{\kappa}$, therefore

$$\frac{|\vec{E}_0|}{\kappa} = |\vec{E}_0| - \vec{E}_i \Longrightarrow |\vec{E}_i| = |\vec{E}_0|\left(1 - \frac{1}{\kappa}\right) = \frac{\sigma_0}{\epsilon_0}\left(1 - \frac{1}{\kappa}\right) \tag{4.94}$$

where in the last step we used the magnitude of the electric field of two plates (strictly speaking infinite, however since we assume that the separation between the plates is much smaller than the area of the plates it is a good approximation). However, the electric field created by the induced charge is $\vec{E}_i = \frac{\sigma_i}{\epsilon_0}$ because the charges induced in the surface of the plates creates two surfaces of electric charge with the same shape of the plates. Therefore,

$$\frac{\sigma_i}{\epsilon_0} = \frac{\sigma_0}{\epsilon_0}\left(1 - \frac{1}{\kappa}\right) \Longrightarrow \sigma_i = \sigma_0\left(1 - \frac{1}{\kappa}\right) \tag{4.95}$$

and since $\kappa > 1$, then we see that the induced electric charge density is lower than the electric charge density in the conductors of the capacitor. The electric charge in the dielectric is called ***bound electric charge*** (because is trapped in the dielectric) and the electric charge in the conductors of the capacitor is called ***free electric charge***.

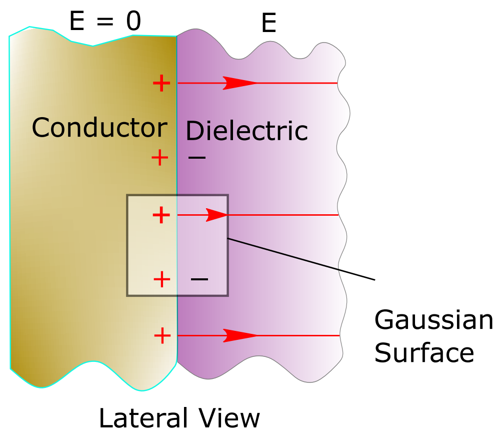


Figure 4.12

The result in equation 4.95 will lead to a beautiful new result! Let´s apply Gauss Law to one of the surfaces of the capacitor as shown in the figure 4.12. So, calculating the flux through the Gaussian surface we have picked, we have

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \tag{4.96}$$

However, the enclosed charge is

$$q_{enc} = \sigma A - \sigma_i A = \left(\sigma - \sigma\left(1 - \frac{1}{\kappa}\right)\right) A = \frac{\sigma A}{\kappa} \tag{4.97}$$

where we used equation 4.95. Notice that σ is just the free electric charge density! So the electric flux only depends on the free electric charge! We used a very specific case, however Gauss law in presence of dielectrics reads as

$$\oint_S \kappa \vec{E} \cdot d\vec{A} = \frac{q_{enc-free}}{\epsilon_0} \tag{4.98}$$

If we want Gauss Law in differential form, lets apply divergence Theorem in the left hand side and write the right hand side the charge in terms of the charge density

$$\int_V \nabla \cdot \left(\kappa \vec{E}\right) dV = \frac{1}{\epsilon_0} \int_V \rho_{free} dV \tag{4.99}$$

and since we are integrating over the same volume, what is inside the integrals must be equal, therefore

$$\nabla \cdot \left(\epsilon \vec{E}\right) = \rho_{free} \tag{4.100}$$

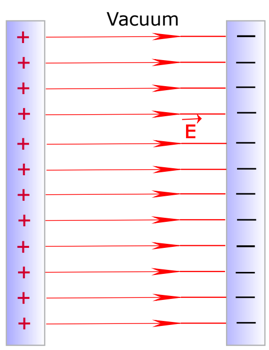
where $\epsilon = \kappa\epsilon_0$. We have been using ϵ_0 all this time, and now we can discuss about it . ϵ is called as permittivity of the material. For vacuum $\kappa = 1$ and that is why ϵ_0 is the **vacuum permittivity** or **free space permittivity**. The permittivity tell us information about the susceptibility of polarization of the material. It depends on the microscopic structure of the material and external factors as temperature.(I would like just to point out that equation 4.100 is not the final equation to consider any electric field in matter. It only applies for very specific materials, which are called *linear dielectrics* (which are the ones we are studying)).

Finally, if the voltage is fixed and the electric charge is not, what happens? Does the equation 4.93 still holds?... Wait! But we said that the voltage is fixed, then the electric field is fixed! Whats is going on?! Exactly! The voltage is fixed, so more electric charge must be in the conductor to maintain a constant electric field. Undoubtedly, a new electric field in the dielectric is created due to polarization of the molecules, and an induced electric charge exists at the surface of the conductor plates. So, in order to cancel out the contribution of the induced electric charge from the dielectric, you need now a free charge κQ_0.

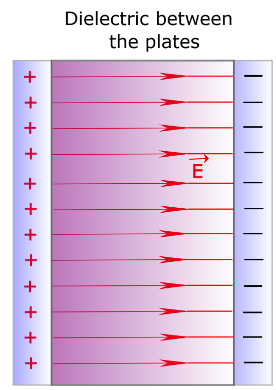
Example 8: Calculating properties of a capacitor with dielectric

The plates in the capacitor of figure 4.13a, each of them has an area 0.2m^{2}and they are separated 1cm. The capacitor has potential difference $\Delta V_0 = 3$kV. Then an insulator is inserted as shown in figure 4.13b and the potential difference reduces to $\Delta V = 1000$V, and the electric charge in the plates remains equal. Calculate

1. Original Capacitance
2. The magnitude of the charge Q in each plate
3. The new Capacitance after the dielectric is introduced
4. The dielectric constant
5. The original electric field
6. The new electric field after the dielectric is introduced



(a)



(b)

Figure 4.13

Solution:
To calculate the capacitance without dielectric , we can use equation 4.11

$$C_0 = \frac{\epsilon_0 A}{d} = \frac{(8.85 \cdot 10^{-12})(2 \cdot 10^{-1} m^2)}{1 \cdot 10^{-2} m} = 1.77 \cdot 10^{-10} F \tag{4.101}$$

So, we have answered question**1**. Now question 2, the electric charge can be obtained by the definition of capacitance.

$$C_0 = \frac{Q_0}{\Delta V_0} \quad \Rightarrow Q = C_0 \Delta V_0 = (1.77 \cdot 10^{-10} F)(3 \cdot 10^3 V) = 5.31 \cdot 10^{-7} F \tag{4.102}$$

where notice that it is indifferent to write Q_0 or Q, because the electric charge before and after inserting is exactly the same. In this exercise what changes is the potential difference. Now question 3, in order to find the new capacitance we could use

$$C = \kappa C_0 \tag{4.103}$$

However, we do not know the dielectric constant. However, the new voltage is given to us. Therefore , we can just calculate

$$C = \frac{Q}{\Delta V} = \frac{5.31 \cdot 10^{-7} C}{1000 V} = 5.31 \cdot 10^{-10} F \tag{4.104}$$

Continuing with question 4, we can easily obtain the dielectric constant once we know the capacitance before and after we inserted the dielectric,

$$C = \kappa C_0 \Longrightarrow \frac{C}{C_0} = \frac{\kappa C_0}{C_0} = \kappa \tag{4.105}$$

Therefore, substituting values

$$\kappa = \frac{5.31 \cdot 10^{-10} F}{1.77 \cdot 10^{-10} F} = 3 \tag{4.106}$$

Question 5, given that we already know the initial potential difference, we assume constant electric field between the parallel plates (as we have done along all this course when dealing with plates). So,

$$E_0 = \frac{\Delta V_0}{d} = 3 \cdot 10^5 \frac{V}{m} \tag{4.107}$$

Finally, for the last question

$$E = \frac{\Delta V}{d} = \frac{1000\text{V}}{1 \cdot 10^{-2}\text{m}} = 1 \cdot 10^5 \frac{V}{m} \tag{4.108}$$

where we could have also found it by

$$E = \frac{E_0}{\kappa} = \frac{3 \cdot 10^5 \frac{V}{m}}{3} = 1 \cdot 10^5 \frac{V}{m} \tag{4.109}$$

4.4.1 Energy in Presence of Dielectrics (Optional)

Unfortunately, at this point you have not studied enough vector calculus to be comfortable with all the derivation shown in this subsection. However, I included it for those curious, brave and bold students who ask me continuously about the energy when there is a dielectric, and why do the energy density now is

$$u = \frac{\epsilon}{2} \vec{E} \cdot \vec{E} \tag{4.110}$$

where $\vec{E}$ is the electric field with the dielectric, not in the vacuum.

So, if you want to know why, follow me , if you do not feel conformable at any step and feel too confused about the calculations. Do not worry, you will be able to understand it with enough comfort at the end of this semester,when you have finished coursing Math IV or Vector Calculus.

So, we begin with

$$\Delta W = \int \Delta \rho_{free} \phi dV \tag{4.111}$$

Given that $\nabla \cdot \left(\epsilon \vec{E}\right) = \rho_{free}$, we have that

$$\Delta \rho_{free} = \nabla \cdot \left(\epsilon \Delta \vec{E}\right) \tag{4.112}$$

Therefore, equation 4.111 becomes

$$\int \nabla \cdot \left(\epsilon \Delta \vec{E}\right) \phi dV \tag{4.113}$$

Now, notice that

$$\nabla \cdot \left[\epsilon \Delta \vec{E} \phi\right] = \nabla \cdot \left(\epsilon \Delta \vec{E}\right) \phi + \epsilon \Delta \vec{E} \cdot (\nabla \phi) \Longrightarrow \nabla \cdot \left(\epsilon \Delta \vec{E}\right) \phi = \nabla \cdot \left[\epsilon \Delta \vec{E} \phi\right] - \epsilon \Delta \vec{E} \cdot (\nabla \phi) \tag{4.114}$$

However, remember that $\vec{E} = -\nabla V$. Therefore,

$$\Delta W = \nabla \cdot \left(\epsilon \Delta \vec{E}\right) \phi = \nabla \cdot \left[\epsilon \Delta \vec{E} \phi\right] + \epsilon \Delta \vec{E} \cdot \vec{E} \tag{4.115}$$

Hence, equation 4.113 reads as

$$\Delta W = \int \left(\nabla \cdot \left[\epsilon \Delta \vec{E} \phi\right] + \epsilon \Delta \vec{E} \cdot \vec{E}\right) dV \tag{4.116}$$

The first integral due to divergence theorem becomes a surface integral, and since we are integrating over all space, it vanishes. So, we are left with

$$\Delta W = \int \epsilon \Delta \vec{E} \cdot \vec{E} dV \tag{4.117}$$

Now, we also have that,

$$\frac{1}{2}\Delta \left(\epsilon \vec{E} \cdot \vec{E}\right) = \frac{1}{2}\Delta(\epsilon |\vec{E}|^2) = \frac{1}{2}\left(\epsilon |\Delta \vec{E}||\vec{E}| + \epsilon |\vec{E}||\Delta \vec{E}|\right) = \epsilon |\Delta \vec{E}||\vec{E}| = \epsilon \Delta \vec{E} \cdot \vec{E} \tag{4.118}$$

Hence, equation 4.117 can be written as

$$\Delta W = \int \frac{1}{2}\Delta \left(\epsilon \vec{E} \cdot \vec{E}\right) dV = \Delta \left(\int \frac{1}{2}\epsilon \vec{E} \cdot \vec{E}\right) dV \tag{4.119}$$

Therefore, getting rid of the change , and taking just the total work we obtain

$$U = \int \frac{1}{2}\epsilon \vec{E} \cdot \vec{E} dV \tag{4.120}$$

Chapter 5

Electric current and Resistance

We have studied so far static electric charges. Let's start our study of electric charges when they flow. Along this chapter we assume that electric charges move with constant mean velocity (no acceleration).

5.1 Electric current

Definition 5.1.1 *The electric current through a cross sectional area A is defined as the total charge that flows through that area per unit of time*

$$I = \frac{dq}{dt} \tag{5.1}$$

The electric charge change can be due to the movement of positive or negative charges, however, by convention the direction of the current is the direction of the movement of the positive charges.

As it is known, the electric current in a conductor is given due to the movement of negative electric charges So, what is the relationship between the electric current and the mean speed of the moving electric charges in the conductor (drift velocity)? Let's define the density of electric charges per volume

$$n = \frac{N}{Vol} = \frac{N}{Ad} \tag{5.2}$$

Therefore, we can say that the number of electric charges that were in the volume Ad crosses the sectional area A in an infinitesimal time dt, so

$$dq = qnAd = nA|v_d|dt \tag{5.3}$$

were v_d is the mean speed of the electric charges or the so called *drift velocity*. Therefore, stating from equation 5.1,

$$I = \frac{nqv_d Adt}{dt} = nqv_d A \tag{5.4}$$

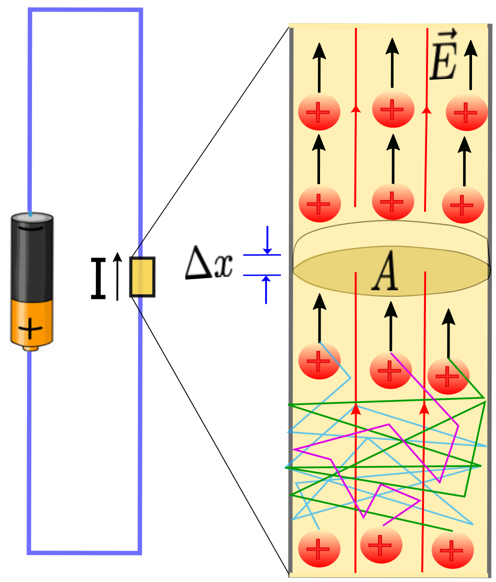


Figure 5.1

Just to remark, actually the electric current is a **vector** and in equation 5.1 and equation 5.4 is just its magnitude. In many of introductory to undergraduate physics, they manage it as a scalar. This is just wrong! The current can flow either to the right, the left , upwards, diagonally, etc and it depends on the velocity of the charges that constitute it, so for sure we need to characterize this in a vector. However, in practice many times (probably in most cases in this course) is not necessary to point out its vector nature and we handle it most of the time just as a scalar.

Now,we define the vector **electric current density** as

$$\vec{J} = \frac{d\vec{I}}{dA} \tag{5.5}$$

which is the electric current per unit of cross sectional area. If the electric current is constant in a cross sectional are, we simply have that the magnitude of the current density is

$$|\vec{J}| = \frac{I}{A} \tag{5.6}$$

Therefore, the magnitude of the current density is

$$|\vec{J}| = nqv_d \tag{5.7}$$

and, the electric current density vector is

$$\vec{J} = nq\vec{v}_d \tag{5.8}$$

Also, we have that

$$\vec{J} = \frac{d\vec{I}}{dA} = \frac{d\lambda\vec{v}}{dA} = \frac{dq}{drdA}\vec{v} \tag{5.9}$$

but $drdA$ is an infnitesimal volume ,

$$\vec{J} = \rho\vec{v} \tag{5.10}$$

where $\vec{v}$ is the instantaneous velocity of the differential amount of charge in the volume $dv = drdA$

Example 1: Drift velocity

A copper wire has a perimeter of 1.02mm. The wire conducts a constant current of 1.67A to a lamp of 200W. The density of the free electrons is 8.5×10^28 electrons per cubic meter. Find the magnitude of

a) The magnitude of the current density

b) The drift velocity.

Solution:
We have from equation 5.6 that the current density magnitude is given by

$$J = \frac{I}{A} \tag{5.11}$$

where we already know the electric current. So ,let's firstly calculate the cross sectional area. The copper wire can be modelled as very large cylinder, so the cross sectional area is

$$A = \pi r^2 = \pi(P/2\pi)^2 = \pi(\frac{1.02 \cdot 10^{-3}m}{2\pi})^2 = 8.27 \cdot 10^{-8}m^2 \tag{5.12}$$

where P stands for perimeter. Therefore, plugging the area in equation 5.11

$$J = \frac{I}{A} = \frac{1.67A}{8.27 \cdot 10^{-8}m^2} = 20,170,933.22\frac{A}{m^2} \tag{5.13}$$

Now, b) the drift velocity can be obtained easily with the electric current density formula

$$J = nqv_d \tag{5.14}$$

So, isolating v_d

$$v_d = \frac{J}{nq} = \frac{20,170,933.22\frac{A}{m^2}}{(8.5 \cdot 10^{28}\frac{1}{m^3})(1.6 \cdot 10^{-19}C)} = 1.48 \cdot 10^{-3}\frac{m}{s} \tag{5.15}$$

5.2 Ohm's Law

We have started this chapter to study electric charges to move, however they cannot move freely. There is always certain friction against their movement. The friction arises from the collision of the electric charges with the walls of the conductor were they are flowing, the collision with other electric charges or molecules.

The property of materials that tell us how difficult is for the charges to move through it is called **resistivity**. Or in other words the measure of the material opposing to the flow of the electric charge. The units of resistivity are Ωm(Ohms meter). We give the letter ρ to resistivity. Unfortunately, we are ran out of letters so do not confuse it with electric charge density!

We call as **conductivity** to the capacity of a material to let electric charges flow, and it is the inverse of the resistivity

$$\sigma = \frac{1}{\rho} \tag{5.16}$$

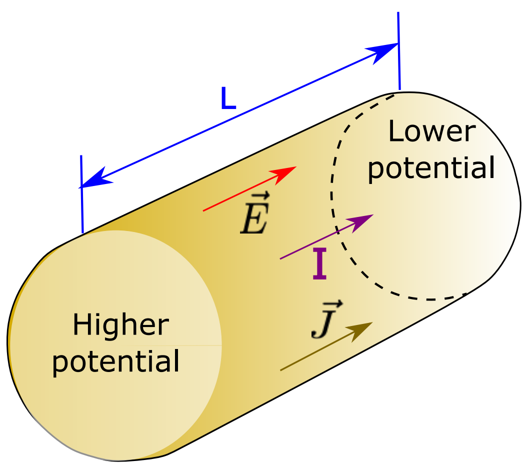


Figure 5.2

Now, for many materials (however not all) , which we call as *ohmic materials* , follow the relationship

$$\boxed{\vec{J} = \sigma \vec{E}} \tag{5.17}$$

which is called as **Ohm's Law**. Mostly most all metals follow Ohm's Law . However, I would like to emphasize that it does not apply to all materials, and therefore it is not a fundamental law of nature. Now, suppose we have wire of ohmic material (copper for instance would work). We model it as a cylinder with cross sectional area A and length L. If that wire is connected to a potential difference ΔV supply as battery, an electric field inside the conductor is created. If we assume the electric field as constant, we have that the magnitude of the potential difference is

$$\Delta V = \vec{E}L \tag{5.18}$$

where we used equation 3.12. Therefore, taking the magnitudes in equation 5.17, we obtain

$$J = \sigma\frac{\Delta V}{L} \Longrightarrow \frac{I}{A} = \sigma\frac{\Delta V}{L} \Longrightarrow \Delta V = \left(\frac{\rho L}{A}\right) I \tag{5.19}$$

where $\rho = \frac{1}{\sigma}$ was used in the last step. Now, we call as **resistance**

$$\boxed{R = \frac{\rho L}{A}} \tag{5.20}$$

where its units are $\Omega \equiv VA$. The resistance also is a measure of opposition of the material to let the charges flow. Greater resistances, more unlikely that electric charge moves through a material. However, notice that resistance depends also on the geometry of the material that we are studying, while the resistivity is an intrinsecal property of the material. So, equation 5.19 becomes

$$\boxed{\Delta V = RI} \tag{5.21}$$

which is a widely used relationship in circuits and probably you have already seen it before. The last equation is not the $Ohm'sLaw$ as many times is referenced as. It is a specific case, using a constant electric field of Ohm's Law. However, since the last relationship is many times used, that commonly it is just referenced as Ohm's Law.

Example 2: Electric current, electric field and resistance

A copper wire has a radius of 0.5cm. The wire length is 4 m and a current of 3.6 A is travelling through it . Find the magnitude of

- Current density
- The electric field in the wire
- The resistance of the wire

Solution:
In order to find the electric current density, we use equation 5.11, so

$$J = \frac{I}{A} = \frac{3.6A}{\pi(0.5 \cdot 10^{-2}m)^2} = 45,836.62\frac{A}{m^2} \tag{5.22}$$

where we already plugged in the values. Notice the huge number! It means lots of electrons are trying to cross an area at the same time. Now,

$$|\vec{E}| = \rho|\vec{J}| \tag{5.23}$$

Therefore,using the resistivity value for copper, we have

$$E = 45,836.62\frac{A}{m^2} \cdot 1.72 \cdot 10^{-8}\Omega \cdot \mathrm{m} = 7.8 \cdot 10^{-4}\frac{v}{m} \tag{5.24}$$

Finally, if we want to know the resistance of the wire, we can use equation 5.20

$$R = \rho\frac{L}{A} = (1.72 \cdot 10^{-8}\Omega \cdot m)\left(\frac{4m}{\pi(0.5 \cdot 10^{-2}m)^2}\right) = 8.75 \cdot 10^{-4}\Omega \tag{5.25}$$

Notice the small value obtained, which makes sense! A wire of copper is a good conductor, therefore the electrons flow easily and henceforth the opposition against their movement is negligible.

5.3 Temperature and Resistance relation

For ohmic materials, within certain ranges of temperature they follow a general rule,

$$\boxed{\rho = \rho_0\left(1 + \alpha(T - T_0)\right)} \tag{5.26}$$

where ρ_0 is the resistivity of the ohmic material at temperature T_0, and ρ is the resistivity at temperature T and α is a constant that depends on each material. Now, the last equation tells us something interesting , if $\alpha > 0$ and $T > T_0$, or in other words if we increase the temperature of the material, the resistivity increases, while if the temperature decreases the resistivity decreases. If you think about it, makes sense! The resistivity is the measure of opposition to electric charges to flow, and at greater temperatures the molecules and charges within the material are mumbling around and causing more collisions than with lower temperatures. However, Is α always greater than zero? Well, it turns out that nature always surprise us. There are certain materials which $\alpha < 1$ and their conductivity increases as temperature increases. These materials are called as *semiconductors*.
Now, given that the resistance is proportional to the resistivity, the resistance follows

$$\boxed{R = R_0\left(1 + \alpha(T - T_0)\right)} \tag{5.27}$$

Example 3: Calculated temperature using platinum thermometer

Suppose that at 20° the resistance of a thermometer of platinum is 164.2Ω (for platinum $\alpha = 3.92 \times 10^{-3}\mathrm{C}^{\circ -1}$). When the thermometer is placed in a particular solution, the resistance is 187.4Ω. What is the temperature of the solution?
Solution: One common application of the change of resistivity of materials, is to calculate the temperature of solutions. A widely used material for thermometers is the platinum, due to its high melting point and high resistance to corrosion. So, how can we calculate the temperature of a solution? Let's use equation, 5.27

$$R = R_0[1 + \alpha(T - T_0)] \tag{5.28}$$

And let's just isolate temperature T (T_0 is the reference temperature, or the initial temperature)

$$T = T_0 + \frac{\frac{R}{R_0} - 1}{\alpha} \tag{5.29}$$

The last equation of course is in general true for certain ranges of temperature. So, for this exercise let's just plug in values.

$$T = 20^0C + \frac{\frac{187.4\Omega}{164.2\Omega} - 1}{3.92 \cdot 10^{-3}(^0C)^{-1}} = 56.04^0C \tag{5.30}$$

Probably you ask yourself "How do I measure the resistance change of the platinum?" Excellent question! There are many ways to achieve this goal, and not necesarilly in a platinum thermometer the circuit mentioned in next chapter is used to determine the resistance. Nevertheless, we will study one electric circuit called *Wheatstone bridge* which is widely used to measure an unknown resistance.

5.4 Electric Power

Suppose certain voltage applied to a certain material (either insulator or conductor). Then, electric charges will start to move due to an electric field. The electric field is doing a *work* to move those electric charges, and so it is transferring electric energy to the material. We know that power is the measurement of energy transfer in certain amount of time, therefore we have that

$$P = \frac{dW}{dt} = \frac{dU}{dt} = \frac{d\left(\Delta V Q\right)}{dt} = \Delta V I \tag{5.31}$$

where equation 4.59 for the potential energy was used. Also, recall that work is $W = U$ when taking as reference frame potential energy $U = 0$ at infinity, and taking infinitesimal changes we have $dW = dU$. Now , for the particular case when there is a resistor

$$P = \Delta V I = \left(RI\right) I = I^2 R \tag{5.32}$$

where I would like to emphasise that last equation only applies to resistors. The last equation is called as **Joule effect**. Energy is transferred to the resistor as internal energy , which is the collection of collisions of the electric charges and molecules inside the resistor. When takes place, the resistor then releases energy as heat. So summarizing

$$\boxed{P = \Delta V I} \quad \text{Power delivered to any component submitted under a voltage} \tag{5.33}$$

and

$$\boxed{P = I^2 R} \quad \text{Power delivered to any resistor submitted under a voltage} \tag{5.34}$$

Now, suppose you want to transfer energy kilometers away with conductor wires. What would you increase? The electric current or the voltage? Well, in reality all materials have certain resistance, no matter if they are conductors, there exist certain collisions of the electric charges with the molecules in the material or impurities. Therefore, the power delivered given certain voltage is given by equation 5.33, however you can think equation 5.34 as the power delivered to the internal energy of the wire as heat. Therefore, we want to increase the delivered power in equation 5.33 and reduce the power delivered to the resistance of the wire in equation 5.34. Therefore, the way to achieve this is by increasing the voltage and decreasing the electric current.

Example 4: Cost of electric energy

An electrical heater uses 15.0A when is connected to a line of 120V. What is the required power and how much does it cost a month (take 30 days) if the heater works during $3.0hrs$ per day and the company charges 0.092¢ (cents) per kWh?
Solution: The power can be easily calculated

$$P = \Delta V I = (120V)(15A) = 1800W \tag{5.35}$$

Now, what an electric company charges you is the energy you have used. The energy can be obtained by multiplying the power by the time of usage. The units kWh are energy, and commonly used in electric companies. Hence, calculating the kwh

$$E = 1800W\left(\frac{3h}{1day}\right)\left(30day\right)(162,000Whr)\left(\frac{1kWhr}{1000Whr}\right)(0.092) = 14.904¢ \tag{5.36}$$

where we made a change of units from seconds to hours.

Example 5: Electric Heater

A heater has to increase the temperature of 1.5 kg of water from 10^0C to 50^0C in 10.0 min, while it works at 110V. What is the required resistance of the heater? (Specific heat of water $c = 4186\frac{\mathrm{J}}{\mathrm{kgC^\circ}}$)

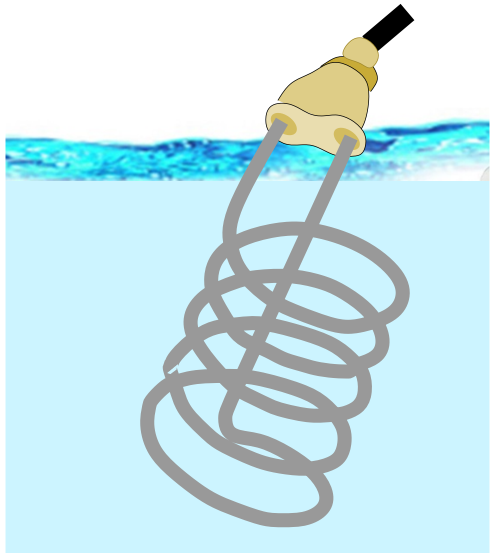


Figure 5.3

Solution:

The energy delivered to the resistor is stored as internal energy. Recall that internal energy is related to the kinetic energy of the molecules that constitute a material. Since the kinetic energy of the molecules of the resistor increases, then the molecules start to collide to each other. From first law of thermodynamics we know that

$$\Delta E_{int} = Q + W \tag{5.37}$$

taking in consideration that all the internal energy is transferred as heat, we have

$$\Delta E_{int} = mc\Delta T \tag{5.38}$$

where the formula for heat calculation was used. Finally, recall that power is in general

$$P = \frac{\text{Energy Transfer}}{\text{Time}} \quad \Rightarrow \tag{5.39}$$

Therefore, if we divide both sides of equation 5.38 by time

$$P = \frac{Q}{t} = \frac{mc\Delta T}{t} \tag{5.40}$$

where P will be the power in equation 5.34, because remember that the energy delivered to the resistor will be stored as internal energy. So , substituing equation 5.34 in P, we have

$$\frac{\Delta V^2}{R} = \frac{mc\Delta T}{t} \quad \Rightarrow R = \frac{\Delta V^2 t}{mc\Delta T}$$

Before we plug in values in the last equation, we convert the minutes to seconds.

$$time = 10min\left(\frac{60s}{1min}\right) = 600sec \tag{5.41}$$

So,

$$R = \frac{(110V)^2(600sec)}{(1.5kg)(4186\frac{J}{kg^0C})(40^0C)} = 28.9\Omega \tag{5.42}$$

Example 6: Burning a fuse

Each device shown in figure 5.4 are connected to the same voltage supply . Calculate the electrical current taken from all the devices . If a fuse is designed for maximum 20 A , Should the fuse burn and would there be current interruption ? (Light bulb 100W, Heater 1800W, Stereo 350W, Hair dryer1200W)

Solution: Given that all devices are connected in parallel, the voltage is exactly the same. Now, we can use equation 5.33 to obtain the electric current that supplies each device

$$P = \Delta VI = \quad \Rightarrow I = \frac{P}{\Delta V}$$

Therefore, we can calculate easily the electric current supplied to each device

$$I_{\text{Light Bulb}} = \frac{P_{\text{Light Bulb}}}{\Delta V} = \frac{100W}{120V} = 0.833\text{A}$$
$$I_{\text{Heater}} = \frac{P_{\text{Heater}}}{\Delta V} = \frac{1800W}{120V} = 15\text{A}$$
$$I_{\text{Stereo}} = \frac{P_{\text{Stereo}}}{\Delta V} = \frac{350W}{120V} = 2.916\text{A}$$
$$I_{\text{Hair}} = \frac{P_{\text{Hair}}}{\Delta V} = \frac{1200W}{120V} = 10\text{A}$$

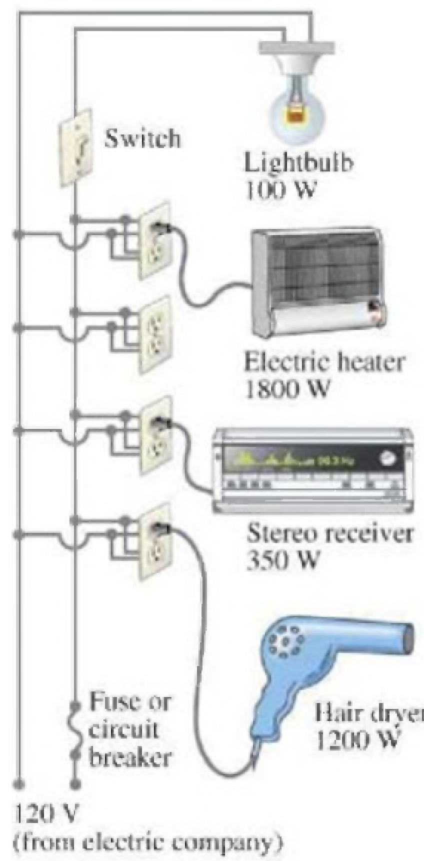


Figure 5.4: Figure taken from reference [8]

The total current is just the sum of all the currents.

$$I_{\text{Total}} = 0.833\text{A}+15\text{A}+2.916\text{A}+10\text{A} = 28.749\text{A}$$

Hence, the fuse will burn ! And the electric current supply will stop immediately,

Chapter 6

Direct Current Circuits

In this chapter , we will make our first circuits with *resistors*, *capacitors* and batteries.

6.1 Electromotive Force (EMF)

Definition 6.1.1 *The electromotive force (abbreviated as emf and represented with the greek letter ε) is the maximum voltage that a battery can supply between its terminals.*

Given that any battery is made of matter, there exists a resistance to the flow of electric charges inside it. This resistance is called as internal resistance (r). When there is a resistance, we know by Ohm's Law that $\Delta V = IR$, so the potential difference due to the resistance of the battery will be $\Delta V = Ir$. Given that the resistance reduces the potential difference between the terminals, we have that the voltage between the terminals of the battery is given by

$$\boxed{\Delta V = \varepsilon - Ir} \tag{6.1}$$

Please do not confuse the emf **with a force!** The units of the emf are volts! Recall the definition, it is the maximum voltage that any battery can supply. Unfortunately for historical reasons it remains in its name $force$ but it does nothing to do with $Newton$ units.

6.1.1 Resistor connections

There are many ways we could connect resistors, but as with capacitors we will study two manners to connect them,

Definition 6.1.2 *When two or more resistors are connected in such a way that the electric current that each resistor carries is exactly the same, we say that they are* ***connected in series*** *(See figure 6.1a)*

Definition 6.1.3 *When two or more resistors are connected in such a way that their voltage ΔV is exactly the same, we say that they are* ***connected in parallel*** *(See figure 6.1b)*

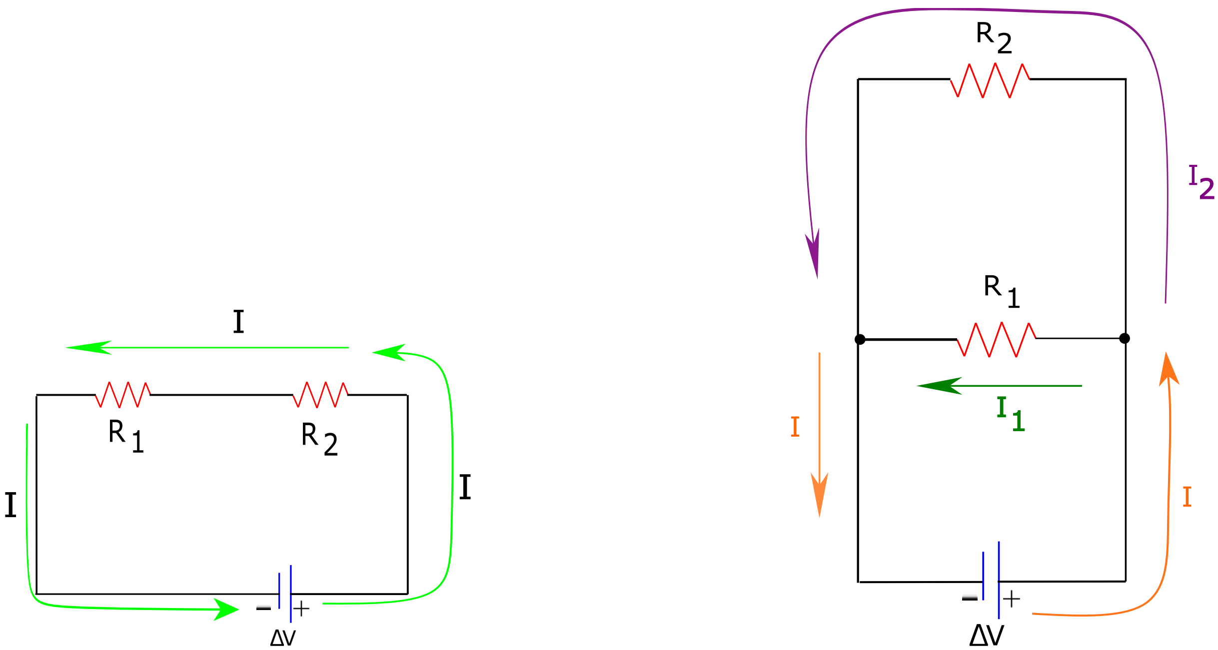


Figure 6.1

From the last definitions, we can calculate equivalent resistances depending how they are connected.

Suppose 2 resistors connected in series as shown in figure 6.1a.The voltage supplied by the battery is the same as the sum of the voltages in each resistor

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{6.2}$$

Now, using Ohm's Law $\Delta V = RI$, therefore

$$R_{eq} I = R_1 I_1 + R_2 I_2 \tag{6.3}$$

However, the electric currents are exactly the same. Hence,

$$R_{eq} = R_1 + R_2 \tag{6.4}$$

The last equation applies for any two pairs of resistors connected in series. In general for any set of N resistors **connected in series, the equivalent resistance is given by**

$$\boxed{R_{eq} = \sum_{i=1}^{N} R_i} \tag{6.5}$$

Now, suppose two resistors connected in parallel as shown in figure 6.1b. The voltage is the same across any resistor. However, the electric current is not exactly the same.

Notice that at any of the two points marked with a black circle (called nodes) in figure 6.1b, the electric charges that are flowing can move in two different directions. Therefore, we can say that the electric current in the right hand conductor that starts in the positive terminal of the battery is splitted in two currents I_1 and I_2, which cross R_1 and R_2 respectively. So, we can say that the electric current that arrives to the right node is the sum of the electric currents that went in both directions

$$I = I_1 + I_2 \tag{6.6}$$

Applying once again Ohm's Law

$$\frac{\Delta V}{R_{eq}} = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2} \tag{6.7}$$

However, the resistors are in parallel, so the voltages are the same. Hence,

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{6.8}$$

The last equation applies for any two pairs of resistors connected in series. In general for any set of N resistors **connected in parallel, the equivalent resistance is given by**

$$\boxed{\frac{1}{R_{eq}} = \sum_{i=1}^{N} \frac{1}{R_i}} \tag{6.9}$$

Example 1: Analyzing a circuit with resistors

Three resistors with resistances 1.60Ω, 2.40Ω, and 4.80Ω are connected in parallel to a battery of 28.0 V with despreciable internal resistance. Find,

1. The equivalent resistance of the combination
2. The current through each resistor
3. The total current through the battery
4. The voltage through each resistor
5. The delivered power in each resistor
6. The delivered power to all the circuit resistors

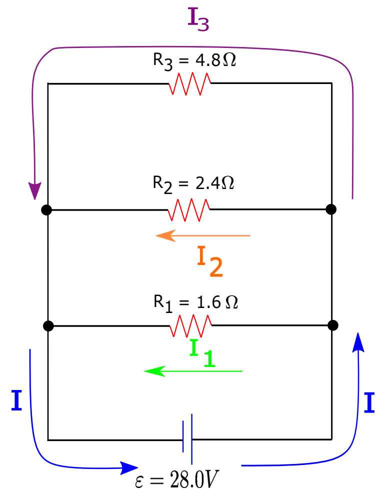


Figure 6.2

Solution:
Given that the three resistors are in parallel, we can use equation 6.9, therefore

$$\frac{1}{Req} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3} \Longrightarrow Req = \frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2} \quad (6.10)$$

Substituting values in the last equation

$$R_{eq} = \frac{1.6\Omega \cdot 2.4\Omega \cdot 4.8\Omega}{2.4\Omega \cdot 4.8\Omega + 1.6\Omega \cdot 4.8\Omega + 1.6\Omega \cdot 2.4\Omega} = 0.8\Omega \quad (6.11)$$

Now, using Ohm's Law

$$V = RI \quad \Rightarrow \quad (6.12)$$

We can obtain easily the electric current carried by each resistor (recall that the voltage in this case will be the same for the three resistors because they are connected in parallel)

$$I_1 = \frac{\Delta V_1}{R_1} = \frac{28.0V}{1.6\Omega} = 17.5A \quad (6.13)$$

$$I_2 = \frac{\Delta V_2}{R_2} = \frac{28.0V}{2.4\Omega} = 11.66A \quad (6.14)$$

$$I_3 = \frac{\Delta V_3}{R_3} = \frac{28.0V}{4.8\Omega} = 5.833A \quad (6.15)$$

The electric current through the battery is the sum of the three electric currents I_1, I_2 and I_3

$$I = I_1 + I_2 + I_3 \Rightarrow I = 17.5\text{A} + 11.66\text{A} + 5.833\text{A} = 34.993\text{A} \quad (6.16)$$

Finally, the power delivered to each resistor can be easily found by using equation 5.33

$$P_1 = \Delta V_1 I_1 = (28.0V)(17.5A) = 490W \text{ , or , } P_1 = I_1^2 R_1 = (17.5\text{A})^2(1.6\Omega) = 490W \quad (6.17)$$

$$P_2 = \Delta V_2 I_2 = (28.0V)(11.66A) = 326.48W \text{ , or , } P_2 = I_2^2 R_2 = (11.66A)^2(2.4\Omega) = 326.48W \tag{6.18}$$

$$P_3 = \Delta V_3 I_3 = (28.0V)(5.833A) = 163.329W \text{ , or , } P_3 = I_3^2 R_3 = (5.833\text{A})^2(4.8\Omega) = 163.329W \tag{6.19}$$

Of course in the last calculation you could have also used $P = \frac{(\Delta V)^2}{R}$. Now,the power delivered to the equivalent resistor can be found by summing the power delivered to each resistor

$$P_{\text{Tot}} = P_1 + P_2 + P_3 \Rightarrow P_{\text{Tot}} = 979.8\text{W} \tag{6.20}$$

Or by using the equivalent resistance, with equation 5.34

$$P = I^2 R_{eq} = (34.993\text{A})^2\,(0.8\Omega) = 979.8\text{W} \tag{6.21}$$

Equivalently, with equation 5.33

$$P = \Delta V I = (34.993\text{A})\,(28.0\text{V}) = 979.8\text{W} \tag{6.22}$$

Example 2: Another equivalent resistance example

Calculate the equivalent resistance of the circuit and find the current passing through each resistor. The battery has negligible internal resistance.

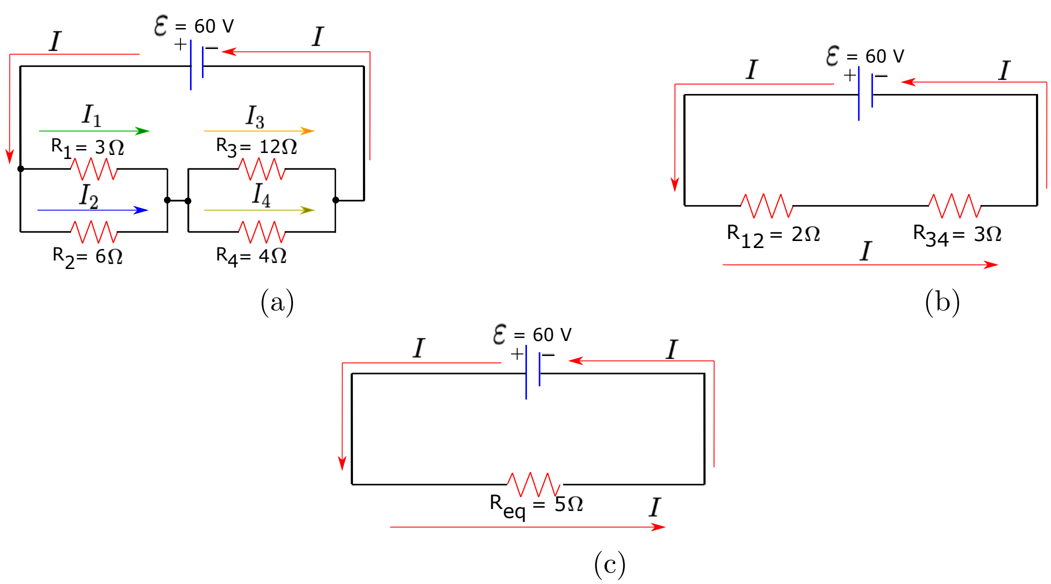


Figure 6.3

Solution:

First we calculate the equivalent resistance. We have that resistorsR_1, R_2 are in parallel

, so

$$\frac{1}{R_{12}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_2 + R_1}{R_1 R_2} \Rightarrow \tag{6.23}$$

$$R_{12} = \frac{R_1 R_2}{R_2 + R_1} = \frac{(3\Omega)(6\Omega)}{9\Omega} = 2\Omega \tag{6.24}$$

And, resistors R_3 and R_4 also are also in parallel, therefore

$$R_{34} = \frac{R_3 R_4}{R_3 + R_4} = \frac{(12\Omega)(4\Omega)}{16\Omega} == \frac{48\Omega^2}{16\Omega} = 3\Omega \tag{6.25}$$

Finally , R_12 and R_34 are in series. Hence, the equivalent resistance is

$$R_{1234} = 3\Omega + 2\Omega = 5\Omega \tag{6.26}$$

Since we have covered all the resistors in the circuit, we can happily say that the equivalent resistance of all the circuit is

$$R_{eq} = 5\Omega \tag{6.27}$$

Now, by using equation 5.21 (Ohm's Law),

$$I = \frac{\Delta V}{R_{eq}} = \frac{60\text{V}}{5\Omega} = 12\text{A} \tag{6.28}$$

So, if we want to know the electric current carried by each resistor, we need to calculate the voltages across each resistor ΔV_1,ΔV_2, ΔV_3 and ΔV_4. However, since R_1 is in parallel with R_2 and R_3 is in parallel with R_4, we have that

$$\Delta V_{12} = \Delta V_1 = \Delta V_2 \qquad \text{and } \Delta V_{34} = \Delta V_3 = \Delta V_4 \tag{6.29}$$

so we first calculate the voltages ΔV_{12} and ΔV_{34}. Now,noticing that resistors R_{12} and R_{34} are in series, the electric currents through resistors R_{12} and R_{34} are the same as the current I of the equivalent resistor R_{eq}

$$I = I_{12} = I_{34} \tag{6.30}$$

Therefore the voltages across resistors R_{12} and R_{34}

$$\Delta V_{12} = R_{12} I = (2\Omega)(12\text{A}) = 24V \text{ , } \Delta V_{34} = R_{34} I = (3\Omega)\,(12\text{A}) = 36\text{V} \tag{6.31}$$

Once you know ΔV_{12}, you could have also calculated ΔV_{34} as

$$\Delta V_{34} = \Delta V - \Delta V_{12} = 60 - 24 = 36\text{V} \tag{6.32}$$

There is not only one way to calculate the voltages and currents, do whatever strategy that comes to your mind (of course while it makes sense and trying to save time!). So, finally

$$I_1 = \frac{\Delta V_{12}}{R_1} = \frac{24V}{3\Omega} = 8A \tag{6.33}$$

$$I_2 = \frac{\Delta V_{12}}{R_2} = \frac{24V}{6\Omega} = 4A \tag{6.34}$$

$$I_3 = \frac{\Delta V_{34}}{R_3} = \frac{36V}{12\Omega} = 3A \tag{6.35}$$

$$I_4 = \frac{\Delta V_{34}}{R_4} = \frac{36V}{4\Omega} = 9A \tag{6.36}$$

6.2 Kirchhoff's Laws

See the circuit shown in figure 6.4. Notice that as far as we know by now, with the knowledge we have discussed of electric circuits, we cannot find currents and voltage across each resistor. So far, we have reduced the resistors to a one single equivalent resistor, then find the electric current I and see how it was splitted in the different branches of the circuit. However, since we have several batteries, we cannot make several series nor parallel combinations to obtain just one equivalent resistor. We do not know how to deal with several batteries and resistors in the same circuit. Let's add to our toolkit a powerful set of laws which will help us to solve other kind of electric circuits, the Kirchhoff Laws.

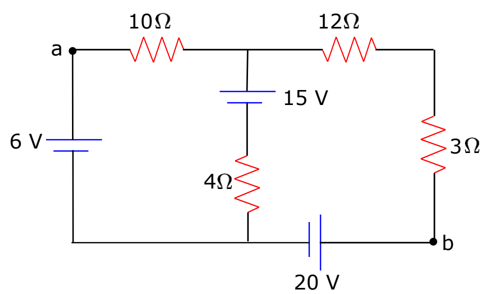


Figure 6.4

Before we establish the Kirchhoff Laws, let's state two important definitions

Definition 6.2.1 *A node is a point where two or more circuit elements meet.*

Definition 6.2.2 *A loop is any closed path in a circuit through selected basic circuit elements, where the traced path is without passing through any intermediate node more than once.*

From the definition of a node, actually we can obtain another definition for a series connection, for any component.

Definition 6.2.3 *When just two elements connect at a single node, the elements are said to be in series.*

Now, we define the Kirchhoff Laws

1. The algebraic sum of the currents in any node is zero. (Electric current that goes in a node is the same as the one that goes out from the node.)
$$\sum_{node} I = 0 \tag{6.37}$$

2. The algebraic sum of the potential differences in any loop must be equal to zero.
$$\sum_{\text{loop}} \Delta V = 0 \tag{6.38}$$

The first rule holds by charge conservation. The charge that enters a node is the same to the one that goes out from the node. The second Kirchhoff rule is due to energy conservation. When having the same potential energy in a beginning, it must be the same if we return to the same point.

Now, when applying Kirchhoff Laws, the following convention must be used

1. Electric current direction is such that it starts from the high-potential end of a resistor toward the low-potential end. Therefore, if a resistor is traversed in the direction of the current, the potential difference across the resistor is $\Delta V = -IR$.

2. If a resistor is traversed in the direction opposite the current, the potential difference is $\Delta V = +IR$.

3. If a source of voltage (assumed to have zero internal resistance, so considering the emf) is traversed from negative terminal to positive terminal, the potential difference $\Delta V = +\varepsilon$

4. If a source of voltage (assumed to have zero internal resistance, so considering the emf) is traversed from positive terminal to negative terminal, the potential difference $\Delta V = -\varepsilon$

Comments on the convention rules mentioned

- Recall firstly that by convention, the electric current is the flow of positive electric charges. Secondly, remember from our discussion of electric potential in chapter 3, positive charges move towards the direction where the voltage is negative (electric potential decreases). So, if a resistor is traversed in the same direction of the current, the voltage must be negative (convention rule 1). Otherwise, if a resistor is traversed in opposite direction the electric current, the voltage must increase (convention rule 2).

- The positive terminal of battery is at higher potential than the negative terminal.

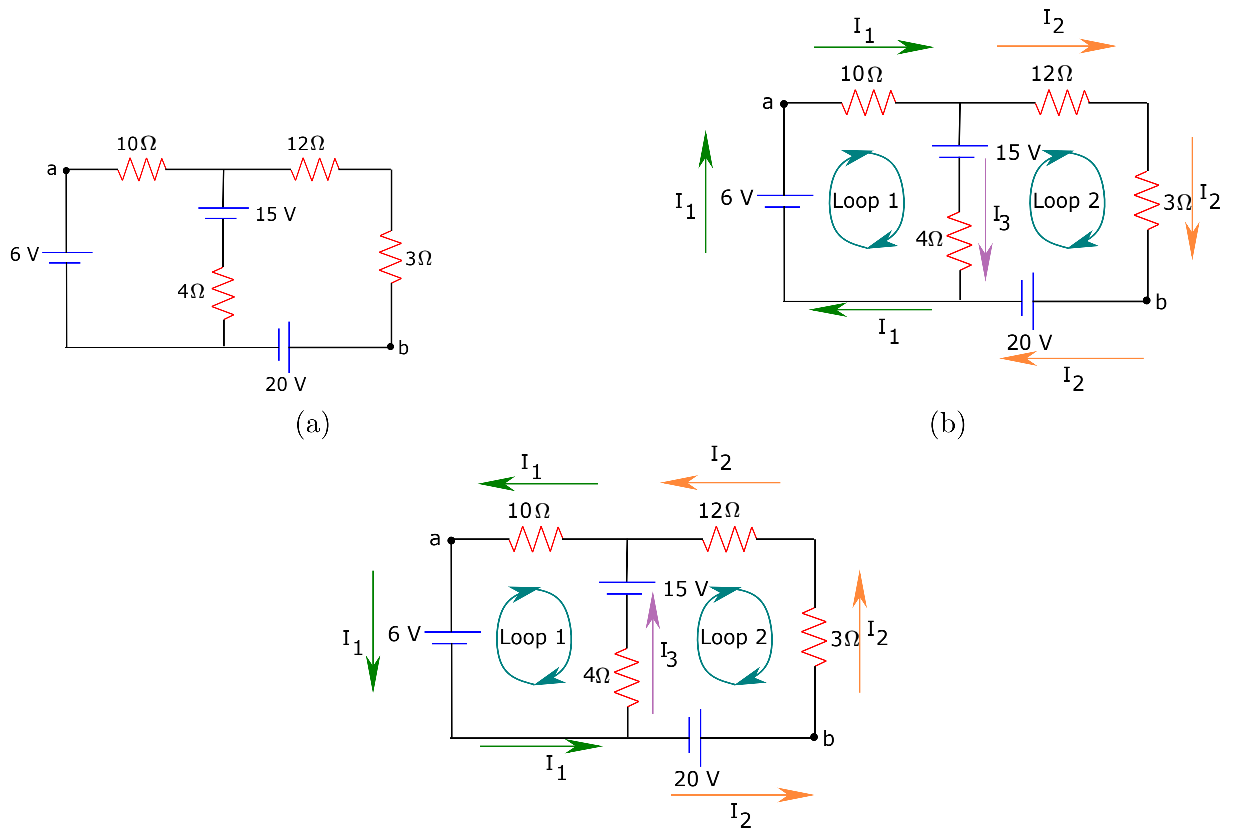
Therefore, if a battery is traversed from negative to positive, you are increasing the potential. Otherwise, if a battery is traversed from positive to negative, you are decreasing potential. The arguments mentioned are the convention rules 3 and 4.

The direction of the electric currents, and the direction you start moving in your loop is arbitrary. Nevertheless, you must respect all the convention rules and the Kirchhoff Laws. **In case you obtain an electric current with negative sign, it means that the electric current is actually flowing in opposite direction**.
Finally, you need N independent equations to solve the circuit with N different electric currents. Once you know the currents, you can use Ohm's Law and obtain the voltage across each resistor.

Let's solve some exercises to exemplify how to use Kirchhoff Laws

Example 3: Applying Kirchhoff Laws example

(a) Calculate the currents that exists in the all branches of the circuit shown in figure 6.5a.
(b) Find the magnitude of the potential difference between the points "a" and "b".



(c)

Figure 6.5

Solution:
From loop 1 (see figure 6.5b), we obtain

$$-10I_1 - 15 - 4I_3 + 6 = 0 \tag{6.39}$$

From loop 2 (see figure 6.5b) , we obtain

$$15 - 12I_2 - 3I_2 - 20 + 4I_3 = 0 \tag{6.40}$$

From the currents in the nodes,

$$I_1 = I_2 + I_3 \tag{6.41}$$

So, we have three equations, three unknown variables. We must be able to solve the problem. Now, just for this exercise I show explicitly the algebra to isolate I_1, I_2 and I_3. For next exercises, since this is just high school algebra, I leave it up to you.

Now, simplifying just quite equation 6.39, we have

$$-10I_1 - 9 - 4I_3 = 0 \tag{6.42}$$

Using I_1 in equation 6.41 and substituting in the last equation

$$-10(I_2 + I_3) - 9 - 4I_3 = 0 \Longrightarrow -10I_2 - 14I_3 - 9 = 0 \tag{6.43}$$

Hence,

$$I_2 = -\frac{14}{10}I_3 - \frac{9}{10} \tag{6.44}$$

Substituting what we have found in the last equation for I_2 in equation 6.40, we have

$$-15\left(-\frac{14}{10}I_3 - \frac{9}{10}\right) - 5 + 4I_3 = 0 \Longrightarrow \left(\frac{15 \cdot 14}{10} + 4\right) I_3 = 5 - \frac{15 \cdot 9}{10} \tag{6.45}$$

Therefore,

$$I_3 = \left(5 - \frac{15 \cdot 9}{10}\right) \Big/ \left(\frac{15 \cdot 14}{10} + 4\right) = -0.34\text{A} \tag{6.46}$$

Once we know I_3, we can obtain I_2 with equation 6.44

$$I_2 = -\frac{14}{10} \cdot (-0.34\text{A}) - \frac{9}{10} = -0.424\text{A} \tag{6.47}$$

Finally, once we know currents I_3 and I_2, current I_1 can be found by using equation 6.41

$$I_1 = I_2 + I_3 = -0.424\text{A} - 0.34\text{A} = -0.764\text{A} \tag{6.48}$$

Summarizing, the currents found were

$$\begin{aligned} I_1 &= -0.764\text{A} \\ I_2 &= -0.424\text{A} \\ I_3 &= -0.34\text{A} \end{aligned} \tag{6.49}$$

All minus signs means that the electric currents actually travel in the opposite direction. So, from the circuit in figure 6.5b where we gave an arbitrary direction to the electric currents to solve the problem, must be corrected to correct direction of the currents shown in the circuit of figure 6.5c
Finally calculating the potential difference. We can start from node a, go through the battery of 6V and then through the battery of 20V. So, using now the correct direction of the currents (figure 6.5c), we obtain

$$\Delta V_{ab} = -6\text{V} + 20\text{V} = 14\text{V} \tag{6.50}$$

Also, you could have calculated the potential difference between node a and b by going through the circuit elements that currents I_1 and I_2 crosses (see figure 6.5c). In such case,

$$\Delta V_{ab} = 10I_1 + 15I_2 = 10\,(0.764) + 15\,(0.424\text{A}) = 14\text{V} \tag{6.51}$$

where notice that we do not use the negative signs any more because now we are taking the correct direction of the currents ! And as expected we obtain the same result by taking either two paths.

Example 4: Another example applying Kirchhoff Laws

Calculate the electric currents I_1, I_2 and I_3 of the following circuit. Find the potential difference ΔV_{ab} and ΔV_{cd}.

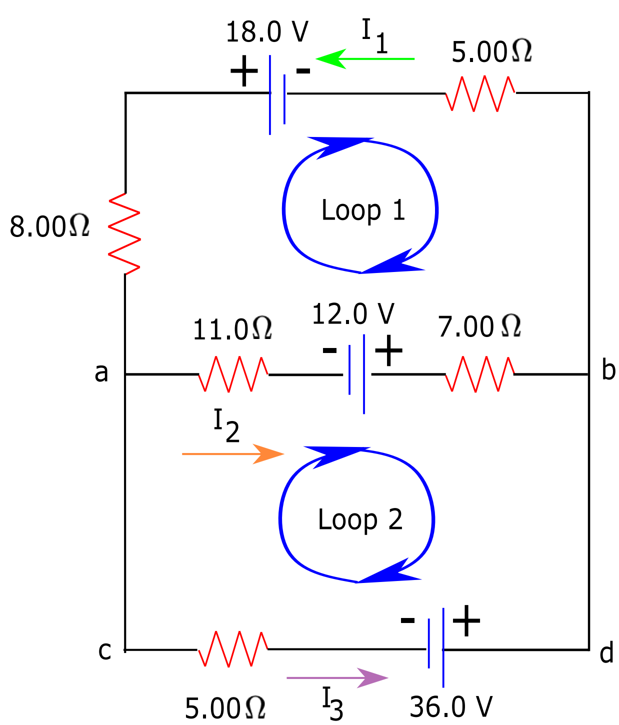


Figure 6.6

Solution:
From loop one we obtain the following equation

$$-18 + 5I_1 + 7I_2 - 12 + 11I_2 + 8I_1 = 0 \tag{6.52}$$

By taking the direction by the blue arrows, as how we will proceed to move in the loop one, we have that when we transverse the battery of $18V$ we go from positive terminal to negative terminal, so we we pick a minus sign (first term in the last equation). After crossing the battery we cross the resistor of 5Ω, given that we go in opposite direction of the electric charge I_1, we pick a positive sign (second term in the last equation $+5I_1$). We continue up to node b and we move to the left to the 7Ω resistor. Now we are moving against the current I_2, so we pick a positive sign (third term of the last equation $+7I_2$). Moving on , we now face the battery of 12V, we move from positive terminal to negative terminal, hence we pick a negative sign (fourth term of the last equation -12). When we transverse the resistor of 11Ω we move against the current I_2, so the sign must be positive (fifth term of the last equation $+11I_2$). Finally, we arrive to node a, and we go up to the 8Ω resistor. Since, we move against the current I_1, we pick a positive sign (last term of the latter equation $+8I_1$). Since we have closed the loop, we have that the sum of all the voltages is zero.

From loop two we obtain the following equation

$$-11I_2 + 12 - 7I_2 - 36 + 5I_3 = 0 \tag{6.53}$$

analogue to the loop 1 , we followed the convention rules to obtain the last equation, and used that the sum of all the voltages is zero.
Now, placing in node a or node b, we obtain the following equation

$$I_1 = I_2 + I_3 \tag{6.54}$$

For instance in node a, the current that goes into the node is electric current I_1, and that current then is separated into two currents I_2 and I_3. In node b, the currents that go into the node are currents I_2 and I_3, which joins into one new electric current I_1 which goes out of the node.

So, we have three independent equations, and three unknown variables (currents I_1,I_2 and I_3). So, we have the enough number of equations to solve the problem. By just basic algebra (isolate one equation and substitute in other), we obtain the following set of electric currents

$$\begin{aligned} I_1 &= 2.884319\text{A}, \\ I_2 &= -0.416452\text{A} \\ I_3 &= 3.3007771\text{A} \end{aligned} \tag{6.55}$$

where of course it is unnecessary and even redundant to keep all decimals, however just for sake that you can compare results, we show the exact result. Now, notice that current

I_2 is negative, which means that its correct direction is opposite to the one that we chose from a first place.

Finally, we are asked to calculate the potential difference between points a and b. If we start from node a, then cross the battery of $12V$ and finally transverse the resistor of 7Ω, we have the following equation

$$\Delta V_{ab} = 11I_2 + 12\text{V} + 7I_2 = 19.49\text{V} \tag{6.56}$$

We could also follow the path going though resistor of 8Ω, the battery of $18V$, and finally cross the resistor of 5Ω to arrive to the node b. If we follow such path,

$$\Delta V_{ab} = 8I_1 - 18 + 5I_1 = 19.49\text{V} \tag{6.57}$$

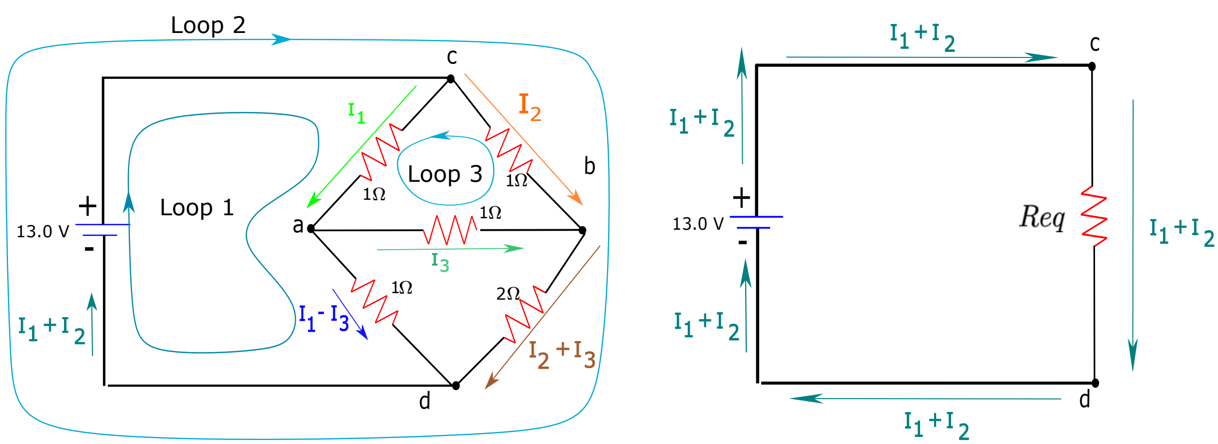
where as expected we have obtained the same result! Finally, potential difference ΔV_{cd} must be equal to ΔV_{ab}, because notice that left side and right side are connected to the same potential. Let's figure out if this is actually true.

$$\Delta V_{cd} = -5I_3 + 36\text{V} = 19.49\text{V} \tag{6.58}$$

So, we obtained what we expected!

Example 5: A nice trick to find equivalent resistance using Kirchhoff Laws

Calculate the equivalent resistance of the resistors shown in the following circuit.



(a) (b)

Figure 6.7

Solution:
We cannot directly calculate the equivalent resistance of the circuit by making arrangements of series and parallel equivalent resistors. Notice that in node c (see figure 6.7a),

the current splits into I_1 and I_2, hence the resistors of 1Ω at the top of the circuit are not in series. They are neither in parallel because another resistor of 1Ω is between their connections. Something similar happens with all resistors, so there is no way to connect them in series or parallel and finish up with just one equivalent resistor.

Even though we cannot reduce directly the set of resistors, we can make a smart move by applying Kirchhoff! Remember that an equivalent resistor is the same as saying that a complete circuit of resistors is substituted by another resistor with a given resistance R_{eq}. So, the trick we are applying is using this idea, we substitute the whole set of resistors with one equivalent resistor. The current that crosses the equivalent resistor is the same as the one that enters in node c (see figure 6.7b). Once we know that current we can know the equivalent resistance by using Ohm's Law $\Delta V = R_{eq} I$.
So , summarizing the procedure, we apply the Kirchhoff rules, and calculate the electric current in the battery I, and then by ohm's law the voltage of the battery is the same as $13\text{V} = R_{eq} I$, so we isolate R_{eq}.

So, let's apply Kirchhoff laws and we obtain the following set of equations

$$\begin{aligned} 13 - I_1 - 1 \cdot (I_1 - I_3) = 0 \qquad & \text{from Loop 1} \\ -1 \cdot I_2 - 2 \cdot (I_2 + I_3) + 13 = 0 \qquad & \text{from Loop 2} \\ -I_1 - I_3 + I_2 = 0 \qquad & \text{from Loop 3} \end{aligned} \tag{6.59}$$

Solving the system of equations we obtain

$$\begin{aligned} I_1 &= 6\text{A} \\ I_2 &= 5\text{A} \\ I_3 &= -1\text{A} \end{aligned} \tag{6.60}$$

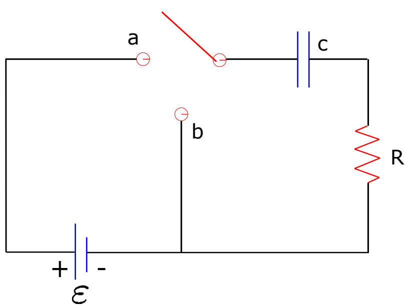
Now, the electric current that goes through the equivalent resistor and the battery is $I = I_1 + I_2$. Therefore, by using Ohm's Law , $13\text{V} = R_{eq}(I_1 + I_2)$, and by isolating the equivalent resistance we have

$$R_{eq} = \frac{13\text{V}}{6\text{A} + 5\text{A}} = 1.18\Omega \tag{6.61}$$

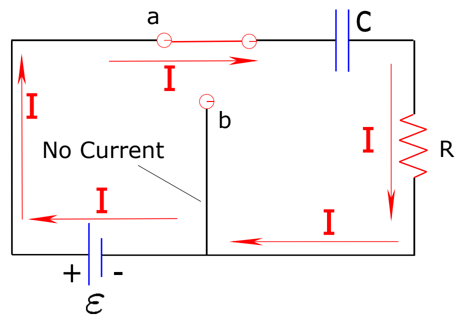
Isn't this trick nice?! And this gets even better! What if I told you that I just invented the battery of 13V to obtain the equivalent resistance R_{eq}? Notice that the equivalent resistance only depends on the resistors that constitute it. No matter what currents crosses it, the resistance is the same! So, this is even better! Whenever you want to know the equivalent resistance of a set of resistors, and you have a complicated circuit, which you cannot split it in series and parallel subsets, or it gets too nasty. Connect the whole circuit to an invented battery of arbitrary emf. Solve the currents with Kirchhoff, and almost by magic you will obtain the equivalent resistance by using at the end Ohm's Law.

6.3 RC Circuits

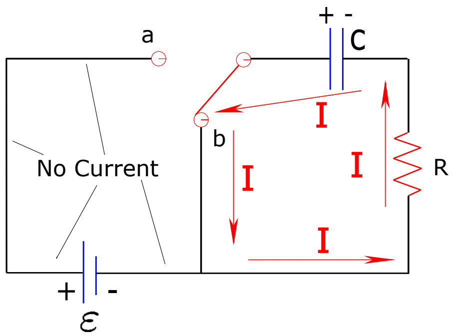
In this section we will study the so called *RC* circuits, where there are resistors and a capacitors in the circuit . We will study two possible scenarios, when there is a battery connected to a resistor and the capacitor in series; and when the circuit only has the resistor and the capacitor in series.



(a) No current in the whole circuit, because all the system is open, no closed path exists in the connections of the elements of the circuit.



(b) No current through conductor b, because that part of the circuit is open. We have a circuit with a resistor and capacitor in series, connected to a emf



(c) No current through the battery, because that part of the circuit is open. We have a circuit with a resistor and capacitor in series, without emf

Figure 6.8: Whenever there is an open circuit (the connections of the circuit do not make a closed path), there is no electric current.

1. When there is a battery in a *RC* circuit as shown in figure 6.8b, at time $t = 0$ when you have just connected the battery to the capacitor and the resistor (closed the switch in point a in figure 6.8a and obtain a configuration as shown in figure 6.8b), the capacitor is neutrally charged and an electric current starts to flow in the circuit. After certain time t the capacitor has electric charge stored now. So, the electric current that is flowing around the circuit now is less! There will come a time that the electric current is zero, when the capacitor is fully charged! Probably the easiest way to visualize this is thinking about the capacitor as a place where you can store electric charges. By no means it is possible to continue storing electric

charge infinitely, there is a maximum electric charge that can be stored. Once the capacitor has reached the maximum amount of electric charge that can store, no more electric charge can continue flowing! Or in other words, an electric current to cross all elements of the circuit. From the perspective of voltage, once the capacitor gets electric charge, between the conductors of the capacitor and the terminals of the battery exists a less potential difference. A decrease of the potential difference means less electric field inside the conductors. A less electric field means less force that moves the charges, less movement of electric charges means less current. Now, is there an equation that describes us what we have mentioned quantitatively? Indeed, there is! Let's use Kirchhoff's Laws to the circuit shown in figure 6.8b. So, we obtain the following equation

$$\varepsilon - \frac{q}{C} - RI = 0 \tag{6.62}$$

where we used the convention rules, and the voltage across a capacitor , by using the formula of capacitance $C = \frac{Q}{\Delta V}$. Now, recall that the electric current is $I = \frac{dq}{dt}$. So , the lat's equation reads as

$$\varepsilon - \frac{q}{C} - R\frac{dq}{dt} = 0 \tag{6.63}$$

so we have obtained a linear, first order, ordinary differential equation, that will be separable. By just some algebra, last equation becomes

$$\frac{dq}{dt} = \frac{\varepsilon}{R} - \frac{q}{RC} \Longrightarrow \frac{dq}{dt} = \frac{\varepsilon C - q}{RC} \tag{6.64}$$

Hence, by separation of variables ,we have

$$\int_0^Q \frac{dq}{q - \varepsilon C} = -\int_0^t \frac{dt}{RC} \tag{6.65}$$

where the limits of integration follow because at time $t = 0$ the electric charge in the capacitor is zero, and after certain time t the capacitor will have certain electric charge Q. Integrating both sides of last equation and doing some algebra

$$\ln \frac{Q - C\varepsilon}{-\varepsilon C} = -\frac{t}{RC} \Longrightarrow \tag{6.66}$$

$$\boxed{Q(t) = C\varepsilon\left(1 - e^{-t/RC}\right) \qquad \text{for charging capacitor}} \tag{6.67}$$

where we have written explicitly that the electric charge is a function of time. Notice that as $t \to \infty$ the capacitor will acquire its maximum amount of electric charge. Wait! So I need to wait my entire life to charge the capacitor ? Well theoretically we need an infinite time to obtain the maximum electric charge, however, as it will turn out the time to obtain a very good approximate to the maximum electric charge is small enough, so that at finite times we can consider that the capacitor has fully charged. So, we can also write equation 6.67 as

$$\boxed{Q(t) = Q_{\max}\left(1 - e^{-t/RC}\right) \qquad \text{for charging capacitor}} \tag{6.68}$$

where

$$\boxed{Q_{\max} = C\varepsilon \qquad \text{for charging capacitor}} \tag{6.69}$$

Now, recalling that the electric current is $I = \frac{dQ}{dt}$, we have that the electric current is now

$$\boxed{I(t) = \frac{\varepsilon}{R}e^{-t/RC} \qquad \text{for charging capacitor}} \tag{6.70}$$

where once again we have written explicitly that the function depends of time. The maximum electric current will be at $t = 0$. when the capacitor is neutrally charged! So the electric charge flows with no opposition in the circuit, and as the electric charge gets stored in the capacitor, less and less electric charge will flow in the circuit. So, we can also write equation 6.70 as

$$\boxed{I(t) = I_{max}e^{-t/RC} \qquad \text{for charging capacitor}} \tag{6.71}$$

where the the maximum electric current is

$$\boxed{I_{max} = I(0) = \frac{\varepsilon}{R} \qquad \text{for charging capacitor}} \tag{6.72}$$

Now, the voltage across the capacitor, recalling that $C = \frac{Q}{\Delta V}$, we have

$$\boxed{\Delta V(t) = \varepsilon\left(1 - e^{-t/RC}\right) \qquad \text{for charging capacitor}} \tag{6.73}$$

The potential difference that is maximum across the capacitor is

$$V_{\max} = \varepsilon \tag{6.74}$$

i.e. the maximum voltage across the capacitor will be emf of the battery !
Finally, we could ask ourselves how does the energy stored in the capacitor changes with respect time? With a substitution using equation 4.59

$$U = \frac{Q^2}{2C} = \frac{C^2\varepsilon^2\left(1 - e^{-t/RC}\right)^2}{2C} \tag{6.75}$$

Therefore,

$$\boxed{U = \frac{C\varepsilon^2\left(1 - e^{-t/RC}\right)^2}{2} \qquad \text{for charging capacitor}} \tag{6.76}$$

where the maximum amount of energy stored in the capacitor, once again will be when the time $t \to \infty$. Hence,

$$\boxed{U = U_{max}\left(1 - e^{-t/RC}\right)^2 \qquad \text{for charging capacitor}} \tag{6.77}$$

where

$$\boxed{U_{max} = \frac{C\varepsilon^2}{2} \qquad \text{for charging capacitor}} \tag{6.78}$$

2. Now, suppose another scenario. Suppose that you charge a capacitor up to a certain electric charge Q_0, so you have connected a battery, a resistor and the capacitor as in figure 6.8b. Then, you remove the battery (or equivalently you close the switch from point a to point b in figure 6.8b and obtain configuration shown in figure 6.8c) and maintain the capacitor connected to the resistor. The electric circuit with no battery is equivalent to the one shown in figure 6.8c, because the battery does not participate in the circuit, the battery part of the circuit is an open circuit, so no current flows through the battery. Now, since there is no battery, there is no electric field pushing the electrons against the negative conductor of the capacitor, and they will start to flow in the circuit. Inevitably, when you have removed the battery, the electric charge will start to flow from the capacitor across the resistor. There will come certain time t when all the electric charge Q_0 is not any more in the capacitor and it becomes neutrally charged. So, using equation 6.63, but considering the case there is no efm we have,

$$-\frac{q}{c} - RI = 0 \Longrightarrow \frac{dq}{dt} = -\frac{q}{RC} \Longrightarrow \int_{Q_0}^{Q} \frac{dq}{q} = -\int_0^t \frac{dt}{RC} \tag{6.79}$$

where the integration limits follow since at time $t = 0$ the capacitor has its initial electric charge Q_0, and at after certain time t the capacitor is left with some electric charge Q. So, integrating last equation we obtain

$$\ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{RC} \tag{6.80}$$

Hence,

$$\boxed{Q(t) = Q_0 e^{-t/RC} \qquad \text{for discharging capacitor}} \tag{6.81}$$

where we have explicitly written the electric charge dependent of time. Now, the electric current (recall $I = \frac{dq}{dt}$)

$$\boxed{I(t) = -\frac{Q_0}{RC} e^{-t/RC} \qquad \text{for discharging capacitor}} \tag{6.82}$$

where the minus sign just means that the electric current must be flowing in the opposite direction that when a capacitor is charging! (see figures 6.8b and 6.8c) We have once again written explicitly that the electric current depends on time. We could write the electric current also as

$$\boxed{I(t) = I_0 e^{-t/RC} \qquad \text{for discharging capacitor}} \tag{6.83}$$

where the initial electric current is

$$\boxed{I_0 = I(0) = \frac{Q_0}{RC}} \tag{6.84}$$

the minus sign will be in general ignored, given that if we shift the direction of the current , it becomes positive.

Now, the voltage across the capacitor, by using $C = \frac{Q}{\Delta V}$

$$\boxed{\Delta V(t) = \frac{Q_0}{C} e^{-t/RC} \qquad \text{for discharging capacitor}} \tag{6.85}$$

where the voltage dependence of time was written explicitly. We could also say that the initial voltage (when $t = 0$) across the capacitor is

$$\Delta V_0 = \frac{Q_0}{C} \tag{6.86}$$

and therefore equation 6.85 becomes

$$\boxed{\Delta V(t) = \Delta V_0 e^{-t/RC}} \tag{6.87}$$

Finally, the energy stored in the capacitor, (using equation 4.59)

$$U = \frac{Q^2}{2C} = \frac{Q_0^2 e^{-2t/RC}}{2C} \tag{6.88}$$

Therefore,

$$\boxed{U = U_0 e^{-2t/RC} \qquad \text{for discharging capacitor}} \tag{6.89}$$

where

$$\boxed{U_0 = \frac{Q_0^2}{2C} \qquad \text{for discharging capacitor}} \tag{6.90}$$

is the initial energy stored in the capacitor.

Now, notice that either for charging or discharging capacitor, there is a constant repeated several times.

$$\boxed{\tau = RC} \tag{6.91}$$

which we call simply as **time constant** of the RC circuit. Notice, that the units of RC are seconds! You can notice this simply by observing that a factor t/RC exists in the exponent of the exponential in many of the last equations, so the units must be cancelled out to be able to calculate the exponential. (You cannot have $e^{\text{any units}}$. The exponent of the exponential must be dimensionless!). Of course also by plugging in the units,

$$[RC] = \Omega \cdot \mathrm{F} = \frac{\mathrm{V}}{\mathrm{A}}\frac{\mathrm{C}}{\mathrm{V}} = \frac{\mathrm{Cs}}{\mathrm{C}} = \mathrm{s} \tag{6.92}$$

The importance of the time constant is that it determines how quickly will a capacitor discharge and how fast it will charge. To notice this let's do a quick calculation, suppose that you connect a capacitor to a battery, after a time $t = \tau$ (using equation 6.68)

$$Q(t = \tau) = Q_{max}\left(1 - e^{-\tau/\tau}\right) = Q_{max}\left(1 - e^{-1}\right) \approx 0.63 Q_{max} \tag{6.93}$$

the capacitor has 63% of the maximum amount of charge it can store!

Now, suppose a capacitor is discharging, no battery is connected in the circuit and after a time $t = \tau$, we have (using equation 6.81)

$$Q = Q_0 e^{-\tau/\tau} = \frac{Q_0}{e} \approx 0.36 Q_0$$

the capacitor has lost more than the 60% of the electric charge that it had stored!

We show in the following table some approximate percentages of the electric charge when discharging or charging a capacitor after certain time in terms of τ. Notice that after 5τ practically we have stored all the electric charge that can be stored in the capacitor! In practice, we say that it takes a finite time to charge the capacitor, even though in principle it would take a infinite time to fully charge it. Also, notice that after a time $t = 5\tau$, the capacitor has practically lost all the initial electric charge that it stored! So, also in practice we say that after certain finite time the capacitor is discharged, when in principle by equation 6.81, the stored charge $Q_0 \to 0$ as $t \to \infty$

Time	% of Q_{max} when charging	% of Q_0 when discharging
1τ	63.2%	36.8%
2τ	86.5%	13.5%
3τ	95.0%	5.0%
4τ	98.2%	1.8%
5τ	99.3%	0.7%
6τ	99.8%	0.3%

Example 6: Solving our first charging RC circuit

The capacitance of the capacitor in the shown circuit is 0.5μF, the resistance of the shown resistor is $30\text{k}\Omega$ and the emf of the battery is $\varepsilon = 15$V. When the switch is closed, determine

1. The time constant τ

2. The maximum charge that can be stored in the capacitor
3. The time that lasts to reach the 95% of its maximum electric charge
4. The electric current I when the charge Q is one quarter of its maximum value
5. The maximum electric current
6. The electric charge Q when I is 40% of its maximum value

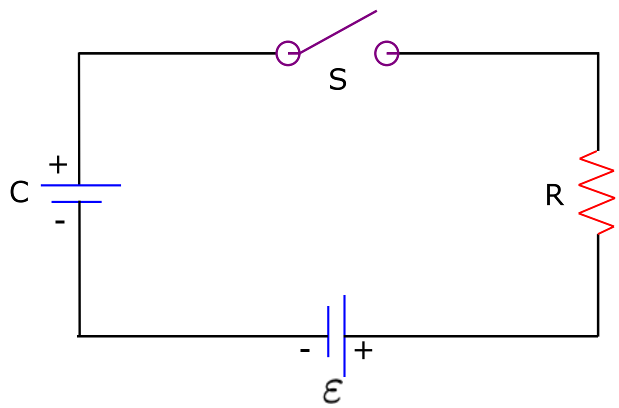


Figure 6.9

Solution:

In order to obtain the time constant of the circuit, we just use equation 6.91 and plug in the values given in the exercise,

$$\tau = RC = \left(30 \cdot 10^3 \Omega\right)\left(0.5 \cdot 10^{-6} F\right) \Rightarrow \tau = 0.015 sec \tag{6.94}$$

Now, the maximum charge that can be stored in the capacitor, is the electric charge Q_{max} in equation 6.69. Therefore,

$$Q_{max} = C\varepsilon = \left(0.5\mu F\right)\left(15V\right) = 7.5\mu C \tag{6.95}$$

Now, for the next question (3) , we want that the electric charge at certain time t be $0.95 \cdot Q_{max}$. In other words, $Q(t) = 0.95 Q_{max}$. Therefore using equation 6.68 and equating to $0.95 Q_{max}$,

$$0.95 C\varepsilon = C\varepsilon\left(1 - e^{\frac{-t}{\tau}}\right) \tag{6.96}$$

where we used the maximum electric charge that can be stored in the capacitor (see equation 6.69). By just some algebra,

$$e^{\frac{-t}{\tau}} = \Bigg(-0.95+1\Bigg) \Longrightarrow \frac{-t}{\tau} = \ln\Bigg(0.05\Bigg) \Longrightarrow t = -\tau ln\Bigg(0.05\Bigg) \tag{6.97}$$

Hence, it takes about

$$t \approx 0.044\text{s} \tag{6.98}$$

to approach almost the maximum electric charge (95% of the maximum)! A tiny amount of time! In principle, the time to obtain the maximum electric charge we need to wait for a time $t \to \infty$. However, to approach to almost the maximum electric charge is a finite time which is so small, that we do not have to worry to obtain the maximum electric charge!
Now, for question 4, nothing stops us to apply voltage Kirchhoff Law, and we have the equation, which is the equation 6.62 that we used to deduce all time dependent functions of electric charge, current , voltage and energy. So,

$$\varepsilon - \frac{q}{C} - RI = 0 \Longrightarrow \varepsilon - \frac{0.25C\epsilon}{C} - RI = 0 \tag{6.99}$$

where we have substituted $1/4 = 0.25$ of the maximum electric charge ($Q_{max} = C\varepsilon$, equation 6.69) . So, finally doing some algebra

$$\varepsilon\Bigg(1-0.25\Bigg) = RI \Longrightarrow I = \frac{\varepsilon\Bigg(1-0.25\Bigg)}{R} \tag{6.100}$$

Therefore, substituting values

$$I = \frac{15\text{V}\cdot 0.75}{30\times 10^3\Omega} = 3.75\cdot 10^{-4}A \tag{6.101}$$

Now, for question 5) , the maximum electric current

$$I = \frac{15}{30\cdot 10^3\Omega} = 5\cdot 10^{-4}A \tag{6.102}$$

Finally, question 6). Nothing stops us to using once again Kirchhoff Voltage Law, so

$$\varepsilon - \frac{Q}{C} - R\Bigg(0.4\frac{\varepsilon}{R}\Bigg) = 0 \Longrightarrow \varepsilon\Bigg(1-0.4\Bigg) - \frac{Q}{C} = 0 \tag{6.103}$$

where we have substituted the 40% of the maximum electric current, i.e. $0.4\frac{\varepsilon}{R}$ (see equation 6.72). Doing some algebra

$$0.6\varepsilon = \frac{Q}{C} \Longrightarrow Q = C\varepsilon(0.6) = Q_{max}(0.6) = (7.5\mu C)(0.6) = 4.5\mu C \tag{6.104}$$

where we substituted the maximum electric charge that we have already found in equation 6.95.

Example 7: Solving our first discharging RC circuit

A capacitor of 5μFand a resistor of resitance 3.5Ω are connected in series without emf. After how much time the stored energy is a third of the initial value?
Solution:
We can apply equation 6.89,

$$U = U_0 e^{-2t/RC} \tag{6.105}$$

However, we want the energy stored in the capacitor U is a third of the of the initial energy. So we want that

$$U = \frac{1}{3} U_0 \tag{6.106}$$

therefore, by using the energy function in equation 6.105, we have that

$$U_0 e^{-2t/RC} = \frac{1}{3} U_0 \Longrightarrow e^{\frac{-2t}{RC}} = \frac{1}{3} \Longrightarrow t = -\frac{RC}{2} \ln\left(\frac{1}{3}\right) \Longrightarrow \tag{6.107}$$

Therefore, by substituting values

$$t = -\frac{\left(5\mu F\right)\left(3.5\Omega\right)}{2} ln\left(\frac{1}{3}\right) \Rightarrow t = 9.61 \cdot 10^{-6} sec \tag{6.108}$$

Example 8: Several resistors and capacitors in a RC circuit

In the circuit that is shown in the figure, each capacitor has initially a charge of 3.50nC in its plates. What will be the current in the circuit in the moment that the capacitor have reached the 75% of its stored energy initially? The values of the resistances of the resistors are : $R_1 = 3\Omega$, $R_2 = 1.5\Omega$ and $R_3 = 4\Omega$. The value of the capacitance of the capacitors are : $C_1 = 2.5\mu F$, $C_2 = 7\mu F$ and $C_3 = 2\mu F$

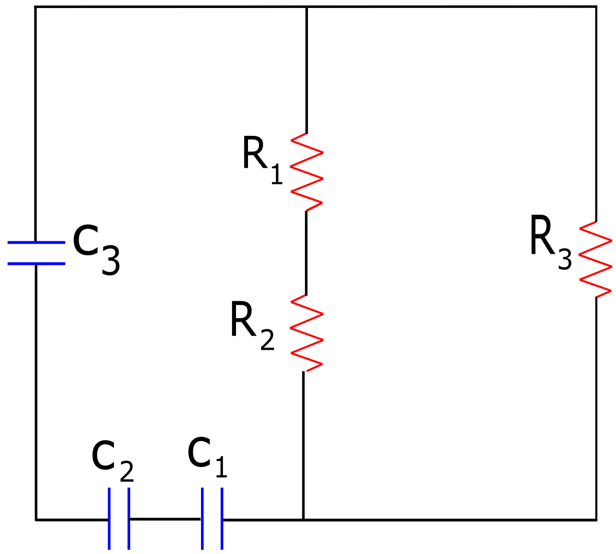


Figure 6.10

Solution:

First we need to reduce the circuit in just one equivalent resistor and one equivalent capacitor. So, calculating firstly the equivalent capacitance , noticing that the capacitors are connected in series, by using equation 4.29, we have that

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \Longrightarrow \frac{1}{C_{eq}} = \frac{C_2C_3 + C_1C_3 + C_1C_2}{C_1C_2C_3} \Longrightarrow C_{eq} = \frac{C_1C_2C_3}{C_1C_2 + C_1C_3 + C_2C_3} \tag{6.109}$$

By plugging in values, we obtain that the equivalent capacitance is

$$C_{eq} = \frac{2.5 \times 10^{-6} \cdot 7 \times 10^{-6} \cdot 2 \times 10^{-6}}{(2.5 \times 10^{-6} \cdot 7 \times 10^{-6}) + (2.5 \times 10^{-6} \cdot 2 \times 10^{-6}) + (7 \times 10^{-6} \cdot 2 \times 10^{-6})} \approx 9.58{\cdot}10^{-7}F \tag{6.110}$$

Now, we obtain the equivalent resistance of the resistors. Now, notice that resistors R_1 and R_2 are in series and we can make a new equivalent resistor R_{12} which will be in parallel with resistor R_3. Hence, the complete circuit equivalent resistance

$$Req = \frac{R_{12} \cdot R_3}{R_{12} + R_3} \tag{6.111}$$

where

$$R_{12} = R_1 + R_2 = 4.5\Omega \tag{6.112}$$

Hence,

$$R_{eq} = \frac{4.5\Omega \cdot 4\Omega}{8.5\Omega} \approx 2.11\Omega \tag{6.113}$$

Now, finally we are left with one resistor and one capacitor in series. So, we can apply the equations we have derived for discharging capacitor. By equation 6.89, we have that the energy

$$U = U_o e^{\frac{-2t}{RC}} \tag{6.114}$$

However, we want the explicit time when the capacitor stores the 75% of its initial energy, $U = 0.75U_0$, hence

$$0.75U_0 = U_0 e^{\frac{-2t}{RC}} \Longrightarrow \frac{-2t}{RC} = \ln(0.75) \Longrightarrow t = -\frac{RC\ln(0.75)}{2} \tag{6.115}$$

So, substituing values we have

$$t = -\frac{(2.11\Omega)\,(9.58\cdot 10^{-7}F)\ln(0.75)}{2} = 2.9\cdot 10^{-7}sec \tag{6.116}$$

Finally, what we are asked is to calculate the electric current when the energy is 75% of the initial energy, We already know the time when that happens (equation 6.116)! So, by using equation 6.83,

$$I = \frac{Q_0}{RC}e^{-\frac{t}{RC}} \tag{6.117}$$

we just plug in values now. We split in steps the calculation so the expression is clear and it does not have all the values at the same time. Calculating firstly the time constant,

$$\tau = R_{eq}C_{eq} = (2.11\Omega)(9.58\cdot 10^{-7}\mathrm{F}) \approx 2.02\times 10^{-6}\mathrm{s} \tag{6.118}$$

So, the exponent in the exponential

$$-\frac{t}{RC} = -\frac{2.9\cdot 10^{-7}\mathrm{s}}{2.02\times 10^{-6}\mathrm{s}} \approx -0.14 \tag{6.119}$$

Hence,

$$I = \frac{3.5\cdot 10^{-9}\mathrm{C}}{2.02\times 10^{-6}\mathrm{s}} \quad e^{-0.14} \approx 1.5\cdot 10^{-3}A \tag{6.120}$$

Part III

Magnetostatics

Chapter 7

Magnetic Field Force

The sources of electric fields are charges, one source of magnetic fields are electric currents, the other is varying electric fields. In this chapter we study how electric charges or currents behave under the influence of a magnetic field. In the next chapter we study how to calculate the magnetic field created by steady currents. This chapter and the following one we consider steady currents which do not vary with time, therefore the magnetic fields are static. Along this chapter, we study several applications of the magnetic force exerted on charges and electric currents.

7.1 Lorentz Force

The Lorentz Force reads as

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{7.1}$$

where the first term is the already known and discussed previously, the electrostatic force. The second term is new, and we can think about it as the magnetic force exerted on a particle that moves with velocity $\vec{v}$.

$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{7.2}$$

I would like to point out that the last equation holds only for electric charges that moves with speeds much smaller than the speed of light ($|\vec{v}| << c \approx 3 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}$). In many examples , you will notice that we find the direction of the magnetic force by using the right hand rule. In general, suppose you operate two vectors $\vec{a}$ and $\vec{b}$ as

$$\vec{c} = \vec{a} \times \vec{b} \tag{7.3}$$

so, in order to know the direction of $\vec{c}$, you can place your index finger in the direction of vector $\vec{a}$, curl your fingers (except the thumb finger) to the direction of the vector $\vec{b}$ and the thumb finger points to the direction of vector $\vec{c}$.

Figure 7.1

I mention the general case, because you will find from now on tons of cross products, and is important to recall how to obtain the direction of the resultant vector of the cross product. In the particular case of the magnetic force,

$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{7.4}$$

the index finger must point towards the direction of the velocity of the charged particle, curl your fingers to the direction of the magnetic field, and the thumb finger points towards the direction of the magnetic force as shown in figure 7.1a. You must be careful of the sign of the charge! If the electric charge q is negative, it flips the direction of the force to the opposite direction. So, if you find for example the direction of the force towards $+y$ using the right hand rule, if the electric charge is negative so actually the force is pointing to $-y$. So, in general using the right hand rule gives the result taking q as positive. If you have negative charges, use the right hand rule, and finally just flip the direction of the force vector to the opposite direction. The right hand rule is very useful, once you know the direction, you just need to calculate the magnitude of the force, by using the general rule

$$|\vec{F}| = |\vec{v} \times \vec{B}| = |\vec{v}||\vec{B}| \sin\theta_{vB} \tag{7.5}$$

where the angle θ_{vB} is the angle between vectors $\vec{v}$ and $\vec{B}$. As you will notice as we do exercises, sometimes the right hand rule is not enough to solve the cross product. The right hand rule helps to obtain the direction of the force direction, but if the direction is not exactly at one direction of an axis , it is just a qualitative direction, because you do not know exactly the components of the vector. However, this information is useful many times to tackle problems when there is symmetry in the system. When, not the symmetry, nor the qualitative information that the right hand rule can provide you is enough to tackle the problem, you must calculate directly the cross product. So, suppose the velocity vector in general as $\vec{v} = (v_x, v_y, v_z)$, and the magnetic field as $\vec{B} = (B_x, By, B_z)$, so the cross product for the magnetic force is

$$\vec{F}_B = q\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} = q[(v_y\cdot B_z - v_z\cdot B_y)\hat{x} - (v_x\cdot B_z - v_z B_x)\hat{y} + (v_x\cdot B_y - v_y B_x)\hat{z}] \tag{7.6}$$

where in the second equality we have the determinant of the shown matrix. The first row of the matrix has the unitary Cartesian vectors. The second row of the matrix has the components of the first vector in our cross product which is the velocity vector, while the third row has the components of the second vector in the cross product, in this case the magnetic field. This is the general form of the cross product of two vectors, many times is not necessary to calculate the cross product from this definition, however in many other occasions as it will turn it out, the last equation is highly useful. For example, when we want to know the dynamics of a charged particle with different magnetic field configurations, as you will learn in example 2, we need to calculate the cross product from the general definition.

Finally, something highly remarkable about the magnetic force is that it does no work! From the definition of the magnetic force, it is perpendicular to the velocity vector of the charged particle to which the force is exerted. In the cross product $\vec{v}\times\vec{B}$ the resultant vector is perpendicular to $\vec{v}$ and $\vec{B}$. So, if we calculate the work done by the magnetic force, we have that

$$W = \int \vec{F}_B \cdot d\vec{l} = 0 \tag{7.7}$$

where the vector $d\vec{l}$ is parallel to the velocity vector. The vector $d\vec{l}$ is a vector tangent to the path of the particle. So, $\vec{F}_B$ and $d\vec{l}$ are always perpendicular to each other, so their dot product is zero! But this becomes even more interesting, if the work done by a magnetic force is zero, then we have that

$$W = \Delta U = 0 \tag{7.8}$$

so the change of potential energy is zero! And by conservation of energy we have

$$\Delta U = -\Delta K \Longrightarrow v_F = v_i \tag{7.9}$$

i.e. the change of kinetic energy is zero! If there is no change of kinetic energy, then the initial speed and final speed of the particle are exactly the same! The magnetic force exerted on a charged particle can not increase its speed! Just deviate it from its original path.

Example 1: Simplest Case Magnetic Field Force Exercise

An electron moves foward (as shown in the figure 7.2) with a speed $5.8\times10^7\frac{\text{m}}{\text{s}}$ in the x axis. There are certain coils that create a magnetic field of 0.08T magnitude , with an angle of 75° with the x axis as shown in the figure

- Calculate the magnetic force exerted on the electron
- If the angle of the magnetic field could be changed. What would be the maximum and minimum magnitude of force exerted on the electron feels?
- What is different if instead of an electron is a proton?

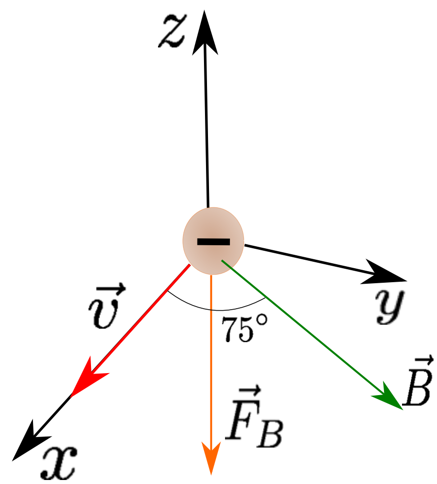


Figure 7.2

Solution:

In this exercise, we consider that there is no electric field, only magnetic field produced by an electric current, which we simplify as constant. So, the Lorentz Force simplifies to the magnetic force exerted by the magnetic field.

$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{7.10}$$

Now, the direction of the force can be determined by the right hand rule. If you place your index finger to the direction of velocity, the middle finger to the magnetic field direction, the thumb finger points to the force direction. The force direction points to $+\hat{z}$. However, that direction is if the charge is positive, if the charge is negative, the direction is flipped to the opposite direction. So, the force direction is to $-\hat{z}$. Once we know the direction of the force, we want to know the magnitude. So,

$$|\vec{F}| = |q||\vec{v}||\vec{B}| \sin\theta_{vB} = (1.6 \times 10^{-19}C)(5.8 \times 10^{7}m/s)(0.085)\sin(75^{0}) \tag{7.11}$$

Where the label "vB'' is just to let you know that it is the angle between vector $\vec{V}$ and $\vec{B}$. This should be obvious from the definition of the magnitude of cross product. We conserve it just to remember you. Finally, from equation 7.5 we know that minimum force is when the angle is zero,when $\vec{v}$ and $\vec{B}$ are parallel or anti parallel. So,

$$|\vec{F}_{\text{min}}| = 0 \qquad \text{when } \theta_{vB} = 0°, 180° \tag{7.12}$$

So, as obviously expected the minimum is when no force exerted. The maximum magnitude of the force is when the magnetic field and the velocity are perpendicular ($\sin 90° = 1$ or $\sin 270° = -1$, we are concerned only by magnitude). So,

$$|\vec{F}_{\text{max}}| = (1.6 \times 10^{-19}\text{C})(5.8 \times 10^7\text{m/s})(0.085\text{T}) = 7.424 \times 10^{-13}\text{N}$$

Example 2: Helix Path

A positive charge q travels with a constant velocity $\vec{v}$. Suddenly, there is a magnetic field perpendicular to $\vec{v}$.

- Determine the path of the particle if moves in two dimensions.
- Determine how would the trajectory be in 3 dimensions if a constant and uniform magnetic field is created to $+x$ direction , suppose the initial velocity in z component is zero.

Solution:

Let's suppose the positively charged particle is at point A at time $t = 0$, as shown in the figure 7.3, with certain velocity $\vec{v}$. The electric charge is immersed in a constant magnetic field towards the page. At time $t = 0$, the exerted magnetic force on the electric charge is perpendicular to the velocity vector, pointing upwards, as you can easily determine with the right hand rule. After certain time t', the particle has deviated its path as shown in figure 7.3 due to the magnetic force and reaches point B. At this new point, the force direction is not any more perfectly directed upwards. If we placed a reference frame at the electric charge in point B, the force would have now x and y components, and its direction would be determined by the right hand rule. The force direction at point B is as it is shown in figure 7.3. So, after another time t'', the particle has deviated again its path and now the force direction is as in point C shown in figure 7.3. If we repeat this process, after a time T, the path has completed a circular path. The magnetic field force acts as a centripetal force! So, we obtain a circular path. We could ask ourselves *"What is the radius of the circular path?"* The centripetal force is the magnetic force, so

$$|\vec{F}_B| = m|\vec{a}_c| = m\frac{v^2}{r} \tag{7.13}$$

where we used the formula of centripetal force and centripetal acceleration. Substituting the magnetic force magnitude

$$qvB\sin\theta_{vB} = m\frac{v^2}{r} \tag{7.14}$$

However, $\vec{v}$ and $\vec{B}$ are perpendicular, so $\sin\theta_{VB} = 1$. Now, isolating the radius, we obtain

$$\boxed{r = \frac{mv}{qB} = \frac{p}{qB}} \tag{7.15}$$

where p stands for momentum. Now, the angular velocity

$$\omega = \frac{v}{r} = \frac{v}{mv}qB = \frac{qB}{m} \tag{7.16}$$

Something interesting! The angular velocity is independent of the velocity of the particle, it depends completely on the strength of the magnetic field, the electric charge and mass.

Finally, we could ask ourselves about the time it takes the electric charge to complete one circular path (the period T). So, we can calculate it easily by

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{\omega r} = \frac{2\pi}{qB}m \tag{7.17}$$

where you can think of it as the distance travelled by the charge (the perimeter of the circle) divided by its velocity, and we used equation 7.16 for ω.

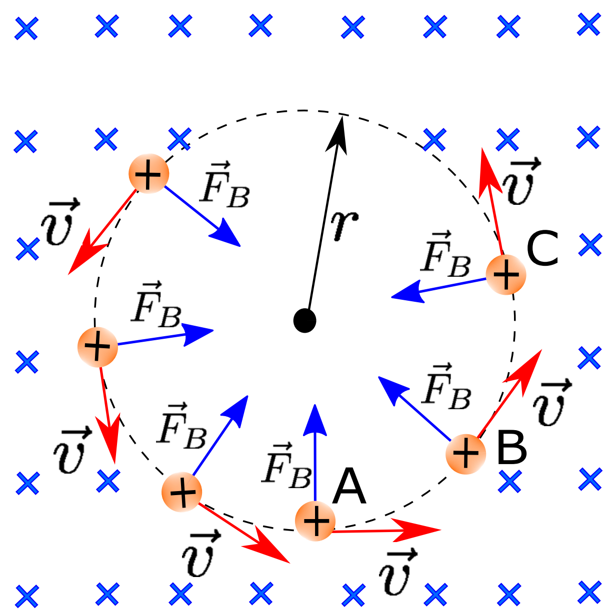


Figure 7.3

Now, let's pump things up! and study an interesting case. We let the particle to move in three dimensions. How does it behave? So, take a constant and uniform magnetic field pointing to $+x$. So, the magnetic force reads as,

$$\vec{F}_B = q\vec{v}\times\vec{B} = q\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ B_x & 0 & 0 \end{vmatrix} \tag{7.18}$$

Therefore, we have that

$$\vec{F}_B = q[(v_y\cdot 0 - v_z\cdot 0)\hat{x} - (v_x\cdot 0 - v_zB_x)\hat{y} + (v_x\cdot 0 - v_yB_x)\hat{z}] = 0\hat{x} + qv_zB_x\hat{y} - q(v_x\cdot 0 - v_yB_xv_z)\hat{z} \tag{7.19}$$

Or, writing it directly as components

$$(ma_x, ma_y, ma_z) = (0, qV_zB_x, -qV_yB_x) \tag{7.20}$$

where we just used $\vec{F} = m\vec{a}$, and wrote the acceleration vector three components a_x, a_y, a_z. Comparing the correspondent components we have the following three equations:

$$ma_x = 0, \qquad ma_y = qv_zB_x, \qquad ma_z = -qv_yB_x \tag{7.21}$$

and, recalling that

$$a_x = \frac{dv_x}{dt}, \qquad a_y = \frac{dv_y}{dt}, \qquad a_z = \frac{dv_z}{dt} \tag{7.22}$$

we have the following three differential equations,

$$\begin{aligned} \frac{dv_x}{dt} &= 0 \\ \frac{dv_y}{dt} &= \omega v_z \\ \frac{dv_z}{dt} &= -\omega v_y \end{aligned} \tag{7.23}$$

where we called $\omega = \frac{qB_x}{m}$ (the name ω is not a coincidence, see that we obtained exactly the same term for the angular velocity in two dimensions). Now, the first equation tells us that the velocity does not change, it remains constant for all time. So,

$$\frac{dx}{dt} = v_{ox} \Rightarrow \int_{x_0}^{x} dx = v_{ox} \int_0^t dt \tag{7.24}$$

where v_{ox} is the initial velocity in the x component, and the limits of integration follow, because at certain time t, the electric charge has certain position x, and at time $t = 0$, the charge would have certain initial position. Therefore,

$$\boxed{x = x_0 + v_{ox}t} \tag{7.25}$$

Now, lets analyze what happens with the y and z components. Notice that the term $(\frac{dv_y}{dt})$, in the second differential equation of 7.23, depends on v_z, and the term $(\frac{dv_z}{dt})$ in the third differential equation of 7.23 depends on v_y. When it happens that in a system of differential equations, one differential equation has explicitly a derivative that depends on a second variable, and that second variable derivative in other differential equation depends explicitly on the first variable, we call the system as "*a coupled system of differential equations*". As it turns out, we can apply a very simple trick to solve the system. Notice that if you derive one equation with respect time, on the right hand side you will obtain a term that is exactly the second equation, i.e.

$$\frac{d}{dt}\left(\frac{dv_y}{dt}\right) = \frac{d}{dt}(\omega v_z) \Rightarrow \frac{d^2v_y}{dt^2} = \omega\frac{dv_z}{dt} \tag{7.26}$$

where we derived with respect time both sides of the second differential equation of 7.23. So now you can substitute the third differential equation of 7.23 into the term $\frac{dv_z}{dt}$ in last equation, i.e.

$$\frac{d^2v_y}{dt^2} = \omega(-\omega v_y) = -\omega^2 v_y \tag{7.27}$$

You have seen last equation before, this must be familiar from *Waves and Oscillations* courses. The equation represents an oscillation which solution is:

$$v_y = A\cos(\omega t + \phi) \tag{7.28}$$

where A is just an amplitude which can be obtained by the initial conditions, and ϕ is a phase constant. So, now we can know the behavior of the path of the particle in the y component

$$\frac{dy}{dt} = A\cos(\omega t + \phi) \;\Rightarrow \int_{y_0}^{y} dy = \int_0^t A\cos(\omega t + \phi)dt \tag{7.29}$$

where the integration limits follow because at time $t = 0$, the particle has certain initial y_0 position and at time t it is at certain position y. Therefore after integration,

$$\boxed{y = y_0 + \frac{A}{\omega}\left(\sin(\omega t + \phi) - \sin\phi\right)} \tag{7.30}$$

Finally, once we know v_y we can solve the differential equation for v_z, by using the third differential equation of 7.23 and using the result in equation 7.28, we have that

$$\frac{dv_z}{dt} = -\omega v_y = -\omega(A\cos(\omega t + \phi)) \;\Rightarrow \int_0^{v_z} dv_z = -\omega A\int_0^t \cos(\omega t + \phi)dt \tag{7.31}$$

where we used the assumption mentioned by the exercise, the initial velocity in the z component is zero. Therefore, after integration

$$v_z = -A\left(\sin(\omega t + \phi) - \sin\phi\right) \tag{7.32}$$

Hence,

$$\frac{dz}{dt} = -A\left(\sin(\omega t + \phi) - \sin\phi\right) \Rightarrow \int_{z_0}^{z} dz = -A\int_0^t \left(\sin(\omega t + \phi) - \sin\phi\right)dt \tag{7.33}$$

Therefore, after integration

$$\boxed{z = z_0 + \frac{A}{\omega}\left(\cos(\omega t + \phi) - \cos\phi\right) - A\sin(\phi)\,t} \tag{7.34}$$

So, we have found finally how each component of the path of the particle behaves with respect time. Writing together the set of equations,

$$\begin{aligned} x &= x_0 + v_{ox}t \\ y &= y_0 + \frac{A}{\omega}\left(\sin(\omega t + \phi) - \sin\phi\right) \\ z &= z_0 + \frac{A}{\omega}\left(\cos(\omega t + \phi) - \cos\phi\right) - A\sin(\phi)\,t \end{aligned} \tag{7.35}$$

So, the last equations are a general set. Let's assume that the particle initial position is $x_0 = y_0 = z_0 = 0$ and that $\phi = 0$. So , when t = 0 from equation 7.28, we have that

$$v_y(0) = v_{oy} = A\cos(0) \Rightarrow A = v_{oy} \tag{7.36}$$

Therefore A is just the initial velocity of the charge in the y component. Hence, finally the equations that determine the motion of the charge are

$$\begin{aligned} x &= v_{ox}t \\ y &= \frac{v_{0y}}{\omega}\sin(\omega t) \\ z &= \frac{v_{0y}}{\omega}\left(\cos(\omega t) - 1\right) \end{aligned} \tag{7.37}$$

The set of equations we obtained describe an helix motion as shown in figure 7.4! Beautiful! Don't you think?

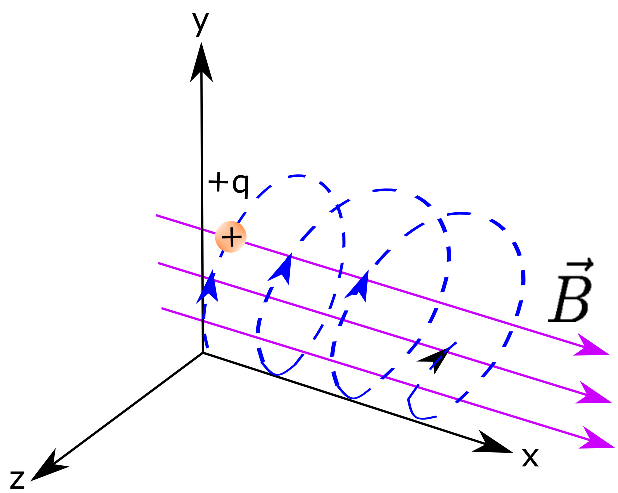


Figure 7.4

Example 3: Cyclotron

The cyclotron is an apparatus that uses electric fields to accelerate charges, and magnetic fields to deviate the path of such particles, so that the cyclotron is able to accelerate those particles in a relatively small space. Nowadays, the accelerators of particles are huge! However, as we will see a cyclotron of radius of 15cm can accelerate a proton to speeds of order $10^7 \frac{\text{m}}{\text{s}}$! A cyclotron is shown in figure 7.5. The cyclotron is composed of two conductors with letter D shape as shown in the figure, sometimes called as *Dees*. When these two conductors are connected to an alternating voltage supply, an electric field is generated between these two *Dees*. Given that at certain time t one *dee* conductor will be positively charged while the other *dee* will be negatively charged, the electric field lines will start from the positively charged *dee* and end with the negatively charged *dee*. The conductor *dees* are hollow, so inside them there is free space so that particles can travel in there. Finally, extremely important, there must be a perpendicular magnetic field to the electric field generated by the charged *dees*. Such magnetic field is generated by the magnetic coils indicated in figure 7.5.

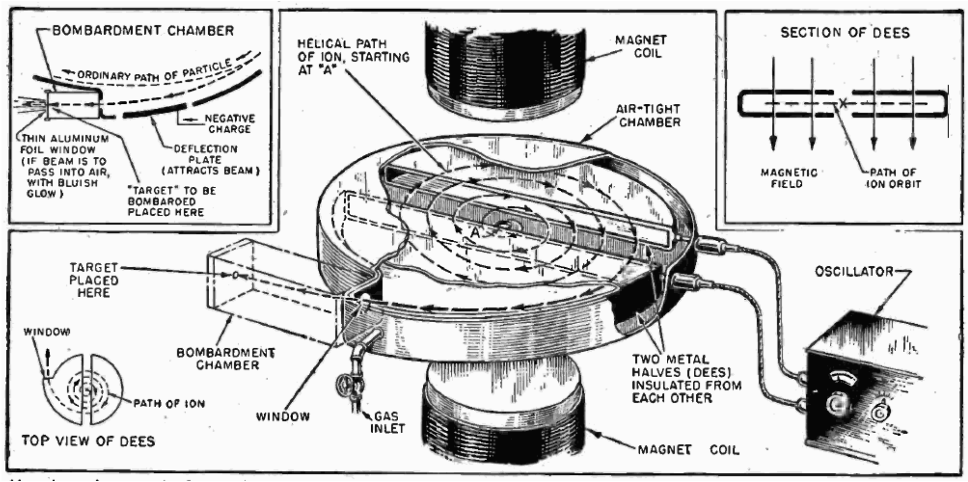
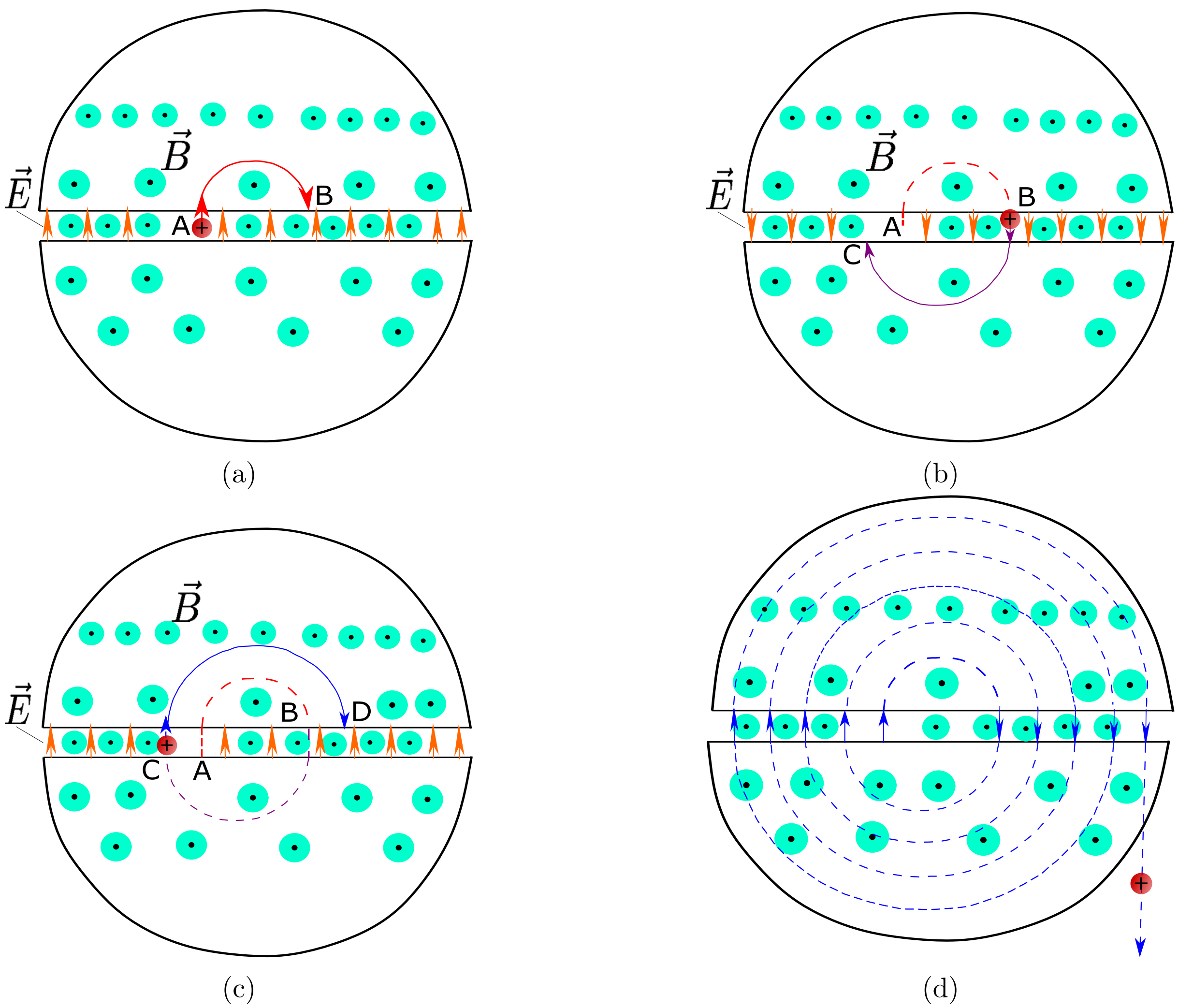


Figure 7.5: Cyclotron diagram. Two conductors in shapes of the letter D are connected to an alternating voltage supply. The dees get the same voltage as the voltage supply, so there is a potential difference between the conductors, and an electric field. The voltage polarity and the direction of the electric field changes with the same frequency as the voltage supply. The dashed arrows represent the path of the charged particles. The magnet coils shown generate a perpendicular magnetic field to the path of the particles and to the electric field generated by the dees. Original picture taken from [13].

Now, how does this work? Suppose a positively charged particle is placed between the two *dees*; and there is a uniform magnetic field pointing out the page as shown in figure 7.6. Suddenly, the voltage supply is turned on and an electric field is created and the charged particle moves as shown in figure 7.6a. The force that accelerated the charge is the electric force, the magnetic force just deviates the charge. Remember that magnetic forces do no work, therefore can not change the magnitude of the speed of the charge. After the charge acquires certain velocity, by using the right hand rule, we notice that the magnetic force exerted on the particle will act as a centripetal force, so that the charge moves in circular motion as we have discussed previously. In figure 7.6a the charge travelled from A to B. Now, when the particle reaches point B when it has traveled half circle, we do not want it to go backwards and loop. We want to accelerate it even more. So, in that precise moment, when the charge is at point B, we need to switch the polarity of the voltage, so an electric field to the opposite direction is created. If we change the polarity at that precise moment, then the particle will travel from point B to point C as shown in figure 7.6b. Notice that when the charge travelled from B to C the radius of the path of the charge has increased. The reason is simple, the charge velocity has increased. Now, when the charge reaches point C, we want to accelerate once again the charge, so we change the polarity of the electric field by switching the voltage. So, now the charge travels from C to D. Once again, the radius of the path of the charge has increased and when the charge reaches point D we will switch once again the direction of the electric field so that we continue accelerating it, and afterwards we continue accelerating the charge by switching the voltage polarity again and again, so on and so forth until the charge has reached the radius of the cyclotron as shown in figure 7.6d. At this stage, we let the charge escape with a tremendous speed as we will see in a simple calculation. However, before that we have a big problem. We want to switch the direction of the electric field every time the charge has reached half of the path of a circle. So, when should we change the polarity of the voltage?



(a) (b) (c) (d)

Figure 7.6

We know that the period of a traveling charged particle in circular motion due to a constant perpendicular magnetic field is

$$T = \frac{2\pi}{qB}m \tag{7.38}$$

so, notice something beautiful! Not intuitive at all! The time it takes to the particle to complete one cycle (the period) is independent of the velocity! So, we do not have to worry if the particle that got into the cyclotron has certain initial speed v_0 or not! Also, if the particle has already cycled n loops, the period is exactly the same as for the first time that the particle will complete a loop! This experimentally is a big relief. If it were dependent of the velocity, it would be extremely difficult to make the voltage to have the correct polarity to make particles accelerate. Nicely, the period that takes a particle to complete a circular path depends only on the charge, the magnetic field and mass! Now, the time it will take the particle to complete half the way of the circle is just half of the period. So, every

$$T' = \frac{\pi}{qB}m \tag{7.39}$$

seconds we have to switch the voltage polarity, so that the electron continues accelerating! So, our voltage supply has to change polarity with a frequency:

$$f' = \frac{1}{T'} = \frac{qB}{\pi m} \tag{7.40}$$

The last equation represents the frecuency that the polarity has to change. However is more common to keep the frequency which the voltage changes polarity two times, i.e. when it goes to one direction, changes direction, and once again has reverted direction. Think about oscillations for instance, the frequency actually takes into account both when goes and come back to the same configuration. So, we say that the frequency of the voltage supply is:

$$f = \frac{1}{2T'} = \frac{qB}{2\pi m} \tag{7.41}$$

and the angular frequency is:

$$\omega_{cyclotron} = 2\pi\left(\frac{qB}{2\pi m}\right) = \frac{qB}{m} \tag{7.42}$$

Notice that it is just the angular velocity of the electric charge! Marvelous! We can adjust our voltage supply of alternate current, by just measuring the magnetic field! (the mass and charge of the particle are already fixed). We can know for sure, what is the needed angular frequency of our voltage supply, by knowing the magnetic field and the kind of particles we want to accelerate. And even better, we can modify such angular frequency, by just changing the magnetic field! Weaker the magnetic field, lower the angular frequency, greater the magnetic field, bigger the angular frequency of our voltage supply to be switching the voltage polarity. Even though it is a difficult task to build the machine, it is not impossible as one would think because the frequency which you must alternate polarities is independent of the velocity and also the radius of the path of the charge! If it were dependent of the radius it will be extremely difficult to know were exactly is placed the particle and change the polarity depending of the radius that its path is making in that exact moment. Now, what will it be the velocity of the particle once it goes out of the cyclotron? From equation 7.15, we have that

$$v = \frac{qBR}{m} \tag{7.43}$$

where R is the radius of the cyclotron.

So, to get a grasp of how fast the particles could get; suppose that the positively charged particle is a *proton*, that the radius of the cyclotron is just 15cm and the magnetic field magnitude is 1.0T. So, the speed with which the proton goes out from the cyclotron is

$$v = \frac{(1.6 \times 10^{-19}C)(1.0T)(0.15m)}{1.67 \times 10^{-27}kg} \approx 1.43 \times 10^7 m/s \tag{7.44}$$

My god! Huge speed has acquired the proton in a relatively small cyclotron. What about its energy?

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(1.67 \times 10^{-27}kg)(1.43 \times 10^7 m/s)^2 \approx 1.7 \times 10^{-13}J \tag{7.45}$$

For particle physics, a common unit of energy that is widely used is the so called eV (electronvolt). An *electronvolt* is defined as

$$1eV = 1.6 \times 10^{-19}J \tag{7.46}$$

which notice the factor 1.6×10^{-19} is exactly the magnitude of the charge of an electron. So, the energy of the accelerated proton is:

$$1.7 \times 10^{-13}J\left(\frac{1eV}{1.6 \times 10^{-19}J}\right) \approx 1.06\ MeV \tag{7.47}$$

where the "M" stands for mega (1×10^6). Today's biggest accelerator in the world, the *LHC* (Large Hadron Collider) accelerates particles almost to the speed of light! (99.99....%) And the energy for instance that the Large Hadron Collider (LHC) achieves are approximately 13 TeV ($T = 1 \times 10^{12}$). Making a fraction to achieve an idea, with our 15 *cm* cyclotron compared to LHC :

$$\frac{1.06 \times 10^6 eV}{13 \times 10^{12} eV} \approx 8.15 \times 10^{-8} \tag{7.48}$$

So,the energy achieved at LHC is huge compared to our accelerated proton with the 15 *cm* radius cyclotron! The LHC has a circumference of 27 km! Now, you probably think,

$$13TeV = 13 \times 10^{12}eV\left(\frac{1.6 \times 10^{-19}\ J}{1eV}\right) = 2.08 \times 10^{-6}\ J \tag{7.49}$$

Therefore,

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2(2.08 \times 10^{-6}\ J)}{1.67 \times 10^{-27}\ kg}} \approx 5 \times 10^{10}\ m/s \tag{7.50}$$

But, wait a minute! that is greater than the speed of light! And Einstein's theory of special relativity establishes that nothing travels faster than the speed of light $c \approx 3\times10^8\ m/s$ (speed of light in vacuum) ! What is going on?! Indeed, nothing travels faster than light, but when an object approaches the speed of light, certain non intuitive things happen. The time is not constant anymore! The time depends on the inertial time of reference. So, things get complicated, and the energy achieved is not only due to increasing the speed of the particle (kinetic energy), as the particles get closer and closer to the speed of light the particles get more massive! As turns out, the mass actually is not constant! All this is beyond the scope of this course and all these effects are included in the special relativity theory of Einstein (we call them as "*relativistic effects*"). These effects take place at LHC, so it is a complex machinery! Beautiful!

Now returning to our cyclotron of 15 cm of radius. How much speed does the proton acquires in each loop? So, in each loop, you increase the kinetic energy two times, each time you change the direction of the electric field.
So, using conservation of energy

$$\Delta U + \Delta K = 0 \Rightarrow \Delta K = -\Delta U \Rightarrow \frac{1}{2}mv_f^2 = -\Delta U + \frac{1}{2}mv_i^2 \tag{7.51}$$

Hence,

$$v_F = \sqrt{\frac{-2\Delta U}{m} + v_i^2} \tag{7.52}$$

Leaving the potential energy difference in terms of potential difference

$$v_F = \sqrt{\frac{-2q\Delta V}{m} + v_i^2} \tag{7.53}$$

where notice that the first term inside of the square root will be positive! Recall that the electric potential decreases (voltage is negative) towards the direction of positive charges move. Also, we could express last equation in terms of electric field. So, if the electric field is constant

$$v_F = \sqrt{\frac{2q|\vec{E}|d}{m} + v_i^2} \tag{7.54}$$

where we used

$$\Delta V = -\int \vec{E} \cdot d\vec{l} = -|\vec{E}|d \tag{7.55}$$

and considered the electric field as constant. If we have a voltage supply which switches its polarization, with constant voltage each time, we can use equation 7.53. Suppose that the alternating voltage supply magnitude is 1000V. So, the first time we accelerate the proton, starting from rest, the new velocity is

$$v = \sqrt{\frac{2(1.6 \times 10^{-19}C)(1000V)}{1.67 \times 10^{-27}\ kg}} \approx 4.37 \times 10^5\ m/s \tag{7.56}$$

when the proton goes around, it is once again accelerated. This time when the proton has completed a travel of 360°, the speed of the proton is

$$v = \sqrt{\frac{2(1.6 \times 10^{-19}C)(1000V)}{1.67 \times 10^{-27}\ kg} + (4.37 \times 10^5\ m/s)^2} \approx 6.18 \times 10^5\ m/s \tag{7.57}$$

The radius of the circular path of the proton is now

$$r = \frac{mv}{qB} = \frac{(1.67 \times 10^{-27}\ kg)(6.18 \times 10^{5}\ m/s)}{(1.6 \times 10^{-19}C)(1.0T)} \approx 6.5\text{mm} \tag{7.58}$$

In such a tiny radius the proton has acquired a huge speed! And so on, and so forth we could calculate what is the described radius of the path of the proton when it has certain velocity.

Example 4: Cathode Rays and Discovering the Electron

In 1897 , Professor Joseph John Thomson discovered the electron with the apparatus shown in figure 7.8a. Before his postulate of the consistency of what he observed; great scientific discussion took place about what was observed in the so called *cathode rays*. Years before other scientists noticed that if in certain glass tube air was pumped out, leaving just a small fraction of it, i.e. securing that the pressure was low inside the tube, if a high voltage was applied a beautiful glow was noticed inside the tube. The physicist Eugen Goldstein came with the name of *cathode rays* back in 1876. The name was because the *rays* (the light seen in such experiments) were emitted from the cathode of the vacuum tube. Now, we now that such *rays* are beams of *electrons*.

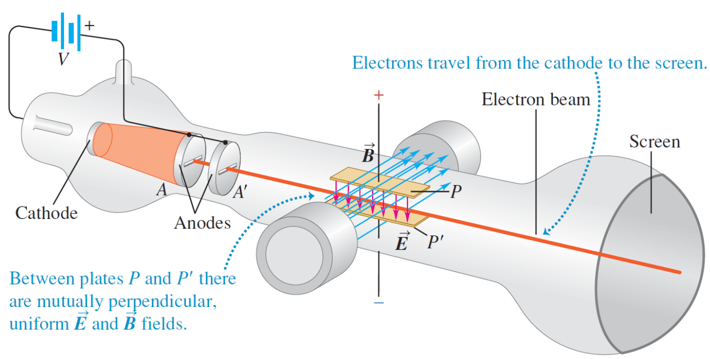
Figure 7.7: Professor J.J. Thomson. Original picture taken from reference [14].

When the high voltage is applied, the remaining air becomes a conductor, so a discharge takes place. When the air becomes conductive, the electrons of some of the atoms are ripped off them and free to move. So, these molecules of air becomes positively ionized, due to loss of electrons. When this happens, the negative electrode (the cathode) accelerates such ions towards them and collides abruptly against the cathode. So, electrons from the cathode are knocked off and accelerated towards the positive electrode (anode). In their way towards the anode, there are more still neutral air molecules, so many electrons collide against them, knocking off electrons of such molecules, and exciting them. When an atom is excited (not in the ground state), the atoms tend to be once again in the ground state by releasing the excessive energy. When an atom returns to its ground state after being excited, photons (light) with certain frequency are released. So that is what we see as the beautiful glowing inside the tube. This beautiful dancing between ions, still neutral atoms, those going to ground state is what is observed. However, this is not intuitive at all. So, at that time, it was thought that cathode rays was light. Heinrich Hertz, a brilliant physicist known for his work demonstrating the existence of electromagnetic waves, tried to observe if the cathode rays was light or something which had electric charge. However, he wrongly concluded that it had to be light, because he used electric fields and such rays didn't seem to be influenced by them. But why did Hertz experiments fail? Because the tube was not enough empty of air molecules. Lower pressures were needed, achieve mostly a vacuum inside the tube. When there is air inside, and it becomes ionized, the molecules generate an electric field, so this electric field attraction to the molecules does not allow the electrons to be deviated easily. So, when Hertz did his experiment, he did not see the rays to deviate because the effects due to electric fields were mostly neutralized. Thomson, himself stated this point years later in 1936.

"The absence of deflection on this view is due to the presence of gas–to the pressure being too high–thus the thing to do was to get a much higher vacuum. This was more easily said than done. The technique of producing high vacua in those days was in an elementary stage. The necessity of getting rid of gas condensed on the walls of the discharge tube, and on the metal of the electrodes by prolonged baking, was not realized. As this gas was liberated when the discharge passed through the tube, the vacuum deteriorated rapidly during the discharge, and the pumps then available were not fast enough to keep pace with this liberation. However, after running the discharge through the tube day after day without introducing fresh gas, the gas on the walls and electrodes got driven off and it was possible to get a much better vacuum. The deflection of the cathode rays by electric forces became quite marked, and its direction indicated that the particles forming the cathode rays were negatively electrified."

— J. J. Thomson

(a) Original figure taken from reference [15].



(b) Original figure taken from reference [7].

Figure 7.8

So, when Thomson did his experiment. he used a high vacuum, in such vacuum then you do not see any glowing inside the tube! Because the glowing is due to the ionized molecules of air. However, at the opposite side of the cathode in the glass itself there could be seen a glowing point. The electrons now free to move all along the tube, when colliding against the glass, light was emitted. However, this effect can be even more noticed if a fluorescent screen is at the glass when the electron collides. A green dot will be seen in there. Now, if we apply a magnetic field, this glowing dot moves to other position! And if we apply an electric field it also deviates. And even more surprisingly, it deviates as a *negative electric charge*! Thomson, carefully once he has noticed this, considering as what he was observing as a charged particle, he was able to determine the fraction of its charge and mass

$$\frac{q}{m} = 1.758820174\,(71) \times 10^{11}\,\frac{\mathrm{C}}{\mathrm{kg}} \tag{7.59}$$

which is the most accurate fraction nowadays of the charge and mass of the electron. Thomson changed the material of the conductors of the cathode and anode and every time he obtained the exact same results! So, whatever the cathode rays were made of, Thomson could observe the following

- They were independent of the material of the cathode and anode.
- They were particles with electric charge and mass, and their fraction no matter what material we used, always the same fraction of q/m is obtained.
- We said that we tried to make vacuum, but even though Thomson vacuum was high much better than other experiments before, the vacuum was not perfect. If we started with different gases inside the tube and then pumped it out, the cathode rays were independent of the remaining gas. In other words, does not matter what gas was used, always the same results were obtained.

From the last observations, then Thomson concluded, whatever these cathode rays are, they are eclectically charged particles, no doubt about that. Using electric and magnetic fields they deviate and behave as negative electrically charged particles. Secondly, given that these particles are there in the cathode rays giving always the same ratio q/m, no matter what conductors we use as electrodes, no matter what is the gas used, then this particle must be present in all atoms that constitute matter! And we can go even beyond. If matter is neutral when there is no high voltage applied, and if these negative particles are present in them, then the atoms must be constituted of something else with exactly the opposite electric charge so that they balance out. In other words, there must be what we know now of course as protons! Mind-blowing! In 1906 Thomson won the Physics Nobel Prize.

Fifteen years later, the physicist Robert Millikan with his graduate student Harvey Fletcher were able to measure the charge of the electron. They used an experiment ionizing drops of oil. Such drops of oil always had an integer multiple of a very specific charge. The charge of the electron! In 1923 , Robert Millikan won the Nobel Prize for such work, Harvey Fletcher did not. While Fletcher was a graduate student, to gave all credit to Millikan even though he did not like the idea, and the paper of the charge of the electron mentioned only Millikan, not Fletcher. The secret was kept until their death.

Now, suppose you have designed an apparatus as the one used by Thomson in the figure. If what is found is an electrically charged particle, then we must coincide experimentally with what is predicted by the Lorentz force. When the coils generate a magnetic field of 5×10^{-3}T and an electric field of $2 \times 10^5 \frac{\mathrm{V}}{\mathrm{m}}$ the trajectory of the cathode rays is a straight line. When you maintain the electric field source turned off, the curvature radius of the cathode rays is 4.55cm.

- Find the fraction $\frac{q}{m}$
- Use the results of Robert Millikan and his graduate student. In their experiment, they found using oil ionized droplets that their charge was always multiples of the electric charge 1.6×10^{-19}C. Does the mass of the cathode rays you find is the same as the mass of the electron ?

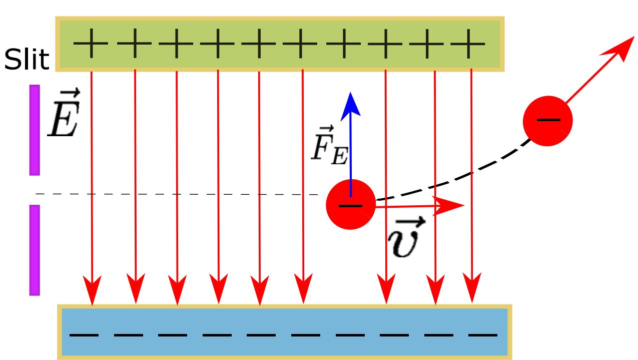
Solution:

We do a smart move. In the apparatus shown in the figure 7.8b, if the magnetic field is turned off, there is an electric field between the plates. This electric field will exert a force on the electric charges $\vec{F}_E = q\vec{E}$. So if the charges are negative, the force is in opposite direction to the electric field! So, the electrons will follow a curved path as shown in figure 7.9a. The electric force will act as a centripetal force, therefore taking the magnitude of such force

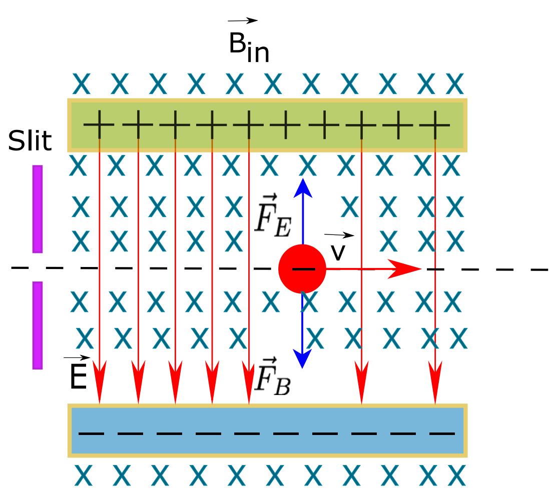
$$m\frac{v^2}{r} = |q||\vec{E}| \tag{7.60}$$

Hence, the charge-mass ratio of the moving particles is

$$\frac{q}{m} = \frac{v^2}{|\vec{E}|r} \tag{7.61}$$



(a) Side view of the electrically charged plates P and P' in figure 7.8b when there is no magnetic field. Electric force drawn and the path of the electron



(b) Side view between the electrically charged plates P and P' in figure 7.8b when there is magnetic field. Electric force and magnetic force drawn and the path of the electron.

Figure 7.9

However, what is the velocity of such particles? The particles are so tiny that we do not see them. And even worse, they go so fast, that even if we could see them, probably we would not be able to measure their velocity! However, here comes the smart move. Remember that magnetic fields do no work! So, if we turn on the magnetic field shown in the figure the velocity of the outgoing electrons will be the same! The magnetic field will not accelerate them, just deviate them! Hence, if the magnitude of the magnetic and electric force exerted on the electrons are equal, then the electrons follow a straight line because the forces cancel out! (we can see this in figure 7.9b). In such scenario, then

$$|\vec{F}_E| = |\vec{F}_B| \Longrightarrow q|\vec{E}| = q|\vec{B}||\vec{v}|\sin\theta_{BV} \tag{7.62}$$

where $\sin\theta_{BV} = 1$ because the velocity vector and the magnetic field are perpendicular . So, we have that

$$|\vec{v}| = \frac{|\vec{E}|}{|\vec{B}|} \tag{7.63}$$

Marvelous ! Now, we know the speed of the electrons! So think about it experimentally, you create the electric field between the plates, then you make a current go in the solenoids. You calibrate both of them until you see a point in straight line from where the electrons are going out. Measure the magnitude of the electric field, the magnitude of the magnetic field and use equation 7.61

$$\frac{q}{m} = \frac{\left(|\vec{E}|\Big/|\vec{B}|\right)^2}{|\vec{E}|r} = \frac{|\vec{E}|}{|\vec{B}|^2 r} \tag{7.64}$$

The radius of the path that the electron describes can also be measured! So you have everything to determine the ratio of charge-mass of one of the fundamental particles of the Universe! If we plug in the values given by the exercise

$$\frac{q}{m} = \frac{2\times 10^5\,\frac{\mathrm{V}}{\mathrm{m}}}{\left(5\times 10^{-3}\mathrm{T}\right)^2\left(4.55\times 10^{-2}\mathrm{m}\right)} \approx 1.75\times 10^{-11}\,\frac{\mathrm{C}}{\mathrm{kg}} \tag{7.65}$$

where we obtained the famous result of the fraction of the charge of the electron and its mass. Now, the exercise gives us what Robert Millikan (and his student) found in the oil droplets experiment. The oil droplets always got a charge multiple of a very specific number.

$$q = -n1.6\times 10^{-19}C \tag{7.66}$$

so n of such elementary particles must have stocked to the droplets of oil. So, taking this experimental result, and arguing that what smashes against the florescent screen in the cathode rays are n of these elementary particles (we can not know if one particle collides against the screen or two, we can say that n of them have smashed when we see the light in the fluorescent screen); then, what smashes against the screen has a mass $m = nm_e$

and a magnitude of charge $q = n1.6 \times 10^{-19}C$
So, plugging in the values given:

$$m_e = \frac{1.75 \times 10^{-11} \frac{\text{C}}{\text{kg}}}{9.1 \times 10^{-31}\text{kg}} \approx 9.1 \times 10^{-31}\text{kg} \tag{7.67}$$

where the label of the electron has been written. The mass of the electron! Congratulations, you have just calculated the mass of a fundamental particle in Nature!

Example 5: Mass Spectrometer

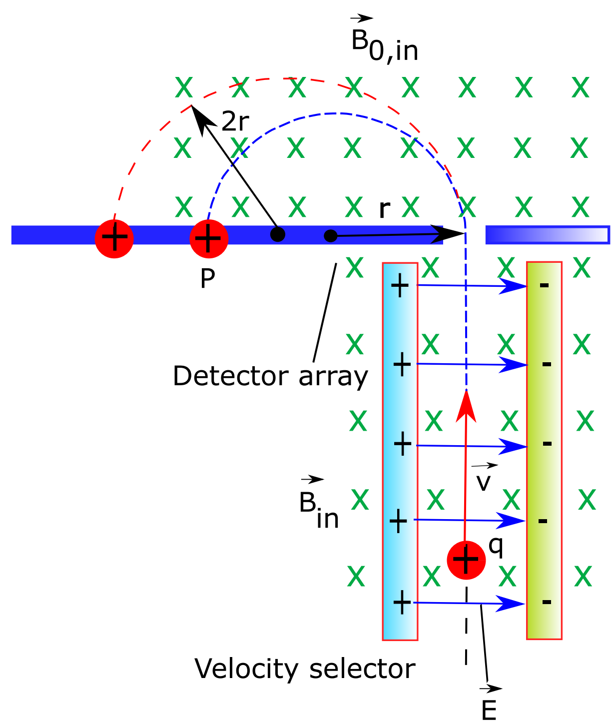


Figure 7.10

A mass spectrometer is a powerful apparatus to determine the chemical composition of any material. It works with magnetic and electric fields to separate the molecules depending their charge-mass ratio q/m. So ,if we are capable of ionizing the gas we want to analyze, and all the molecules obtain the same electric charge, we can separate them then by their mass. There are many different designs for a mass spectrometer, but the physical basis is the same. One simple configuration is shown in figure 7.10. The molecules go into a section of the spectrometer with certain velocity $\vec{v}$ where there are a magnetic and an electric field perpendicular one to each other (in the figure case the magnetic field goes into the page while the electric field to the right). This section of the mass spectrometer is called as *speed selector*. Actually, notice that the configuration of the fields in this part of the spectrometer of masses is exactly the same as the section between the plates P and

P' in figures 7.8b and 7.9b studied in the last exercise. So, the name follows because only the molecules with certain speeds will go in a straight line. If the speed is lower or greater than the speed where the magnetic and electric force balances out, the molecules will simply deviate and collide with the plates. Now, after the speed selector, the molecules with the appropriate speed go through a section where there is a magnetic field (see figure 7.10). Here, depending of their mass and electric charge the molecules will be deviated. However, assuming all the molecules have the same electric charge, then they deviate more or less depending on their mass. How exactly does this work? Notice that at this stage, is exactly the same what happened when when we analyzed the helix path in two dimensions, i.e. the molecules are under the influence of the magnetic force, which acts as a centripetal force so

$$|\vec{F}_B| = |q||\vec{B}||\vec{v}| = m\frac{v^2}{r} \tag{7.68}$$

So, the radius that each molecule describes is

$$r = \frac{m|\vec{v}|}{|q||\vec{B}|} = \frac{|\vec{p}|}{|q||\vec{B}|} \tag{7.69}$$

which is dependent only of the electric of the mass (or momentum) if all the molecules have the same velocity and electric charge. If it is the case, the heavier the particle greater the radius and we can separate the elements in a gas for instance. This is extremely powerful! It has so many applications the mass spectrometer, ranging from anthropological applications and medical to warlike applications.

One particular application of the mass spectrometer is for separating isotopes from a material, as scientists did during the Second World War II. Professor Ernest Lawrence, from University of California in Berkeley, built a huge mass spectrometer to separate isotopes of Uranium, specifically ^{235}U and ^{238}U. Isotopes are variants of a particular chemical element which differ in neutron number, and consequently in nucleus number. All isotopes of a given element have the same number of protons but different number of neutrons on each atom. For example, neutrally charged Uranium has 92 electrons, therefore also 92 protons. However, ^{235}U has 143 neutrons, while ^{238}U has 146 neutrons. As it turns out, the uranium obtained from mines is mostly composed of ^{238}U (approximately 99.28%) and only about the 0.7% of ^{235}U. Both are radioactive, however ^{235}U needs much lower

Figure 7.11: Professor Ernest Lawrence with a cyclotron. He was the inventor of the cyclotron and the so called calutron. He used calutrons to separate Uranium isotopes to build the first atomic bombs. Original picture taken from reference [16].

energy to establish a nuclear chain reaction, i.e. if we fire a neutron to the nucleus of the atom ^{235}U it becomes ^{236}U which is highly unstable and it breaks into atoms with lower number of protons. Specifically, it could break into $^{89}_{36}Kr$ and $^{144}_{56}Ba$ (this is not the only possibility, but one of the most common in these nuclear reactions). However, notice that we are missing neutrons. The $^{89}_{36}Kr$ has 53 neutrons while $^{144}_{56}Ba$ has 88 neutrons, and ^{236}U has 144 neutrons. The remaining three neutrons are now free, and all with enough energy to collide with another ^{235}U nucleus and cause more atoms to break. That's why we say it is a chain reaction. The break of one ^{235}U atom, due to the free neutrons breaks another three, these three break another nine! those nine break 27! those 27 another 81 and so on! This is shown schematically in figure 7.12

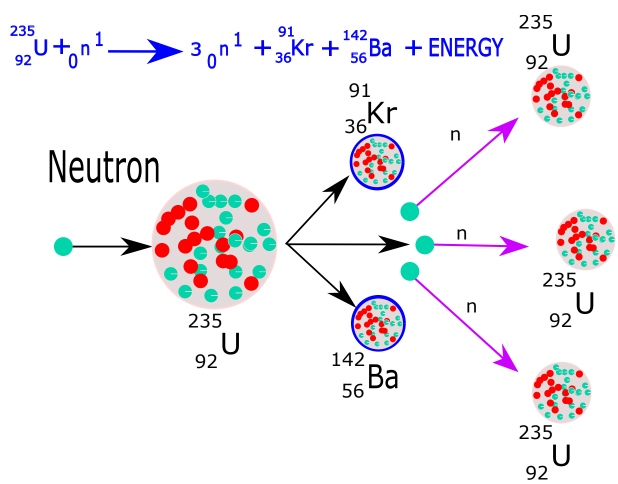


Figure 7.12

Professor Lawrence mass spectrometer, used a cyclotron to accelerate the isotopes to great speeds. Then, he used a speed selector and separated the isotopes due to their mass difference. The obtained isotopes of ^{235}U were used to build the first atomic bombs that were detonated in Hiroshima and Nagasaki in 1945. The combination of the mass spectrometer and the cyclotron was named as *Calutron*. The name is due to *California University Cyclotron* , honoring the University of California Berkeley.

So, lets make some simple calculations to get a better grasp of how to separate ^{238}U and ^{235}U by using a mass spectrometer. First, let's calculate the mass of one single ^{235}U atom and also one single atom mass of ^{238}U. So, we have have

$$m_{^{235}U} = (235.04)(1.66 \times 10^{-27} kg) \approx 3.9 \times 10^{-25}\text{kg} \tag{7.70}$$

$$m_{^{238}U} = (238.051)(1.66 \times 10^{-27} kg) \approx 3.95 \times 10^{-25}\text{kg} \tag{7.71}$$

where we used the respective atomic masses of ^{238}U ^{238}U and the conversion factor $1.66 \times 10^{-27} kg$ to obtain their masses in kilograms. So, taking a magnetic field of $B = 1.0T$ and suppose we ionize the uranium isotope with the electric charge of the proton, i.e. we pull out one electron of the isotope uranium atoms. So that their electric charge is the electric charge of the proton $q = 1.6 \times 10^{-19}C$. Therefore, using equation 7.69, the radius that the path of the ^{235}U makes is

$$r_{^{235}U} = \frac{(3.9 \times 10^{-25}kg)(2 \times 10^{6}m/s)}{(1.6 \times 10^{-19}C)(1.0T)} \approx 4.87\ m \tag{7.72}$$

while for ^{238}U:

$$r_{^{238}U} = \frac{(3.95 \times 10^{-25}kg)(2 \times 10^{6}m/s)}{(1.6 \times 10^{-19}C)(1.0T)} \approx 4.93\text{m} \tag{7.73}$$

Notice that the distance from the slit where the atoms go out to the place where they collide is $2r$. So,

$$2r_{^{238}U} - 2r_{^{235}U} \approx 9.86\text{m} - 9.74\text{m} \approx 0.12\text{m} \tag{7.74}$$

So about just $12cm$ of separation! of course this is a big mass spectrometer! But it is feasible to separate the uranium isotopes. To have smaller radius we can increase the magnetic field.

7.2 Exerted forces on electric currents

Imagine a steady current in certain wire where there is a magnetic field with arbitrary direction as shown in figure 7.13. So, this current is made of bunch of electric charges moving with certain velocity $\vec{v}$ and all and each of them are influenced under a magnetic field force. So, if we define as the charge density of the electric charges that **move** as λ, we can write the electric current as

$$\vec{I} = \lambda\vec{v} \tag{7.75}$$

where $\vec{v} = d\vec{l}/dt$. In other words, what we say is that the charges that move in an infinitesimal part of the wire in certain infinitesimal time, constitute the electric current. Notice something we have not payed much attention before. As we mentioned in chapter 5, the electric current is a **vector**. We have mostly worked with the magnitude of it and in cases on defining a direction, we just picked a $+$ or $-$ sign. Now , the force exerted on a segment of the wire

$$\vec{F} = \int dq\vec{v} \times \vec{B} = \int \lambda dl\vec{v} \times \vec{B} = \int \vec{I} \times Bdl \tag{7.76}$$

where we took that each infinitesimal length carries electric charge dq; also we used that $\lambda = \frac{dq}{dl}$. However, the direction of the vector $d\vec{l}$ is the same as the electric current, so we can write the last equation as

$$\boxed{\vec{F} = \int Id\vec{l} \times \vec{B}} \tag{7.77}$$

where I is the magnitude of the electric current vector. If the electric current is constant in magnitude along the wire in the last equation we can take the current out of the integral.

For the particular case when the current, the magnetic field are constant and the wire is completely straight, we have that the force exerted on the straight wire is

$$\boxed{\vec{F} = I\vec{L} \times \vec{B}} \tag{7.78}$$

where the magnitude of $\vec{L}$ is the length of the wire and its direction is the direction of the electric current.

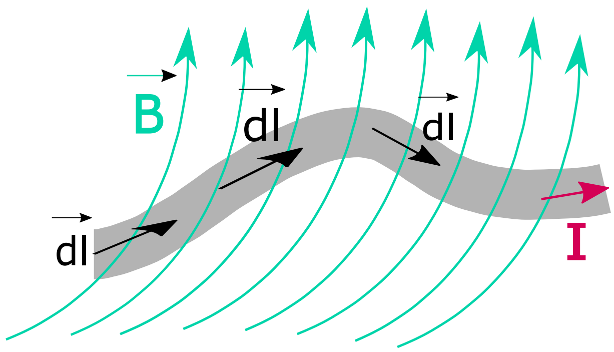


Figure 7.13

Example 6: Force on a straight wire exercise

A thin wire of 3m transports an electric current of 20A, it lies in in the y axis and the current direction is to $-y$. The wire is perpendicular to a magnetic field. A magnetic force of $0.50N$ is exerted over the rod in $+z$direction .

- Determine the magnitude of the magnetic field
- Determine the direction of the magnetic field

Solution:
This one is quite easy. The magnitude of force is just:

$$|\vec{F}| = IL|\vec{B}|\sin\theta \tag{7.79}$$

So, isolating the magnetic field magnitude and substituing the values

$$|\vec{B}| = \frac{|\vec{F}|}{Il\sin\theta} = \frac{0.5N}{(20A)(3m\sin 90)} = 8.33 \cdot 10^{-3}T \tag{7.80}$$

where the angle is 90^0 because the exercise mentions that magnetic field and the wire are perpendicular to each other.

Finally, using the right hand rule, we see that the magnetic field direction is to $+\hat{i}$. Therefore:

$$|\vec{B}| = 8.33 \cdot 10^{-3}T\hat{i} \tag{7.81}$$

Example 7: Force exerted on a semicircle shaped wire

In the figure the magnetic field is uniform with magnitude $|\vec{B}| = 2.5$T. The conductor wire transports an electric current $I = 4.5$A. Find the magnetic force exerted on the wire $L = 2.5m$ and $R = 1.2$m.
Solution:

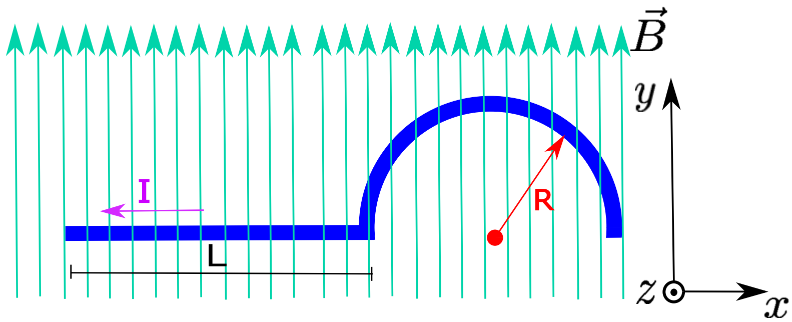


Figure 7.14

We solve the exercise in two parts. Let's see first what is the exerted force on the semicircle curve. By using the right hand rule, we notice that the force all along the semicircular wire points to the $-\hat{z}$ direction. So, we already know the direction, let's see the magnitude of the force. To obtain it we integrate over all the semicircular wire:

$$|\vec{F}_s| = \int I|d\vec{l} \times \vec{B}| = \int I|\vec{B}| \sin\theta_{IB} dl \tag{7.82}$$

where θ_{IB} stands for the angle between the electric current and the magnetic field. However notice that the angle between the x - axis and the radius R, where there is an electric current and an element $d\vec{l}$ is exactly the same angle as the angle between the electric current and the magnetic field ! So, we can use the arc length of the differential angle as:

$$dl = Rd\theta \tag{7.83}$$

And the integral becomes:

$$|\vec{F}_s| = \int_0^{\pi} IR|\vec{B}| \sin\theta d\theta \tag{7.84}$$

where the limits of integration follow because we start from zero up to half of a circle. So,

$$F_1 = -IR|\vec{B}| \cos\theta \Big|_0^{\pi} = -IRB(-1-1) = 2IR|\vec{B}| \tag{7.85}$$

Substituting values and writing the direction of the force obtained with the right hand rule,

$$\vec{F}_s = 2(4.5A)(1.2m)(2.5T)((-\hat{k}) = -27N\hat{k} \tag{7.86}$$

Now,the force exerted on the horizontal wire can be found easily. By right hand rule, the force also points to $-\hat{k}$ and the magnitude is

$$|\vec{F}_H| = \int I|d\vec{l} \times \vec{B}| = I|\vec{B}| \int dl = I|\vec{B}|L \tag{7.87}$$

So, substituting values and writing the direction of the force

$$\vec{F}_H = (4.5A)(2.5T)(2.5m)(-\hat{k}) = -28.125\hat{k} \tag{7.88}$$

Hence, the total force exerted on the wire

$$\vec{F}_T = \vec{F}_H + \vec{F}_s = -28.125N\ \hat{k} - 27N\ \hat{k} = -55.125N\ \hat{k} \tag{7.89}$$

Example 8: Force exerted on a wire with two components

A rigid wire carrying a current $I = 5$A, consists of a semicircle of radius $R = 0.7$m and a straight portion of 1m as shown in figure 7.15. The wire lies in a plane perpendicular to a uniform magnetic field of magnitude $|\vec{B}| = 3.5$T. Determine the net force exerted on the wire due to the magnetic field.

Solution:

Once again, we split solve problem in two parts. We first see what's going on with the curved wire. We can think of it as a quarter of a complete circle. This time, notice that if we use the right hand rule, the force vector does not point to just one direction. If we move all along the curved part of the wire, the force vector has x and y components. So, we need to calculate:

$$F_x = \int dF_x = \int dF\cos\phi \qquad F_y = \int dF_y = \int dF\sin\phi \tag{7.90}$$

where, the angle ϕ is the angle between the perpendicular axis and the direction of the differential force as shown in figure 7.15. It is important that you keep in mind that this is not the angle that rises from the cross product in dF. Also, we have a $\cos\phi$ and a $\sin\phi$ respectively, where they are there in the corresponding component of the force. Now,

$$dF = I|d\vec{l} \times \vec{B}| = Idl|\vec{B}|\sin\theta_{IB} \tag{7.91}$$

where explicitly the angle is labelled so that you keep in mind which angle we are talking about. This

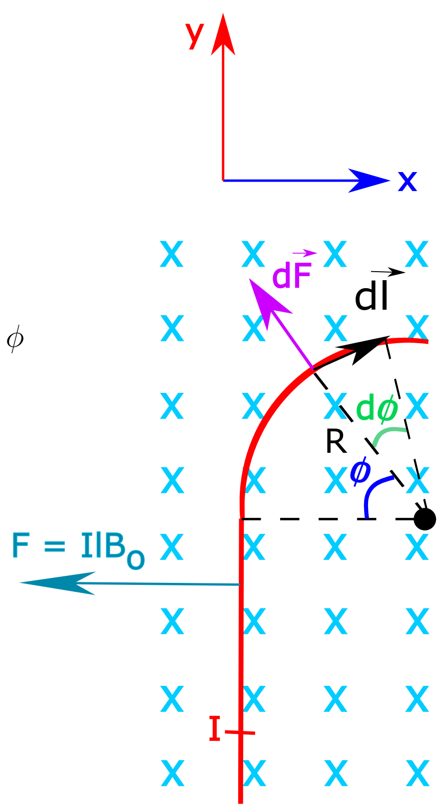


Figure 7.15

angle is the one between the current and the magnetic field which rises from the cross product magnitude. However, notice that the current and magnetic field are perpendicular all the time. Therefore,

$$dF = I|\vec{B}|dl \tag{7.92}$$

Now, using the arc length:

$$dl = Rd\phi \tag{7.93}$$

we have that:

$$dF = I|\vec{B}|Rd\phi \tag{7.94}$$

Hence,

$$F_x = \int_0^{\pi/2} I|\vec{B}|R\cos\phi d\phi = I|\vec{B}|R\sin\phi\Big|_0^{\frac{\pi}{2}} = I|\vec{B}|R \tag{7.95}$$

and

$$F_y = \int_0^{\pi/2} IBR\sin\phi d\phi = -IBR\cos\phi\Big|_0^{\frac{\pi}{2}} = -IBR[0-1] = IBR \tag{7.96}$$

where the limits of integration follow because we are integrating over a quarter of a circle. Hence, the force vector (written in components) exerted on the semicircular part of the wire is

$$\vec{F}_s = (-I|\vec{B}|R, I|\vec{B}|R) \tag{7.97}$$

where notice that we include a minus sign in the x component. The reason is that all contributions go to $-x$ direction. The integration result is the magnitude, you need to include also the direction! Now the force exerted on the vertical straight wire is just

$$\vec{F}_v = -I|\vec{B}|L\hat{x} = -(5A)(3,5T)(1m) = -17.5N\hat{x} \tag{7.98}$$

So, the total force

$$\vec{F}_T = \vec{F}_s + \vec{F}_v = (-I|\vec{B}|(L+R), I|\vec{B}|R) \tag{7.99}$$

So, by plugging just the values, we obtain that the total force is

$$\vec{F}_T = -((5A)(3.5T)(1m+0.7m)\ ,\ (5A)(3.5T)(0.7m)) = (-29.75N\ ,\ 12.25N) \tag{7.100}$$

7.3 Torque on conducting wires

Consider the scenario shown in figure 7.16a. An electric current flows in a wire with the direction shown. Given that there is a magnetic field, a force is exerted on the wire, except to the wire at the top where the current direction is parallel to the magnetic field and the bottom wire where the current direction is anti-parallel to the magnetic field. These two parts of the circuit are not affected by a magnetic field force, because for either case the exerted force magnitude is zero, i.e.$|\vec{F}| = I|\vec{L}||\vec{B}|\sin(180^0) = 0$ and

$\vec{F} = I|\vec{L}||\vec{B}|\sin(0) = 0$. The forces, $\vec{F_1}$ applied to the left arm of the wire and $\vec{F_2}$ applied to the right arm of the wire, are

$$\vec{F_1} = -Ia|\vec{B}|\hat{z} \qquad , \qquad \vec{F_2} = Ia|\vec{B}|\hat{z} \tag{7.101}$$

where we used the right hand rule and the magnitude of the cross product $I\vec{L}\times\vec{B}$,noticing that $|\vec{L}| = a$. Now, recall that torque is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{7.102}$$

So, the generated torques by each force are

$$\vec{\tau}_1 = -\frac{b}{2}F_1\hat{j} = -Ia\frac{b}{2}|\vec{B}|\hat{j} \qquad , \qquad \vec{\tau}_2 = -\frac{b}{2}F_2\hat{j} = -Ia\frac{b}{2}|\vec{B}|\hat{j} \tag{7.103}$$

where we used the right hand rule, and magnitude of $\vec{r} \times \vec{F}$, noticing that $|\vec{r}| = b/2$. So the total torque is:

$$\vec{\tau}_{Tot} = \vec{\tau}_1 + \vec{\tau}_2 = -Iab|\vec{B}|\hat{j} \tag{7.104}$$

So, given that there is a total torque different of zero, a rotational motion will take place. Notice that the product ab is the area enclosed by the wire loop. In general, there could be N loop wires, carrying each a current I. Therefore, in general the total torque is

$$\vec{\tau}_{Tot} = -NIA|\vec{B}|\hat{j} \tag{7.105}$$

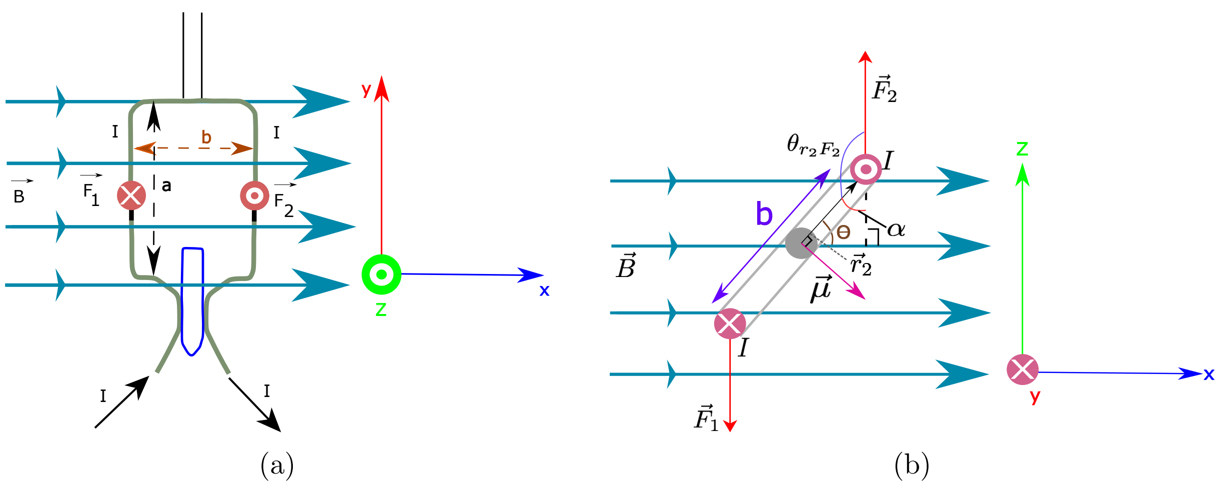


Figure 7.16

We define a vector which we call as *dipole magnetic moment*

$$\boxed{\vec{\mu} = NI\vec{A}} \tag{7.106}$$

where the vector $\vec{A}$ is perpendicular to the area that the loops encloses and its magnitude is the area of the loop. So,

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{7.107}$$

where the magnetic moment direction is found by using the right hand. You curl your fingers in the direction of the current, so your thumb finger is the $\vec{\mu}$ direction as shown in figure 7.17. Also, we have removed the label total in τ, but of course is the total torque that the wire experiences.

Now, probably you think, what happens if the wire has already rotated and is not exactly as in figure 7.16a, and now looks the configuration of the circuit as in figure 7.16b. Is the torque still given by equation 7.107? As it turns out, it still holds. To demonstrate this fact, place yourself to the $x - z$ plane, so, you see the figure as the one shown in 7.16b. By using the right hand rule, notice that the torque of the forces $\vec{F_1}$ and $\vec{F_2}$ point outwards the page (to -$\hat{y}$ according to our reference frame). So, the torques are:

$$\vec{\tau}_1 = -\frac{b}{2}|\vec{F_1}|\sin\theta_{bF_1}\hat{j} = -\frac{b}{2}Ia|\vec{B}|\sin\theta_{r_1F_1}\hat{j} \quad , \quad \vec{\tau}_2 = -\frac{b}{2}|\vec{F_2}|\sin\theta_{bF_2} = -\frac{b}{2}Ia|\vec{B}|\sin\theta_{r_2F_2}\hat{j} \tag{7.108}$$

where the label $\theta_{r_1F_1}$ means the angle between the $\vec{r}_1$ vector and the force vector $\vec{F_1}$, and $\theta_{r_2F_2}$ the angle between the $\vec{r}_2$ vector and the force vector $\vec{F_2}$. However, these angles are exactly the same

$$\theta_{r_1F_1} = \theta_{r_2F_2} \tag{7.109}$$

and we will remove the label and keep the angle just as θ_{rF}. Now probably you think what happens to the top and bottom parts of the wire. Do they contribute to the total torque? The force exerted on the top part of the wire and the bottom is not zero any more. Well indeed, however notice that their $\vec{r}$ vectors are parallel and anti-parallel respectively to the force they experience, so $\sin\theta = 0$ in the cross product of the magnitude of the torque $\tau = \vec{r}\times\vec{F}$. Therefore, they do not contribute to the total torque. So, the total torque is:

$$\vec{\tau}_{Tot} = \vec{\tau}_1 + \vec{\tau}_2 = -\frac{b}{2}IA|\vec{B}|\sin\theta_{rF}\hat{j} - \frac{b}{2}IA|\vec{B}|\sin\theta_{rF}\hat{j} = -IA|\vec{B}|\sin\theta_{rF}\hat{j} \tag{7.110}$$

Once again, in general there could be N wires transporting a current I. So, the total torque becomes

$$\vec{\tau}_{Tot} - NIA|\vec{B}|\sin\theta_{rF}\hat{j} \tag{7.111}$$

The last equation becomes equation 7.107 if $\sin\theta_{rF} = \sin\theta_{\mu B}$. Let's show that indeed is the case. See figure 7.16b, $\vec{\mu}$ is a perpendicular vector to the area of the loop with the direction shown in the figure. From the figure notice that:

$$\theta_{rF_2} + \alpha = 180^0 \qquad , \qquad \theta + \alpha + 90^0 = 180^0 \tag{7.112}$$

where in the second equality we used that the sum of angles in a triangle is 180^0. Therefore equating both relations,

$$\theta_{rF_2} = \theta + 90^0 \Rightarrow \sin(\theta_{rF_2}) = \cos\theta \tag{7.113}$$

where we just used the trigonometric identity that for any angle β, $\sin(\beta + 90^0) = \cos\beta$. However,

$$\theta_{\mu B} = 90^0 - \theta \Rightarrow \sin\theta_{\mu B} = \cos\theta = \sin(\theta_{rF_2}) \tag{7.114}$$

where we just used the trigonometric identity that for any angle β, $\sin(90^0 - \beta) = \cos\beta$. Therefore, finally we have shown that in general

$$\vec{\tau}_{Tot} = -NIA|\vec{B}|\sin\theta_{rF_2}\hat{j} = -NIA|\vec{B}|\sin\theta_{\mu B}\hat{j} \tag{7.115}$$

So, in general for any loop with current, the torque experienced by the loop is given by

$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}} \tag{7.116}$$

where $\vec{\mu} = NI\vec{A}$ where the area vector can be found with the right hand curling all your fingers except the thumb in the direction of the current, and the thumb finger points to the direction of $\vec{\mu}$ as shown in figure 7.17

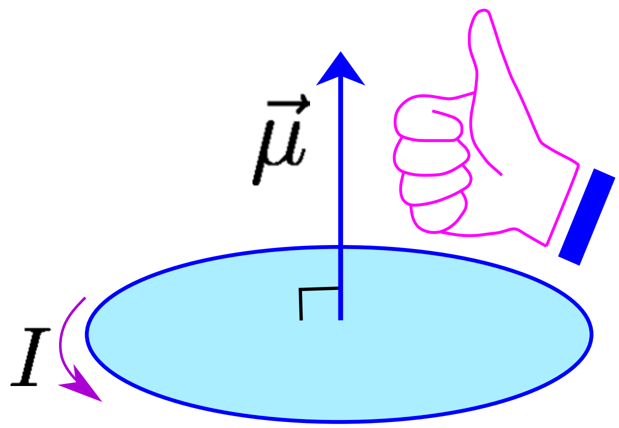


Figure 7.17

Example 9: Torque on a circular wire

A circular coil has a diameter of 40.0cm. The number of loops is 7. The electric current in each loop is 3.00A , and the coil is located in an external magnetic field of 2.0T . Determine the magnitude of the

- magnetic dipole moment
- the maximum and minimum torque exerted on the coil by the field.

Solution:
The torque magnitude can be easily found by:

$$|\vec{\tau}| = |\vec{\mu}||\vec{B}|\sin\theta \tag{7.117}$$

where:

$$|\vec{\mu}| = NI|\vec{A}| = 7 \cdot (3A)(\pi \cdot (10 \cdot 0.20m)^2) = 2.638\text{Am}^2 \tag{7.118}$$

Now, the torque will be a maximum or a minimum when the $\sin\theta$ is maximum or minimum in magnitude. The maximum value of the $\sin\theta$ function is 1 and minimum is 0. Therefore:

$$|\vec{\tau}|_{max} = (2.638Am^2)(2.0T) = 5.27Nm \tag{7.119}$$

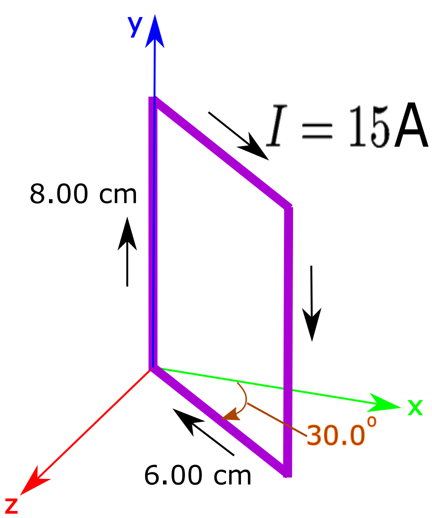
and of course

$$|\vec{\tau}|_{min} = 0Nm \tag{7.120}$$

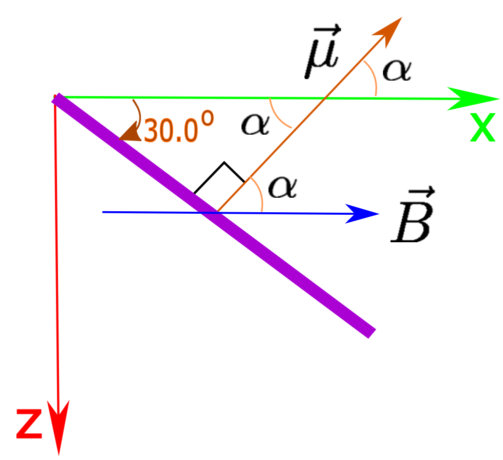
Now, this is quite important for direct current motors, where there is a magnetic field to produce the rotation, every $n\pi$, the motor will stop running smoothly. When the angle is $n\pi$, there is no torque forcing the wire to rotate. So, at that moment it rotates due to its inertia. To solve this problem, in a motor you need several loops with different $\vec{\mu}$ directions, so that when one loop torque is zero, another one is different of zero and the movement is smooth as possible.

Example 10: Torque on a pivoted rectangular loop

A rectangular loop is pivoted about the $y-axis$ and carries a current of 15A in the direction shown in figure 7.18a. If the loop is immersed in a uniform magnetic field with magnitude $0.48T$ in the $+x$ direction, find the torque of an external agent required to hold the loop in the position shown.



(a)



(b)

Figure 7.18

Solution:

By using the right hand to determine the direction of $\vec{\mu}$, we see that it's direction is as shown on the figure 7.18b. In such figure we moved to the x - z plane and we see the wire from the top. Now, the angle between the magnetic field and the $\vec{\mu}$ vector is α not 30^0! Be careful! So, the magnitude of the torque that the magnetic field exerts is

$$|\vec{\tau}_B| = \mu B\sin(60^0) = (15\text{A})(6\cdot 10^{-2}\text{m})(8\cdot 10^{-2}\text{m})(0.48\text{T})\sin(60^\circ) \tag{7.121}$$

So,

$$|\vec{\tau}_B| = -0.027Nm \quad \hat{j} \tag{7.122}$$

where we used the right hand rule to determine the direction of the torque. However the exercise asks for the torque that certain external agent needs to exert so that the wire does not rotate! So, the torque that needs to be applied to cancel out $\vec{\tau}_B$, must be of the same magnitude but opposite direction. Hence,

$$\vec{\tau}_E = 0.027Nm \quad \hat{j} \tag{7.123}$$

where the label “E” stands just for external agent.

Chapter 8

Sources of Magnetic Field

In this chapter, we study how the magnetic fields are generated by steady electric currents, so we study static magnetic fields. We study different configurations of conducting wires and their corresponding generated magnetic fields.

8.1 Bio-Savart Law

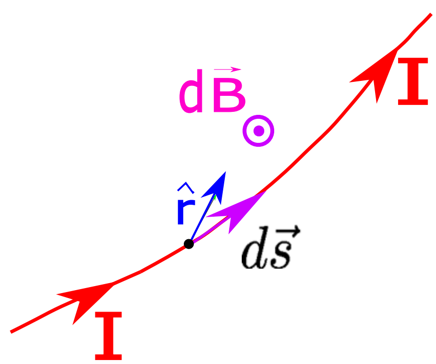


Figure 8.1

The Bio-Savart Law is given by

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{s} \times \hat{r}}{r^2}} \tag{8.1}$$

where I is the electric current that generates the magnetic field, r is the distance from the infinitesimal element of length $d\vec{s}$ to the point where we want to calculate the magnetic field, $\hat{r}$ is a unitary vector pointing from the infinitesimal element to the point of interest, and

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \frac{\mathrm{T} \cdot \mathrm{m}}{\mathrm{A}}} \tag{8.2}$$

The units T (*Tesla*) are the units for magnetic fields (the international system of units). The name for the units of magnetic field is in honour of one of the greatest inventors of all times, *Nikola Tesla*. Another highly used unit for magnetic fields are the so called *Gauss*(G). The conversion from Tesla to Gauss units is

$$\boxed{1\text{G} = 1 \times 10^{-4}\text{T}} \tag{8.3}$$

The equation 8.1 was forged experimentally. In April 1820, the Danish physicist and chemist Hans Christian Ørsted discovered that a needle in compass would deviate if an electric current was placed nearby. Few months later, he discovered the circular configuration of a magnetic field around a conducting wire. Not much time passed, for two great French scientists to get highly interested in the experimental results of Ørsted. Jean-Baptiste Biot and Félix Savart discovered the following

- The magnetic field generated by an infinitesimal element in the wire that carries electric current in a conductor is perpendicular to the electric current direction.

- The magnetic field mentioned previously is perpendicular to the vector that joins an infinitesimal element that carries current in the wire to the point where the magnetic field is analyzed.

- The magnetic field is proportional to $\frac{1}{r^2}$, where r is the distance from an infinitesimal element that carries electric current in the wire to the point where the magnetic field is analyzed.

- The magnetic field is proportional to the angle between the electric current in an infinitesimal element of the wire and the vector that points from the infinitesimal element to the point we are analyzing the magnetic field.

The previous observations, led to equation 8.1 when all the contributions of the infinitesimal elements that carries an electric current in a conducting wire are taken into account. The Ørsted experiment, and the Bio-Savart Law were the first experimental and theoretical discoveries that connected electric phenomena with magnetism, and from then our world would never be the same.

Example 1: Magnetic Field of Finite and Infinite Thin Conductor carrying steady current

Find the magnetic field at a point P with coordinates $(0, a)$ due to a perpendicular thin and straight conductor which carries an electric current I as shown in figures 8.2a and 8.2b.

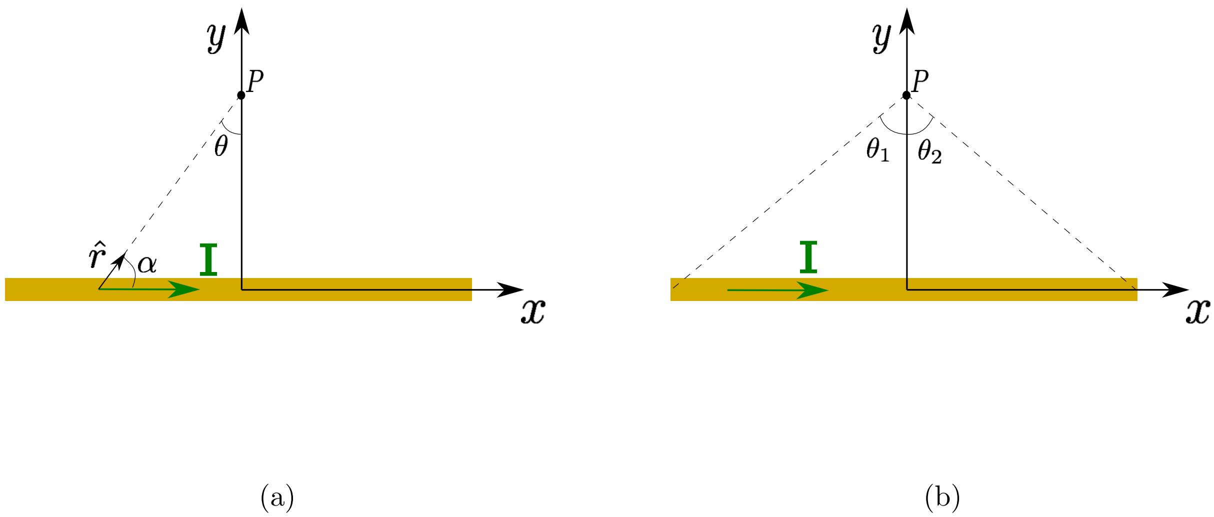


Figure 8.2

Solution:
We have to calculate

$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{d\hat{s} \times \hat{r}}{r^2} \tag{8.4}$$

at point P. So, first let's see what happens with the cross product $d\vec{s} \times \hat{r}$. Recall that $d\hat{s}$ direction is the direction of the electric current and $\hat{r}$ is a unitary vector starting from $d\vec{s}$ and pointing towards where we want to calculate the magnetic field. By using the right hand rule ((index finger) the first vector in the cross product, middle finger the second vector in the cross product, and thumb finger the vector as result of the cross product) we have that the result vector of the cross product points to +z (out of the page). While, its magnitude

$$|d\hat{s} \times \hat{r}| = ds|\hat{r}| \sin\alpha = ds \sin\alpha \tag{8.5}$$

where α is the angle between the vector d$\vec{s}$ and $\hat{r}$. From the triangle formed by the extension of the unitary vector $\hat{r}$, the y axis and the electric current shown in the figure 8.2b, we have that

$$\alpha + \theta + \pi/2 = \pi \rightarrow \alpha = \theta - \pi/2 \tag{8.6}$$

where we just used that the sum of the angles in a triangle is 180^0 (π radians). Therefore, the magnitude of $d\vec{s} \times \hat{r}$ becomes

$$|d\vec{s} \times \hat{r}| = ds \sin(\theta - \pi/2) = ds \cos\theta \tag{8.7}$$

where we just used the trigonometric identity $\cos\theta = \sin(\theta - \pi/2)$. So, substituting $d\vec{s} \times \hat{r}$ in the Bio-Savant Law

$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{ds \cos\theta}{r^2} \hat{k} \tag{8.8}$$

Now, we have in our integral the non constant elements, ds, $\cos\theta$ and r. We need to leave all of them in terms of just one variable to integrate. So, before proceeding to the integral, from figure 8.2a, we have that

$$\cos\theta = \frac{a}{r} \Rightarrow \qquad r = \frac{a}{\cos\theta} \tag{8.9}$$

Once again from figure 8.2a, we have that

$$\tan\theta = \frac{s}{a} \tag{8.10}$$

where s will be a distance element in the $-x$ axis. Given that $s < 0$ and $\tan\theta > 0$ according to our reference frame; then ,

$$\tan\theta = -\frac{s}{a} \Rightarrow s = -a\tan\theta \Rightarrow ds = -asec^2 d\theta \tag{8.11}$$

Hence, substituting what we obtained for r in equation 8.9 and what we obtained for ds in equation 8.11 into equation 8.8, we have that

$$\vec{B} = -\frac{\mu_0}{4\pi} I \int (a\sec^2\theta d\theta)\left(\frac{\cos\theta}{a}\right)^2 \cos\theta\, \hat{k} = -\frac{\mu_0}{4\pi a} I \int \cos\theta d\theta\, \hat{k} \tag{8.12}$$

where we used that $\sec\theta = 1/\cos\theta$. Finally, we can say that the angle from the y axis to the end points of the conductor, starts from certain angle θ_1 up to certain angle θ_2 as shown in figure 8.2b. So,

$$\vec{B} = -\frac{\mu_0}{4\pi a} I \int_{\theta_1}^{\theta_2} \cos\theta d\theta\, \hat{k} = \frac{\mu_0}{4\pi a} I[\sin\theta_1 - \sin\theta_2]\, \hat{k} \tag{8.13}$$

The last result applies for any wire, however what if the wire is infinitely long? So, $\theta_1 \to +\pi/2$ and $\theta_2 \to -\pi/2$, hence, for an infinite wire

$$\vec{B} = \frac{\mu_0}{4\pi a} I(+1+1)\, \hat{k} = \frac{\mu_0}{2\pi a} I\, \hat{k} \tag{8.14}$$

Example 2: Magnetic Field due to a conductor with two straight sections and a curved section

Calculate the magnetic field at point O, generated by the electric current that is carried by the conductor wire shown in the figure 8.3. The wire consists of two straight portions and a semicircle with radius a, which extends an angle θ.

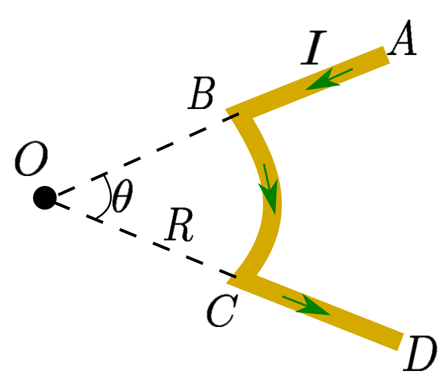


Figure 8.3

Solution:
Let's see what is the contribution of the magnetic field at point O by segments $A-B$, $B-C$ and $C-D$ independently.
For segment $A-B$,

$$|d\vec{s}\times\hat{r}| = ds|\hat{r}|\sin 0 = 0 \tag{8.15}$$

For segment $C-D$

$$|d\vec{s}\times\hat{r}| = ds|\hat{r}|\sin 180 = 0 \tag{8.16}$$

For both segments $A-B$ and $C-D$, the currents are parallel o anti parallel to the point of interest, so the magnetic field due to the current transported in these segments is zero.

Now for the segment $B-C$, the vectors $\hat{r}$ and $d\vec{s}$ are perpendicular because $d\vec{s}$ is tangent to the semicircle curve and $\hat{r}$ is along the radii of the semicircle. Also, using the right hand rule, we know that the direction of $d\vec{s}\times\hat{r}$ is $-\hat{k}$. So,

$$|d\vec{s}\times\hat{r}| = ds \tag{8.17}$$

Hence, the magnetic field generated by the current transported along the segment $A-C$ is

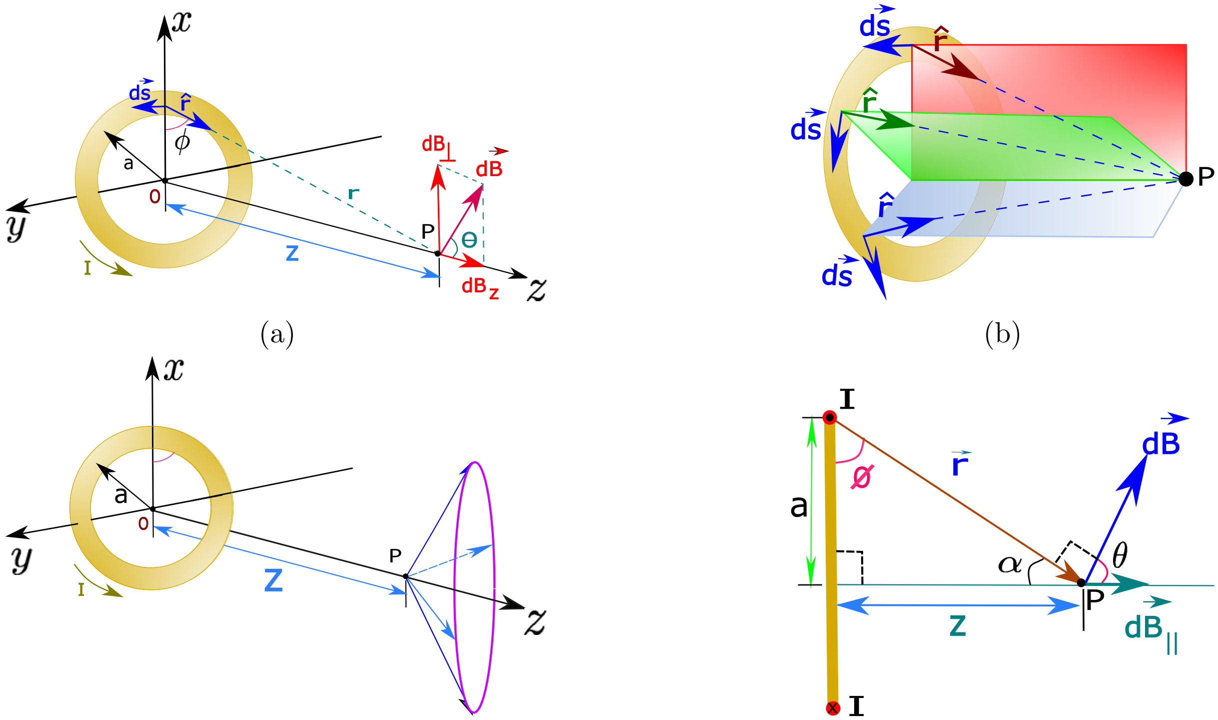
$$\vec{B} = \frac{\mu_0}{4\pi}I\int\frac{d\vec{s}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}I\int\frac{ds}{r^2}\,(-\hat{k}) = \left(-\frac{\mu_0}{4\pi R^2}I\int ds\right)\hat{k} \tag{8.18}$$

where we used that the distance r from any infinitesimal segment in the segment $B-C$ is $r=R$ (and obviously constant because is the radii of the semicircle). Also, we included the direction already. Now, using arc length $ds = Rd\theta$

$$\vec{B} = \left(-\frac{\mu_0}{4\pi R}I\int_0^\theta d\theta\right)\hat{k} = \left(-\frac{\mu_0}{4\pi R}I\theta\right)\hat{k} \tag{8.19}$$

Example 3: Magnetic field generated by a ring carrying steady current

Consider a circular wire of radius a placed in the $y-z$ plane. The wire carries a stable electric current I as shown in the figure. Calculate the magnetic field at point P shown in the figure.



(a) (b) (c) (d)

Figure 8.4

Solution:
We apply once again the Bio-Savart law

$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{s} \times \hat{r}}{r^2} \tag{8.20}$$

and we need to know how to tackle the cross product in the integral. By using the right hand rule, if we move along all along the ring, we would notice that a cone of magnetic field vectors is formed at point P due to the contribution of all the infinitesimal segments in the ring as shown in figure 8.4c. So, do we need to calculate three integrals in the x, y and z direction? Fortunately, all contributions in $\hat{x}$ and $\hat{y}$ component are cancelled out. For each vector, exists an opposite vector with opposite sign contribution in the x and y component and all z components sum, since they go to the same direction. So, we have to calculate just one integral, the z component magnetic field

$$B_z = \frac{\mu_0}{4\pi} I \int \frac{|d\vec{s} \times \hat{r}|}{r^2} \cos\theta \tag{8.21}$$

where we multiplied by *cos* θ, since we want to sum the contributions in the z component (see figure 8.4a). Now, the magnitude of the cross product in general is

$$|d\vec{s} \times \hat{r}| = ds \sin\theta_{sr} \tag{8.22}$$

where the label sr stands just to be clear that is the angle between the $d\vec{s}$ and $\hat{r}$ vectors. However, notice that vector $d\vec{s}$ and $\hat{r}$ lie in two different perpendicular planes as shown in the figure 8.4b. So, all vectors along the ring $d\vec{s}$ are perpendicular to $\hat{r}$. So, we just have that

$$|d\vec{s} \times \hat{r}| = ds \tag{8.23}$$

So, the integral in equation 8.21 becomes

$$B_z = \frac{\mu_0 I a}{4\pi} \int \frac{ds}{r^2} \cos\theta \tag{8.24}$$

Now, what is $\cos\theta$? Can it be written in other terms? Is it dependant of ds? Or can we just take it out from the integral? So, first lets show that actually angles θ, ϕ in figure 8.4a are exactly the same $\theta = \phi$. If we move to the $z - x$ plane we see the ring from one side as shown in figure 8.4d. So, we have from figure 8.4d that

$$\phi + \alpha + 90° = 180° \qquad \text{and,} \qquad \theta + \alpha + 90° = 180° \tag{8.25}$$

where the first equation follows because we summed the angles in a triangle which must be equal to 180°, and the second equation from the fact that we have 180° starting from the z axis up to the same z axis but in opposite direction. Therefore, equating both last equations, we have that

$$\phi = \theta \tag{8.26}$$

Probably you think "*Why did we do that? What was the purpose?*" Well, now we can express the $\cos\theta = \cos\phi$ in terms of the radius of the ring. So, using figure 8.4d, we have that

$$\cos\phi = \frac{a}{r} \Longrightarrow \cos\theta = \frac{a}{r} \tag{8.27}$$

because $\phi = \theta$. So, we have that $\cos\theta$ is just a constant and we have that the integral in equation 8.24 is

$$B_z = \frac{\mu_0 I a}{4\pi r^3} \int ds = \frac{\mu_0 I a}{4\pi r^3}(2\pi a) = \frac{\mu_0 I a^2}{2\left(z^2 + a^2\right)^{3/2}} \tag{8.28}$$

where we substituted

$$r = \sqrt{z^2 + a^2} \tag{8.29}$$

This result holds for any ring carrying stable current, at a symmetrical point P that is perfectly perpendicular to the ring. Notice, that the generated magnetic field is always perpendicular to the ring. So, we can simply say now that the total magnetic field is

$$\vec{B} = \frac{\mu_0 I a^2}{2\left(z^2 + a^2\right)^{3/2}}\ \hat{k} \tag{8.30}$$

because we already now that the other components at the end cancel out.

Now suppose we go extremely far from the ring, a distance much bigger than the ring radius ($z >> a$). What will be the magnitude of the magnetic field? So, if $z \gg a$, we can approximate.

$$|\vec{B}| \approx \frac{\mu_0 I a^2}{2z^3} \tag{8.31}$$

where we ignored a in the denominator, because it is negligible in comparison to z in this approximation. If we use the magnetic moment $\mu = IA = I\pi a^2$, we have that far away

$$|\vec{B}| \approx \frac{\mu_0 \mu}{2z^3} \tag{8.32}$$

The last result looks very similar to the dipole electric field from far sources.

$$|\vec{E}| = \frac{kp}{2z^3} \tag{8.33}$$

8.2 Magnetic Force between parallel conducting wires

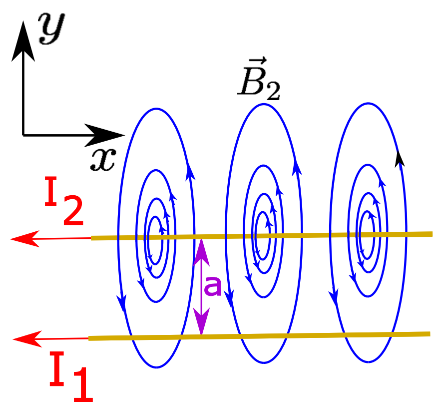


Figure 8.5

From the previous chapter, we already know how to calculate the force exerted on a conducting wire that carries an electric current given that there is an external magnetic field present. So, if the source of the magnetic field is a conducting wire with current I_2. What is the force exerted on another conducting wire that carries a current I_1? In other words we are interested to know what is

$$\vec{F}_{21} = \int I_1 d\vec{s} \times \vec{B}_2 \tag{8.34}$$

where $\vec{B}_2$ is the magnetic field generated by the current I_2 and exerts a force F_{21} on the wire with conducting current I_1. This can be seen in figure 8.5, the source of the magnetic field is the current I_2. Given that such magnetic field is present where I_1 is, we want to know the exerted force on the conducting wire carrying I_1. So, to simplify our analysis, we consider both wires that have stable currents I_1 and I_2 as infinite. So, the magnetic field generated by the current I_2 is given by equation 8.14. So, the magnitude of the magnetic field generated by I_2 is

$$|\vec{B}_2| = \frac{\mu_0}{2\pi a} I_2 \tag{8.35}$$

Therefore, the magnitude of the force is

$$|\vec{F}_{21}| = I_1 \frac{\mu_0}{2\pi a} I_2 \int ds \tag{8.36}$$

where the integral should extend from 0 up to ∞ , because we said that the wires where infinite! Indeed! The force exerted from an infinite source of magnetic field should exert an infinite force on a infinite body! That makes total sense. However, recall this was just an approximation, and it works very well in wires which are extremely long. So, in the integral we keep just a finite length. So,

$$|\vec{F}_{21}| = I_1 \frac{\mu_0}{2\pi a} I_2 L \tag{8.37}$$

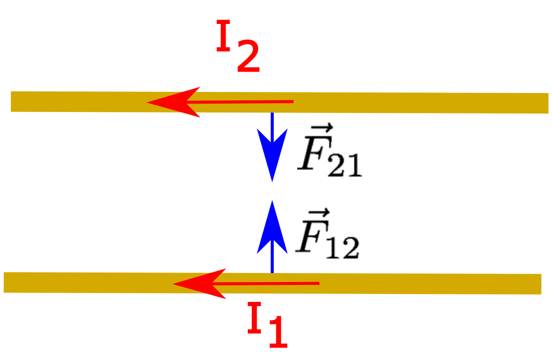
where L is the length of the wire conducting I_2 or also, it could be just certain length of the total length of the wire, in which case you would be calculating the force exerted in just that section of the wire. By right hand rule we have that the force is exerted to $+y$ direction. So, using the reference frame shown in the figure 8.5, we have that

$$\vec{F}_{21} = \mu_0 l \frac{I_2 I_1}{2\pi a} (+\hat{y}) \tag{8.38}$$

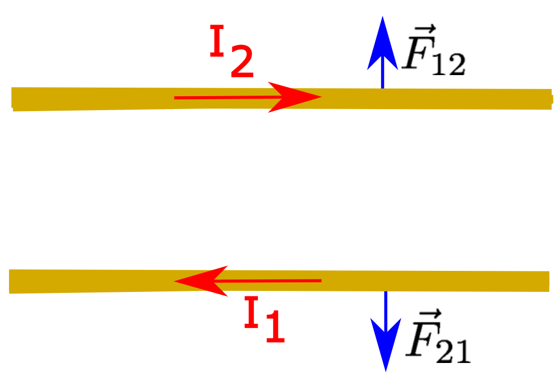
Finally, by third Newton's law, we have that the force that the wire with current I_1 exerts on the wire with current I_2 has the same magnitude but opposite direction. Therefore

$$\vec{F}_{21} = \mu_0 l \frac{I_2 I_1}{2\pi a} \, (-\hat{y}) \tag{8.39}$$

Now, notice the following, if the currents have the same direction as we assume from the beginning, the force is such that the wires will attract each other. while if the currents have opposite directions the force will be such that they repeal.



(a)



(b)

Figure 8.6

- Parallel conducting wires carrying electric currents with the same direction attract each other.
- Parallel conducting wires carrying electric currents with opposite direction repeal each other.

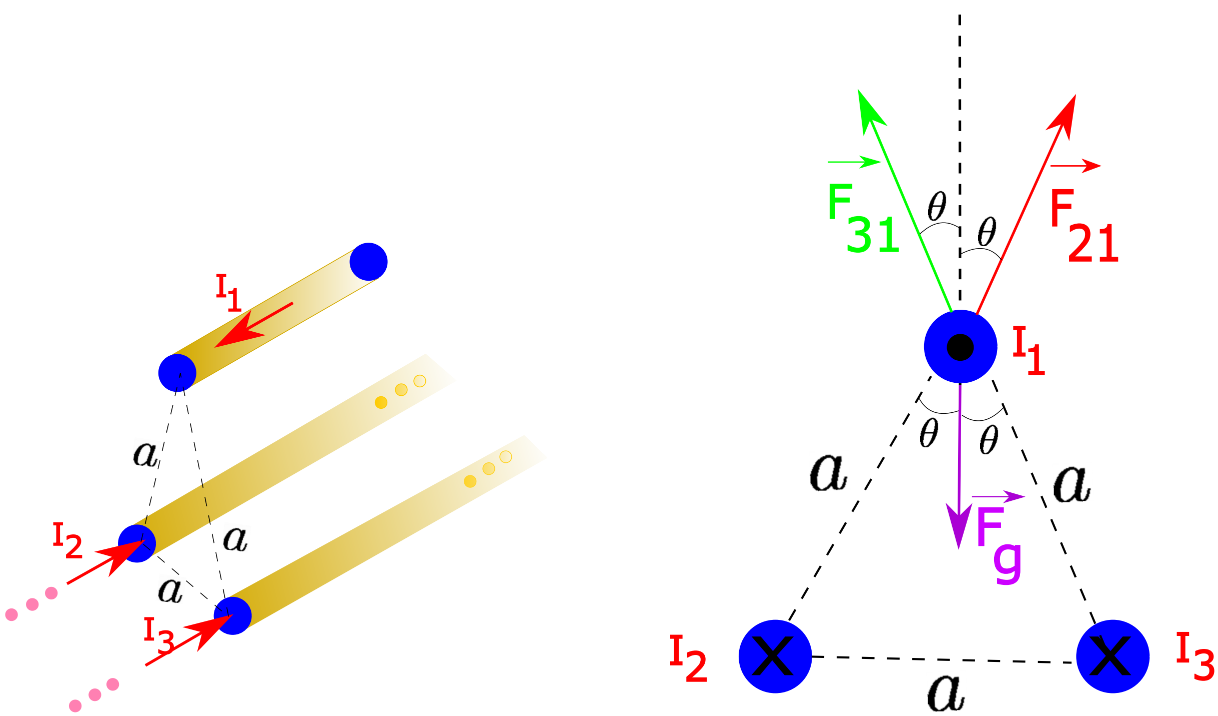
For both cases the magnitude of the force per unit length is

$$\frac{|\vec{F}|}{L} = \mu_0 \frac{I_1 I_2}{2\pi a} \tag{8.40}$$

where a is the distance between the conducting wires.

Example 4: Levitating Wire

Two infinitely long conducting wires are separated a distance $a = 1cm$, as shown in figure 8.7a. A third conducting wire with length $L = 10m$ and $400g$ of mass, carries a current $I_1 = 100$A and levitates. It is horizontally at a middle point between the two other conducting wires. The two infinitely long wires carry the same electric current with the same direction, opposite to the finite wire that levitates. So, in figure 8.7a $I_2 = I_3$. What is the electric current of the infinitely long wires?



(a) (b)

Figure 8.7

Solution:

From what we have learned, when electric currents have opposite directions they repeal each other. Therefore, the acting forces on the third wire which carries an opposite direction to the wires with currents I_2 and I_3 are as the one shown in figure 8.7b. Now, since $I_2 = I_3$ and they are placed at the same distance from the third wire at the equilibrium point, the forces exerted on the third wire are of the same magnitude. So, as it can be seen in the figure 8.7b the x (horizontal) contributions cancel out without giving us extra information. However, in the y (vertical) component there is also a gravitational contribution, so we can extract information from the forces sum in this component (we do not have an exact cancellation). So, concerning then about the forces in the y component

$$\sum F_y = -F_g + F_{B,21y} + F_{B,31y} = 0 \tag{8.41}$$

where the sum is equated to zero because the system is in equilibrium. Doing some algebra

$$F_{B,21y} + F_{B,31y} = F_g \tag{8.42}$$

$$\mu_0 L \frac{I_2 I_1}{2\pi a} \cos\theta + \mu_0 L \frac{I_3 I_1}{2\pi a} \cos\theta = F_g \Rightarrow \tag{8.43}$$

$$2\mu_0 L \frac{I_2 I_1}{2\pi a} \cos\theta = mg \Rightarrow I_2 = 2\frac{\pi a m g}{\mu_0 L I_1 \cos\theta} \tag{8.44}$$

where we used $I_2 = I_3$ and the $\cos\theta$ because is the y component we are interested (see figure 8.7b). Finally, seen from the front view the three conducting wires form an equilateral triangle. If it were not the case, one of the forces that exerts the wires with equal currents would be greater and the system would not be in equilibrium. So, using the fact that the triangle shown in figure 8.7b is equilateral, then we can easily see that $\theta = 30^0$. Hence, substituting values

$$I_2 = \frac{\pi(0.01m)(0.4kg)(9.81m/s^2)}{(4\pi \times 10^{-7}Tm/A)(10m)(100A)\cos 30^0} = 113.27A \tag{8.45}$$

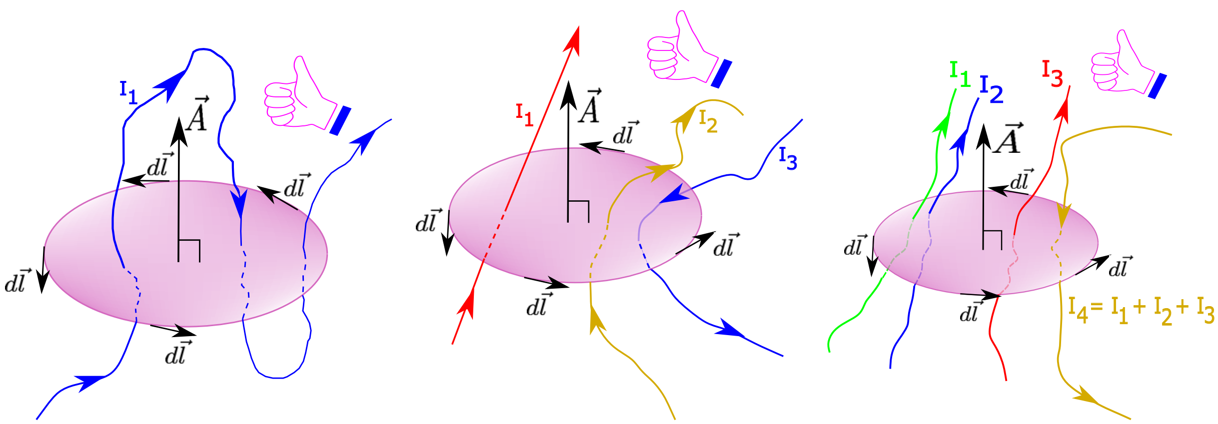
8.3 Ampère's Law

The Ampère's Law is a powerful tool to calculate the magnetic field in highly symmetrical systems. And its relevance is such that when we study the Maxwell equations, we will see that this equation with a slight (but theoretically profound) modification becomes one of the pillars of the electromagnetism.

Choosing certain closed path, Ampère's Law states that

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}} \tag{8.46}$$

where I_{enc} is the electric current that crosses or penetrates the area of the closed path and $d\vec{l}$ is a tangent vector to the closed path. The current I_{enc} is called as the *enclosed current*, but this name could be quite misleading. Unfortunately, the name has been used for so long time that you just have to get used to it. You should picture in your mind that is just the current that crosses the area that encloses the closed loop. So, to get a better grasp about how to use the Ampère's Law, let's see the examples shown in figure 8.8.



(a) (b) (c)

Figure 8.8

- If we calculate the closed loop integral $\oint \vec{B} \cdot d\vec{l}$ in the case of figure 8.8a, we have that the enclosed current is $I_{enc} = I_1 - I_1 + I_1 = I_1$. Therefore, the integration result will be $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_1$

- For the figure 8.8b, the enclosed current is $I_{enc} = I_1 + I_2 - I_3$. So, the integration result will be $\oint \vec{B} \cdot d\vec{l} = \mu_0 (I_1 + I_2 - I_3)$.

- For the figure 8.8c, the enclosed current is $I_{enc} = I_1 + I_2 + I_3 - I_4 = 0$ because $I_4 = I_1 + I_2 + I_3$. So, the integration result will be $\oint \vec{B} \cdot d\vec{l} = 0$

The closed path that we use to calculate $\oint \vec{B} \cdot d\vec{l}$ is called as *Amperian Loop*. Such Amperian Loop encloses the area that we use to calculate the enclosed current. Now, some **warnings** about Ampère's Law.

- The integration $\oint \vec{B} \cdot d\vec{l} = 0$ **does not mean that the magnetic field is zero**. It just means that if you sum infinitesimally the dot product $\vec{B} \cdot d\vec{l}$ all along the closed loop, the contributions cancel out. In some exercises, you will see that by using Ampère's Law we conclude that the magnetic field is zero given that $\oint \vec{B} \cdot d\vec{l} = 0$. However in such cases, there must necessarily be something else in the system that we analyze that can lead us to such conclusion. In general, by just knowing that $\oint \vec{B} \cdot d\vec{l} = 0$ you cannot conclude that the magnetic field is zero.

- In the last three examples where we calculated the enclosed current, we used a convention. And you must stick to it, so that you do not have misleading results when applying Ampère's Law. You can choose a direction for $d\vec{l}$ as you wish, either clockwise or anti-clockwise. However,when giving the direction, curl your fingers of the right hand in the direction of $d\vec{l}$, and the direction of the thumb finger we say is the direction of the area vector as we did in figures 8.8a, 8.8b and 8.8a. If the electric current when penetrating the area of the loop has an angle lower than 90° with the area vector, then the current is taken as positive, otherwise as negative. So, for example in figures 8.8a, 8.8b and 8.8c, if you curl your fingers of the right hand in the direction of $d\vec{l}$ you notice that the thumb finger points upwards. That's why in the last three calculations of the enclosed currents we took some currents as positive and others as negative. If you obtain a negative enclosed current, it only means that the direction of the current is opposite to what the convention establishes (the direction of your thumb finger).

Ampère's Law is powerful as we have mentioned. It will be as useful to Gauss Law when we wanted to find the electric field of symmetrical objects. It is of course not the same nor analogous, because in Ampère's Law we calculate a path integral in a closed loop, while Gauss Law is an integral of area in certain closed surface. In the exercises, it will become clearer how we use Ampère's Law .

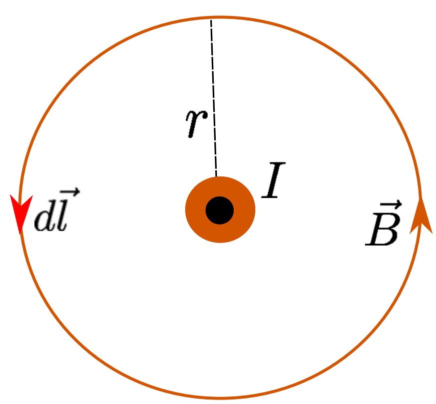
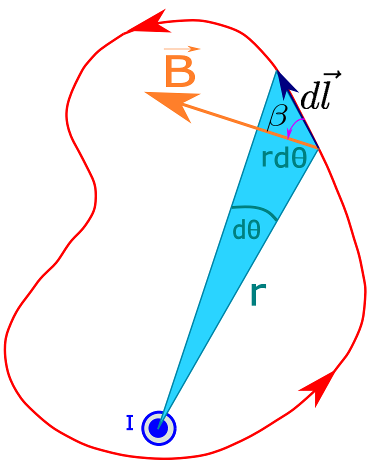


Figure 8.9

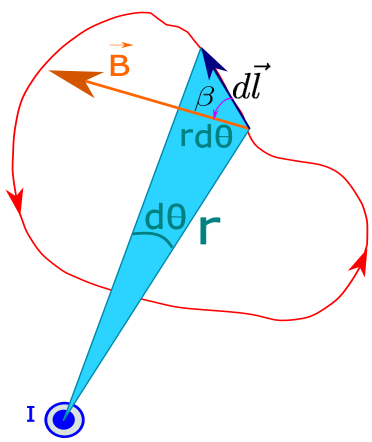
Finally, we will show that Ampère's Law is true. To get a grasp about its truthness, lets start with the simplest case. Taking a circle as our closed loop, as shown in figure 8.9 it is centered where there is an infinite conductor wire which carries a stable current I. The magnetic field generated by this electric current is given by equation. 8.14. At every point along the closed loop the magnetic field is parallel to $d\vec{l}$. Therefore,

$$\oint \vec{B} \cdot d\vec{l} = |\vec{B}| \oint dl = |\vec{B}| 2\pi r = \left(\frac{\mu_0 I}{2\pi r}\right) 2\pi r = \mu_0 I \tag{8.47}$$

where we took the magnetic field out of the integral because it is constant at certain radius r. So, this agrees with Ampère's Law because the enclosed current is I.



(a)



(b)

Figure 8.10

Let's complicate things now a little more. We used a circle as closed loop, however let's show that for any shaped closed loop Ampère's Law still holds. We keep our infinite wire as the conductor that carries a stable electric current I. From figure 8.10a we have that

$$\vec{B} \cdot d\vec{l} = |\vec{B}| dl \cos\beta \tag{8.48}$$

However, from the triangle formed with the angle β as shown in figure 8.10a, we have that

$$dl \cos\beta = r d\theta \tag{8.49}$$

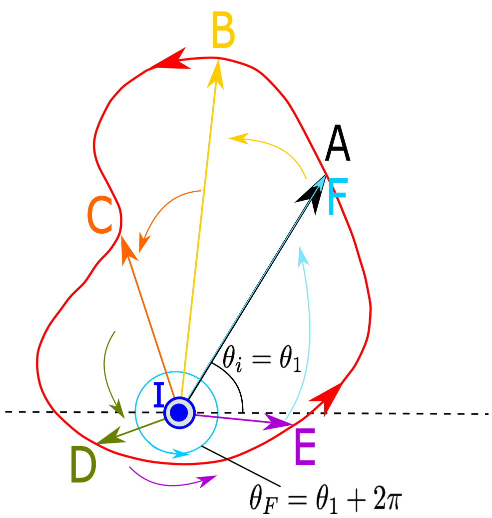
We have that such side of the triangle is $rd\theta$, because $\sin\theta \approx \theta$ for extremely small angles (in this case $d\theta$ is an infinitesimal angle so this holds). If you use $\sin(d\theta) \approx d\theta$, you obtain using the triangle formed with $d\theta$ in figure 8.10a that the projection of the vector $d\vec{l}$ onto the magnetic field vector must be $rd\theta$. So, we have that

$$\oint \vec{B} \cdot d\vec{l} = \oint |\vec{B}| r d\theta = \oint \left(\frac{\mu_0 I}{2\pi r}\right) r d\theta = \frac{\mu_0}{2\pi} \oint d\theta = \frac{\mu_0 I}{2\pi} 2\pi = \mu_0 I \tag{8.50}$$

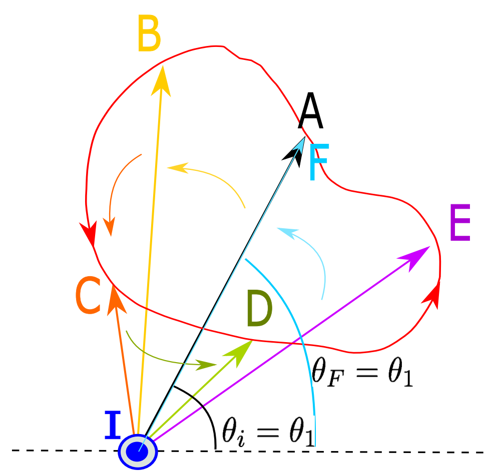
Good! Even if the shape isn't a circle, Ampère's Law holds. What about if the electric current is now outside the loop. According to Ampère's Law , the integral $\oint \vec{B} \cdot d\vec{l}$ must be zero because no current is penetrating the area, so the enclosed current is zero. So, let's see if this is true. From figure 8.10b, we have that

$$\oint \vec{B} \cdot d\vec{l} = \oint |\vec{B}| r d\theta = \oint \left(\frac{\mu_0 I}{2\pi r}\right) r d\theta = \frac{\mu_0}{2\pi} \oint d\theta \tag{8.51}$$

where we used equation 8.49. However, this time the closed integral will not be 2π. When the electric current is outside the closed loop, when we run all over the path of the closed loop, we start with certain initial angle $\theta_i = \theta_1$ and finish with $\theta_F = \theta_1$!(see figure 8.11b to understand why). So, the closed integral is zero! This is not the case when the current is inside the closed loop, because when you run all along the closed loop, you started from certain arbitrary initial angle $\theta_i = \theta_1$ and finished with a final angle $\theta_F = \theta_1 + 2\pi$ that coincides in the same point. You ran the path in such way that you arrived to the exact same point cycling over 360° degrees(2π radians), arriving where you started(see figure 8.11a to understand why). So, when the electric current is inside the loop, even though the point you started with angle θ_1 coincides with the same final point with angle $\theta_1 + 2\pi$, the initial and final angles differ in 2π.



(a) Showing schematically what happens when you integrate over the closed path when the current is inside the Amperian Loop. Starting from point A, moving all along the path, passing though point B, then C, afterwards D,consequently point E and finishing at point F. Points F and A are exactly the same point. However, notice that in this case the final angle turned a complete cycle of 2π radians.



(b) Showing schematically what happens when you integrate over the closed path when the current is outside the Amperian Loop. Starting from point A, moving all along the path, passing though point B, then C, afterwards D,consequently point E and finishing at point F. Points F and A are exactly the same point. However, notice that in this case the final angle is exactly the same as the initial angle.

Figure 8.11

Therefore, when the current is outside the loop,

$$\oint \vec{B} \cdot d\vec{l} = 0 \tag{8.52}$$

as Ampère's Law stablishes. Nice! So far, Ampère's Law looks to work even with random shapes of Amperian loops.

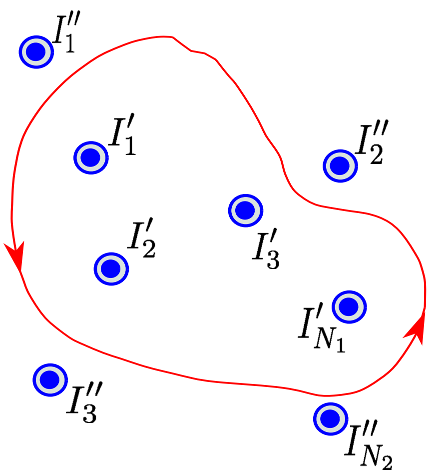


Figure 8.12

Finally, what if we have N electric currents. Is all what we have discussed still true? Inded, if Ampère's Law holds for one electric current, then it holds for N electric currents; and the demonstration is quite trivial. Nevertheless, let's do it! Suppose that N_1 electric currents do not pentrate the area enclosed by the randomly shaped closed loop, and N_2 electric currents do penetrate the area enclosed by the Amperian loop. If there are N currents in total, well obviously $N = N_1 + N_2$. Now, the total magnetic field at any point due to all the N electric currents is just the vector sum of each magnetic field of the electric currents

$$\vec{B} = \sum_{i=1}^{N} \vec{B}_i = \sum_{i=1}^{N_1} \vec{B}_i' + \sum_{i=1}^{N_2} \vec{B}_i'' \tag{8.53}$$

where the last equality is the sum of magnetic fields due to electric currents that does not cross the loop area (labelled with $\vec{B}''$) and the ones does (labelled with $\vec{B}'$). Therefore, we have that

$$\oint \vec{B} \cdot d\vec{l} = \oint \sum_{i=1}^{N} \vec{B}_i \cdot d\vec{l} = \oint \sum_{i=1}^{N_1} \vec{B}_i' \cdot d\vec{l} + \oint \sum_{i=1}^{N_2} \vec{B}_i'' \cdot d\vec{l} \tag{8.54}$$

so let's analyze each integral. For the magnetic fields that does cross (penetrate) the loop area, we have that

$$\oint \sum_{i=1}^{N_1} \vec{B'}_i \cdot d\vec{l} = \oint \vec{B}_1' \cdot d\vec{l} + \oint \vec{B}_2' \cdot d\vec{l} + ... + \oint \vec{B_{N_1}'} \cdot d\vec{l} \tag{8.55}$$

however, for each of the integrals in the last equation, we have the closed loop integral of a single electric current magnetic field. And we already know that for one electric current that does not penetrate the loop area, its magnetic field follows equation 8.52. Therefore, each of the terms are exactly equal to zero. Now, for

$$\oint \sum_{i=1}^{N_2} \vec{B''}_i \cdot d\vec{l} = \oint \vec{B}''_1 \cdot d\vec{l} + \oint \vec{B}''_2 \cdot d\vec{l} + ... \oint \vec{B}''_{N_2} \cdot d\vec{l} \tag{8.56}$$

each of the integrals correspond to one single magnetic field generated by one electric current that penetrates the area of the loop. So, from equation 8.50, we know that each integral will be equal to the corresponding electric current that generates the magnetic field times μ_0, i.e.

$$\oint \sum_{i=1}^{N_2} \vec{B}_i \cdot d\vec{l} = \mu_0 I_1 + \mu_0 I_2 + ... + \mu_0 I_{N_2} = \mu_0 \left(I_1 + I_2 + ... + I_{N_2}\right) \tag{8.57}$$

Therefore, we have obtained that for N electric currents

$$\oint \vec{B} \cdot d\vec{l} = 0 + \mu_0 \left(I_1 + I_2 + ... + I_{N_2}\right) = \mu_0 I_{enc} \tag{8.58}$$

where the currents I_1 up to I_{N_2} correspond to the enclosed current, because those currents do penetrate the amperian loop area. Nice! So, indeed Ampère's Law holds for N electric currents. So, we are starting to get confident that Ampère's Law is true. We have shown that for infinite conducting wires with stable currents I, it is true. However, probably you could get obsessed to see actually a demonstration for any shape of the conducting wire that carries an stable electric current I. Since this demonstration requires more vector calculus than what is expected for the reader for this book, we won't do the demonstration. However, with what we have shown in this section must be enough to give you the grasp that Ampère's Law for steady currents is actually correct.

Example 5: Magnetic Field inside and outside a wire with uniform electric current density

An extremely long and straight wire of radius R carries an stable current I that is distributed uniformly across the cross sectional area of the wire (see figure 8.13). Calculate the magnetic field a distance r from the center of the wire in the regions

- $r \geq R$
- $r < R$

Solution:

We center a circumference at the middle point of the wire as shown in the figure 8.13). We take advantage of the symmetry of the magnetic field. We solve the exercise in two parts, when $r \geq R$ and $r < R$.

For $r \geq R$:
At every point of the circumference of radius r, the magnetic field is parallel to $d\vec{l}$. Therefore,

$$\oint \vec{B} \cdot d\vec{l} = |\vec{B}| \oint dl = |\vec{B}| 2\pi r \tag{8.59}$$

where we took out the magnetic field out of the integral because it is constant at certain radius r. Also, we used that integration result is the perimeter of the loop. The enclosed electric current is the total electric current I carried by the conductor. Therefore, using Ampère's Law

$$|\vec{B}| 2\pi r = \mu_0 I \Rightarrow |\vec{B}| = \frac{\mu_0 I}{2\pi r} \tag{8.60}$$

nothing new. The formula obtained is the one we already knew from equation 8.14.

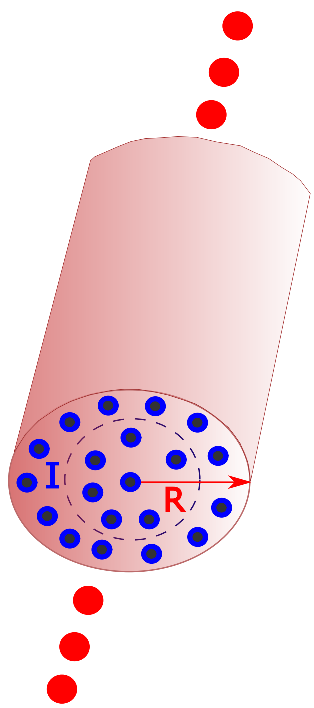


Figure 8.13

Now, what happens inside ($r < R$)? Depending of r, we will be enclosing more or less electric current. When $r = R$, we enclose all the electric current that goes through the wire. However, as $r \to 0$ the electric current enclosed $I \to 0$. So, we need an equation that tell us how much current is enclosed depending on r. Recalling that $\vec{J}$ is the current density, we can obtain the enclosed electric current with different r's with

$$I_{enc} = \int |\vec{J}| dA = \int_0^r |\vec{J}| 2\pi r dr \tag{8.61}$$

where we used the differential area of a circle. If we assume that the charge density is uniform as mentioned in the exercise (this is not necessarily true in real life), we can take $|\vec{J}|$ out from the integral, so

$$I_{enc} = |\vec{J}|\pi r^2 = \left(\frac{I}{\pi R^2}\right)\pi r^2 = I\left(\frac{r}{R}\right)^2 \tag{8.62}$$

where we used that for uniform charge density

$$|\vec{J}| = \frac{I}{A} \tag{8.63}$$

where A is the cross sectional area of the wire. Now, we have that

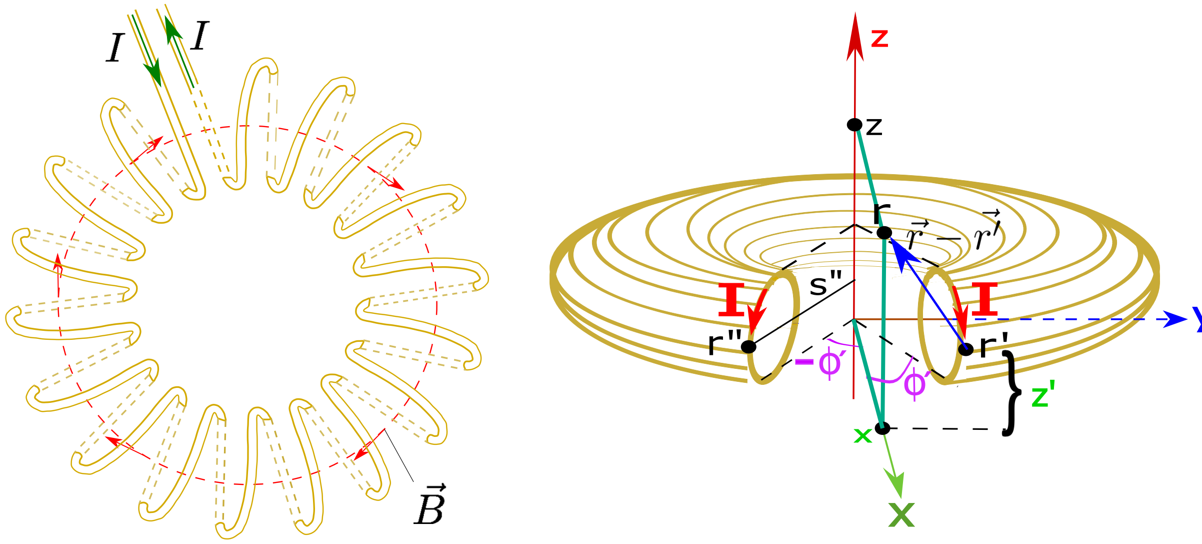
$$\oint \vec{B}\cdot d\vec{l} = |\vec{B}|\int_0^{2\pi r} dl = |\vec{B}|2\pi r \tag{8.64}$$

where once again by symmetry $d\vec{l}$ is always parallel to $\vec{B}$. So, substituting equation 8.64 and equation 8.62 in Ampère's Law, we have that

$$|\vec{B}|2\pi r = \mu_0 I_{enc} = \mu_0 I\left(\frac{r}{R}\right)^2 \Longrightarrow |\vec{B}| = \frac{\mu_0 I}{2\pi R^2}r \tag{8.65}$$

Example 6: Magnetic Field inside and outside a Toroid

The device shown in the figure 8.14a is called *toroidal solenoid* or simply *toroid*. The beauty about this device is that it creates a constant magnetic field inside, and the magnetic field outside of it is completely zero if there are no gaps between one coil ring and then next! The device consists of a conducting wire that is winded in loops without gaps between the loops in a donut shape (in mathematics, more formally we say is a *toroid shape*). In real toroids, there are gaps between the winded loops, small probably but present. In such cases, what we calculate now is an approximation. Even though, the approximation is highly accurate, inside the magnetic field is practically and outside negligible. For a toroid with N loops of wire , calculate the magnetic field inside and show that it is zero outside. Use figure 8.14b for such calculations. In that figure, there are no gaps, and the winded loops are so close to each other that we can consider it as a continuum surface.



(a) (b)

Figure 8.14

Solution:

We will use Ampère's Law to determine the magnitude of the magnetic field. However, in order to use it we need to know what is the behavior of the magnetic field. It will turn out that it has circular shape as shown in figure 8.14a. So, we will use smartly the information given in figure 8.14b to obtain the behaviour of the magnetic field. At point r, by Biot - Savat Law, the magnetic field contribution by the electric current element at $\vec{r''}$ is given by:

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{s} \times \left(\vec{r} - \vec{r'}\right)}{|\vec{r} - \vec{r'}|^3} \tag{8.66}$$

Now, we will do a quite smart move. (I am being modest. A beautifully smart move!). We place r, with no "y" component as shown in the picture. And even though it is at the $x - z$ plane, the result will be general. Just be patient. So, we say that:

$$\vec{r} = (x, 0, z) \tag{8.67}$$

Hence

$$\vec{r} - \vec{r'} = (x - R\cos\phi, -R\sin\phi, z - z') \tag{8.68}$$

Now, in general we can say that:

$$d\vec{s} = (dx, dy, dz) = (|d\vec{s}|\cos\phi, |d\vec{s}|\sin\phi, dz) \tag{8.69}$$

Therefore,

$$d\vec{s} \times \left(\vec{r} - \vec{r'}\right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ |d\vec{s}|\cos\phi & |d\vec{s}|\sin\phi & dz \\ x - R\cos\phi & -R\sin\phi & z - z' \end{vmatrix} \tag{8.70}$$

$$d\vec{s} \times \left(\vec{r} - \vec{r'}\right) =$$
$$[\sin\phi(|d\vec{s}|(z - z') + Rdz)]\ \hat{x} + [dz(x - R\cos\phi) - |d\vec{s}|\cos\phi(z - z')]\ \hat{y} + [|d\vec{s}|x\sin\phi]\ \hat{z} \tag{8.71}$$

Now, from figure 8.14b notice that there exists a symmetrically situated point r'' which is at the same distance from r as r', same electric current passes through that point, but the angle is with opposite sign $-\phi$. So, we have that for such point

$$d\vec{s''} \times \left(\vec{r} - \vec{r''}\right) =$$
$$[\sin(-\phi)(|d\vec{s}|(z - z') + Rdz)]\ \hat{x} + [dz(x - R\cos(-\phi)) - |d\vec{s}|\cos(-\phi)(z - z')]\ \hat{y} + [|d\vec{s}|x\sin(-\phi)]\ \hat{z} \tag{8.72}$$

where we used that infinitesimal segments are equal $d\vec{s''} = d\vec{s}$. So, notice that what we obtained is exactly the same as $d\vec{s} \times \left(\vec{r} - \vec{r'}\right)$ but with angle $-\phi$. Now, here comes the magic. Given that $\sin(-\phi) = -\sin\phi$ and $\cos(-\phi) = \cos\phi$, we have that

$$d\vec{s} \times \left(\vec{r} - \vec{r'}\right) + d\vec{s''} \times \left(\vec{r} - \vec{r''}\right) = (0, 2[dz(x - R\cos\phi) - |d\vec{s}|\cos\phi(z - z')], 0) \tag{8.73}$$

i.e. the components $\hat{x}$ and $\hat{z}$ cancel out (check that indeed this is true). So, the contribution to the calculation of the magnetic field at point r due to the electric currents at r'' and r' leaves just one component. Why? Because when you sum their contributions to the calculation of the magnetic field at point r due to both currents at r' and r'', what you are summing is

$$\frac{\mu_0}{4\pi} I \frac{d\vec{s} \times \left(\vec{r} - \vec{r'}\right)}{|\vec{r} - \vec{r'}|^3} + \frac{\mu_0}{4\pi} I \frac{d\vec{s} \times \left(\vec{r} - \vec{r''}\right)}{|\vec{r} - \vec{r''}|^3} = \frac{\mu_0 I}{4\pi|\vec{r} - \vec{r'}|^3}\left(d\vec{s} \times \left(\vec{r} - \vec{r'}\right) + d\vec{s} \times \left(\vec{r} - \vec{r''}\right)\right) \tag{8.74}$$

where we used that the magnitudes

$$|\vec{r} - \vec{r'}| = |\vec{r} - \vec{r'''}| \tag{8.75}$$

are exactly the same because points r' and r'' are equally distanced from point r. Hence, after taking the contribution of all currents in the toroid to the magnetic field at point r, the only contribution will be in the y component! For each current at each point in the toroid, there is a symmetrically opposite point which current generates a magnetic field that cancels out the x and z component! Now, you probably say "*Well, not big deal! That is true at point r with components* $(x, 0, z)$. *What about the other points?*". Well, indeed we calculated for point r. However our reference frame is arbitrary and this applies for any point! If this is still not clear why, rotate the reference frame, so that a new point r''' has no y component. The contributions of the magnetic field, due to all currents in the toroid will have only y again! Just exactly as it happened with point r before you rotated the reference frame. Shift once again the reference frame for another point r'''' to be where our points r and r''' exactly were. It will hold once again that all contributions of the magnetic field due to all currents will cancel out in x and z component, leaving only the y component. And if you keep repeating the rotation of the reference frame, it will hold again and again and again, due to symmetry! Beautiful! In general, we said that the field has only $\hat{\phi}$ component, so drawing all these magnetic field lines will look as shown in figure 8.14a. Concentric circumferences, of magnetic field. Beautiful! Now, that we have shown that the magnetic field has circular shape, we can proceed to obtain the magnitude of the magnetic field with Ampère's Law.

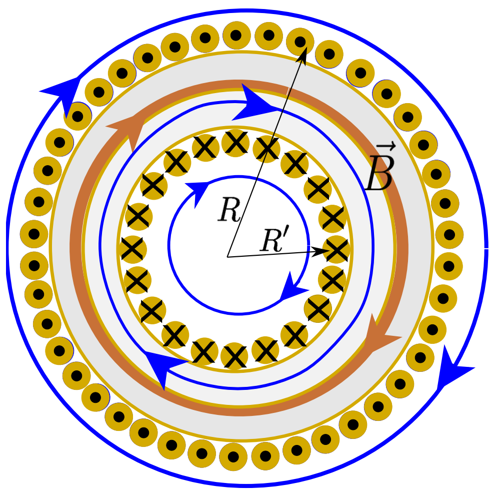


Figure 8.15: Cross sectional view from the top of the toroid. The toroid has inner radius R and outer radius R'. The three blue arrowed paths are the three Amperian loops we create to study the magnitude of the magnetic field generated by the toroid.

We create first a circular Amperian loop such that $r < R'$ (the inner blue arrowed path in figure 8.15). For such case

$$\oint \vec{B} \cdot d\vec{l} = |\vec{B}| \int_0^{2\pi r} dl = |\vec{B}| 2\pi r = \mu_0 I_{enc} \tag{8.76}$$

Since there is no enclosed current

$$\vec{B} = 0 \qquad for\ r < R' \tag{8.77}$$

Wait a minute! Didn't we say that if the path integral is zero, not necessarily means that the magnetic field is zero? Indeed! However, notice that the magnetic field is constant at certain radius r. So, that assures us that it must be zero. Probably this is trivial (but have helped to get the idea to many students) with a very simple example you can visualize this. Suppose that you have the following

$$\sum_{i=1}^{N} iA(i) = 1A(1) + 2A(2) + 3A(3) + ... + NA(N) = 0 \tag{8.78}$$

Now, if $A(i) = A$ is constant for all i, well this trivially becomes

$$\sum_{i=1}^{N} iA = 1A + 2A + 3A + ... + NA = A(1 + 2 + 3 + ... + N) = A\left(\sum_{i=1}^{N} i\right) = 0 \quad (8.79)$$

so in the multiplication of whatever number gives the sum times the constant A to be zero, then A is zero necessarily. No discussion. This is just what happened in the moment that in equation 8.76 the magnetic field was constant (recall that an integral is an infinitesimal continuous sum). Now if A in equation 8.78 changes its value when i iterates, then of course there is no way to say that $A(i)$ is zero. In less words, in the moment you could get the magnetic field in equation 8.76 out of the integral, it assured you that if the enclosed current was zero, there is no way that the magnetic field is not zero.

Now, taking any point such that $R' \leq r < R$ (using as Amperian loop the blue arrowed path where there is magnetic field in figure 8.15)

$$\oint \vec{B} \cdot d\vec{l} = |\vec{B}| \int_0^{2\pi r} dl = |\vec{B}| 2\pi r = \mu_0 I_{enc} \Rightarrow \quad (8.80)$$

$$|\vec{B}| = \mu_0 \frac{I_{enc}}{2\pi r} \quad (8.81)$$

Now, what is the enclosed current? There are N wraps of coil. So N times the current I in each loop of wire is passing at the same time. Therefore:

$$|\vec{B}| = \mu_0 \frac{NI}{2\pi r} \qquad for\ R' < r \leq R \quad (8.82)$$

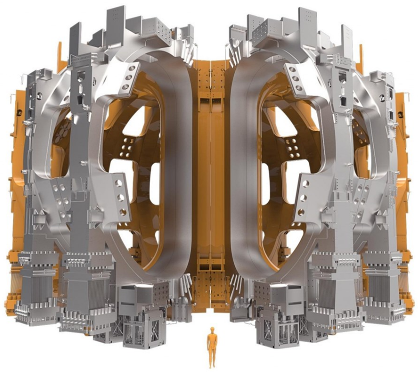
Finally, let's see outside the toroid when $r > R$ (using as Amperian loop the outter blue arrowed path in figure 8.15). Using once again Ampère's Law:

$$|\vec{B}| = \mu_0 \frac{I_{enc}}{2\pi r} \quad (8.83)$$

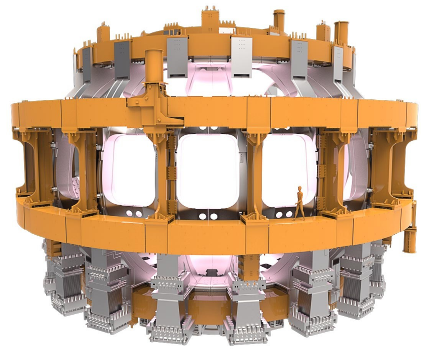
However this time what is the enclosed current? Probably you are tempted to say again that $I_{enc} = NI$. However, this time notice that for each current going outwards the page, there is a current going inwards the page (see figure tal). Recall that depending how we picked the direction of $d\vec{l}$ in the contour in the loop, N currents will contribute positively and N negatively, cancelling completely. So the enclosed current actually is $I_{enc} = 0$. So, we have found that

$$\vec{B} = \left\{ \begin{array}{cc} \frac{\mu_0 NI}{2\pi r}\hat{\phi} & \text{for points inside the toroid} \\ 0 & \text{for points outside the toroid} \end{array} \right\} \quad (8.84)$$

One extremely interesting application of the toroid is the construction of the so called *Tokamaks* (The term *tokamak* comes from a Russian acronym that stands for "Toroidal Chamber with Magnetic Coils."). These devices are huge toroids that create strong magnetic fields to confine hot plasma for fusion energy production. Nowadays, the use of



(a)



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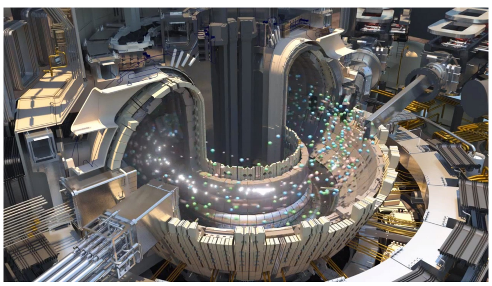

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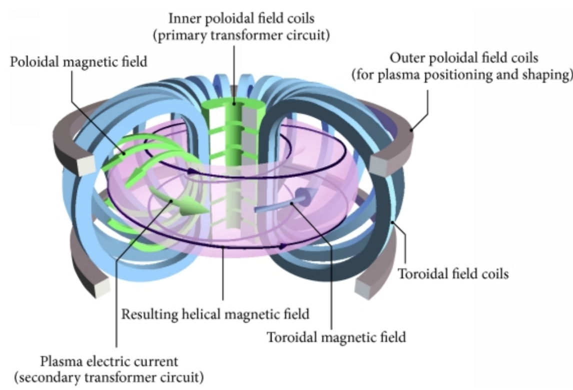


(d)

Figure 8.16

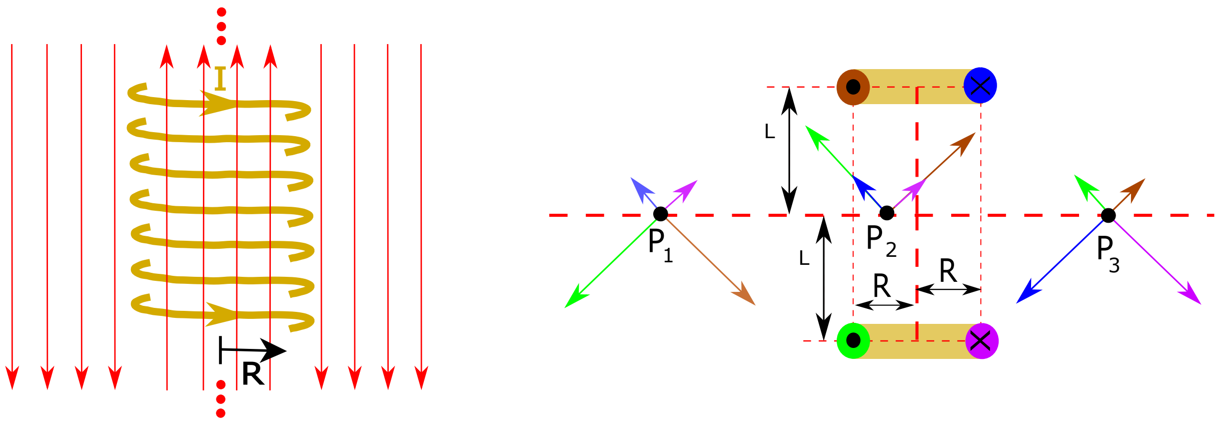
fusion energy to produce electricity still is under research, and hopefully one day we will be using this kind of clean energy. The International Thermonuclear Experimental Reactor (ITER) is an experiment that is under construction in *Saint-Paul Lez-Durance, France* and it will be the biggest Tokamak in the world. The first plasma at ITER is scheduled for December 2025, and has been designed to prove the feasibility of fusion as a large-scale and carbon-free source of energy based on the same principle that powers our Sun and stars.[27]. The fusion reactions take place when atomic nuclei are close enough for the enough time so that the *nuclear force* pulling them exceeds the electrostatic force trying to separate them. The result is new heavier nuclei, where the two nuclei fused into one. When this process takes place energy is released. However, in order to achieve this, the atoms must gain extremely high kinetic energy. So, the temperatures must be extremely high, and when reaching such point, a *plasma* is created. A plasma is a state of matter when electrons are separated from nuclei and fusion reactions can take place. The design of ITER toroidal coils to create the magnetic fields to confine the plasma are shown in figures 8.16a,8.16b and 8.16c (figures taken from reference [27]). The general function of any Tokamak is shown in figure 8.16d (figure taken from reference[23]).

The most efficient fusion reaction in the laboratory setting is the reaction between two hydrogen isotopes deuterium (D) and tritium (T). The fusion of these light hydrogen atoms produces a heavier element, helium, and one neutron (...) Approximately 80 percent of the energy produced is carried away from the plasma by the neutron which has no electrical charge and is therefore unaffected by magnetic fields. The neutrons will be absorbed by the surrounding walls of the tokamak, where their kinetic energy will be transferred to the walls as heat. In ITER, this heat will be captured by cooling water circulating in the vessel walls and eventually dispersed through cooling towers. In the type of fusion power plant envisaged for the second half of this century, the heat will be used to produce steam and—by way of turbines and alternators—electricity.(...) A fusion reaction is about four million times more energetic than a chemical reaction such as the burning of coal, oil or gas and four times as much as nuclear fission reactions (at equal mass). While a 1000 MW coal-fired power plant requires 2.7 million tonnes of coal per year, a fusion plant of the kind envisioned for the second half of this century will only require 250 kilos of fuel per year, half of it deuterium, half of it tritium. Only a few grams of fuel are present in the plasma at any given moment. This makes a fusion reactor incredibly economical in its fuel consumption and also confers important safety benefits to the installation. (...) Fusion fuels are widely available and nearly inexhaustible. Deuterium can be distilled from all forms of water, while tritium will be produced during the fusion reaction as fusion neutrons interact with lithium. (Terrestrial reserves of lithium would permit the operation of fusion power plants for more than 1,000 years, while sea-based reserves of lithium would fulfil needs for millions of years.) (...)Fusion doesn't emit harmful toxins like carbon dioxide or other greenhouse gases into the atmosphere. Its major by-product is helium: an inert, non-toxic gas.(...)A Fukushima-type nuclear accident is not possible in a tokamak fusion device. It is difficult enough to reach and maintain the precise conditions necessary for fusion—if any disturbance occurs, the plasma cools within seconds and the reaction stops. The quantity of fuel present in the vessel at any one time is enough for a few seconds only and there is no risk of a chain reaction.

— ITER Official webpage [27]

Example 7: Magnetic field of an infinite solenoid

A solenoid is a wrapped conducting wire in the configuration shown in figure 8.17a. Each loop of wire is extremely closed to each other, trying to make no gaps between the conducting wire loops. In this problem actually we assume that there is no gap between the conducting wires. In real solenoids, there is certain gap between the conducting wire loops, however our derivation still is a good approximation. Now, suppose that the solenoid is infinite (once again this is an approximation, however for very long solenoids this coincides quite accurately with reality). Calculate the magnetic field inside and outside the solenoid.



(a) (b)

Figure 8.17

Solution:

First of all, we need to know the behaviour of the magnetic field before we apply Ampère's Law. In figure 8.17a the magnetic field lines are already drawn. However, how do you know actually that they behave in such way? So, we make a simple qualitative analysis that will lead to a powerful conclusion. Using the direction of the electric current carried by the solenoid in figure 8.17a, the magnetic field lines inside the solenoid point upwards, while outside downwards. So, in figure 8.17b is shown the transverse section of just two loops of the solenoid (imagine you cut in half the solenoid and you visualise two loops). Points P_1 and P_3 lie outside the solenoid, while P_2 is inside the solenoid. Given that the solenoid is infinite, whatever direction of the magnetic field we obtain from this picture applies to any point inside and outside the solenoid. Why? Because every point, can be considered to be in a symmetrical point in the **vertical axis** of the infinite solenoid. At any point P, you have an infinite number of contributions from the loops above point P, and equally an infinite number of contributions from the loops below point P.

Let's start analyzing the magnetic field contributions at point P_2. Using the right hand rule we obtain the direction of the contributions shown in figure 8.17b. Every color arrow represents the direction of the magnetic field of each current colored with the same color. Also, the arrows are drawn in scale of their magnitude (larger greater the magnetic field contribution). Notice that all the contributions will lie upwards in the vertical direction, because the components of the magnetic fields of each current in the horizontal contribution cancel out! Notice that the electric currents going into the page, are equally separated from point P_2, so the magnetic field magnitude is exactly the same, with opposite horizontal direction, so their horizontal contributions cancel out! The same happens with the magnetic field contribution of the currents colored in brown and green. Therefore, counting all contributions, the resultant magnetic field points upwards!

Now, for points outside the solenoid, what is the direction of the magnetic field? Let's analyze point P_1 (something similar happens to point P_3 and you can check it out in analogy to point P_1 by yourself). The magnetic field vectors colored in pink and blue are shorter than the other two vectors because the in-going towards the page currents colored in blue and pink are further to point P_1 than the other two currents. So, necessarily the magnetic field magnitudes generated by these two currents are smaller. Once again notice that the horizontal contribution of the magnetic field are cancelled by pairs; the magnetic field represented in pink color, its horizontal component its cancelled by the blue vector magnetic field, and also happens the same with the other two vectors. Now, at point P_1, the in the vertical direction, there will be contribution of the magnetic field upwards and downwards. However, the magnetic field contributions upwards are smaller than the contributions downwards. So, the resultant magnetic field is downwards!

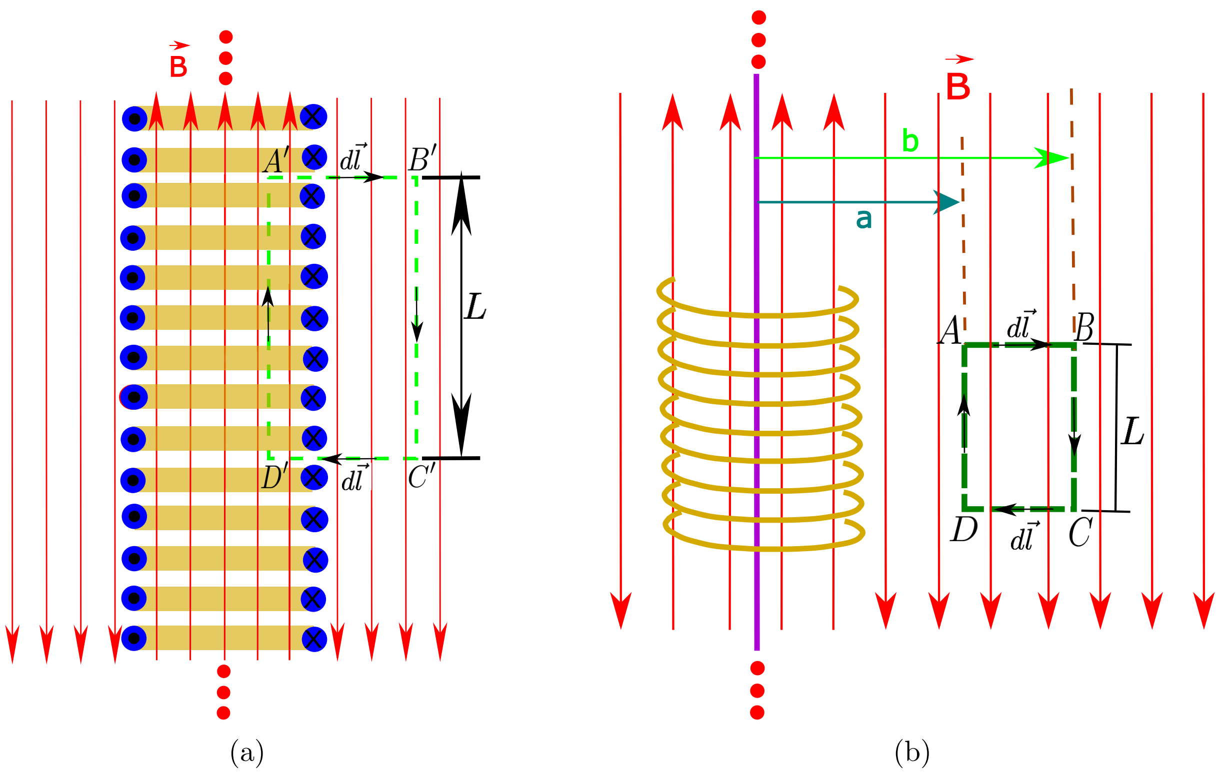


Figure 8.18

We have found then, that outside at any point, the magnetic field direction is downwards as shown in figure 8.17a and at any point inside the solenoid the magnetic field direction is upwards as also shown in figure 8.17a. If the current direction were opposite to what we have assumed, then the magnetic lines inside would be pointing downwards, and outside the magnetic field lines would be pointing upwards!

So, now let's apply Ampère's Law to know what is the magnitude of the magnetic field inside and outside the infinite solenoid. Let's start with the magnitude of the magnetic field outside the solenoid. Before we apply Ampère's Law, let's point out that at infinity, the magnetic field magnitude generated by the solenoid must be $|\vec{B}| = 0$. Recalling that the contribution of the magnetic field of any current, by Bio-Savart Law decreases as $1/r^2$, then as $r \to \infty$ the contribution must vanish. So, here comes the magic! Let's now apply Ampère's Law to the Amperian loop shown at figure 8.18b. Following the direction of path drawn with $d\vec{l}$ (clockwise), we have that

$$\oint \vec{B} \cdot d\vec{l} = \int_A^B \vec{B} \cdot d\vec{l} + \int_B^C \vec{B} \cdot d\vec{l} + \int_C^D \vec{B} \cdot d\vec{l} + \int_D^A \vec{B} \cdot d\vec{l} \tag{8.85}$$

However, given that the magnetic field direction is downwards, it is perpendicular to $d\vec{l}$ in paths $A-B$ and $C-D$; furthermore $\vec{B}$ and $d\vec{l}$ are parallel in path $B-C$, and anti-parallel in path $D-A$. Hence,

$$\oint \vec{B} \cdot d\vec{l} = \int_B^C \vec{B} \cdot d\vec{l} + \int_D^A \vec{B} \cdot d\vec{l} = |\vec{B}(b)| \int_B^C dl - |\vec{B}(a)| \int_D^A dl \tag{8.86}$$

where we are left with just two integrals because as mentioned, $d\vec{l}$ are perpendicular to $\vec{B}$ in paths $A-B$ and $C-D$ so their dot product is zero. In the other two integrals we took out the magnetic field out of the integral because they are constant all along a path if the radial distance from the central axis of the solenoid is fixed. The vectors $\vec{B}(b)$ and $\vec{B}(a)$ mean that they are the magnetic fields evaluated at certain distance b and a from the central axis of the solenoid respectively. Now, applying Ampère's Law, we have that

$$\oint \vec{B} \cdot d\vec{l} = |\vec{B}(b)|L - |\vec{B}(a)|L = \mu_0 I_{enc} \tag{8.87}$$

where we already evaluated the path integrals, that their result is just the length (L) of the paths $B-C$ and $D-A$. However, given that the Amperian loop is completely outside the solenoid, the enclosed current is zero. Therefore,

$$|\vec{B}(b)| = |\vec{B}(a)| \tag{8.88}$$

The last result tells us that the magnetic field does not depend on the distance from the solenoid! We could have arbitrary chosen a and b. So, nothing stops us to make $b \to \infty$. But, recall that at infinity the magnetic field magnitude is zero! So,

$$\vec{B}(\infty)| = |\vec{B}(a)| \Rightarrow 0 = |\vec{B}(a)| \tag{8.89}$$

and given that a is also any arbitrary distance outside the solenoid, then everywhere outside the solenoid the magnetic field is zero! Beautiful!

Now, let's see what happens inside the solenoid. We take the Amperian Loop shown in figure 8.18a. Therefore, following the path shown in the figure (clockwise), we have that

$$\oint \vec{B} \cdot d\vec{l} = \int_{A'}^{B'} \vec{B} \cdot d\vec{l} + \int_{B'}^{C'} \vec{B} \cdot d\vec{l} + \int_{C'}^{D'} \vec{B} \cdot d\vec{l} + \int_{D'}^{A'} \vec{B} \cdot d\vec{l} \tag{8.90}$$

Now, for trajectory $D' - A'$, the vector $d\vec{l}$ is parallel to $\vec{B}$, while for all other trajectories the integration is zero, either because the magnetic field is perpendicular to $d\vec{l}$ or because we are outside the solenoid and the magnetic field is zero. Therefore,

$$\oint \vec{B} \cdot d\vec{l} = \int_{D'}^{A'} \vec{B} \cdot d\vec{l} = |\vec{B}|L = \mu_0 I_{enc} \Rightarrow |\vec{B}| = \frac{\mu_0 I_{enc}}{L} \tag{8.91}$$

where the magnetic field is constant all along the path $D' - A'$. The enclosed current is NI, because there are N wrapped wire loops, each carrying current I. Notice that N currents are penetrating the area enclosed by our Amperian loop. So, we can define $n = N/L$ as the number of loops density per unit length. Hence,

$$|\vec{B}| = \mu_0 n I \tag{8.92}$$

Therefore, we have found that the magnetic field of the infinite solenoid is given by

$$\vec{B} = \left\{ \begin{matrix} \mu_0 n I \hat{y} & \text{inside the solenoid} \\ 0 & \text{outside the solenoid} \end{matrix} \right\} \tag{8.93}$$

where we gave as $\hat{y}$ the direction of the field because the magnetic field points upwards. Once again, if the electric current goes in opposite direction, the magnetic field switches its direction. This was for an infinite solenoid, in real solenoids, there is a magnetic field outside. As, we build larger solenoids the magnetic field outside tends to zero and the magnetic field magnitude inside tends to a constant and uniform magnetic field, with magnitude $\mu_0 n I$. So, this is a good approximation for long solenoids.

Part IV

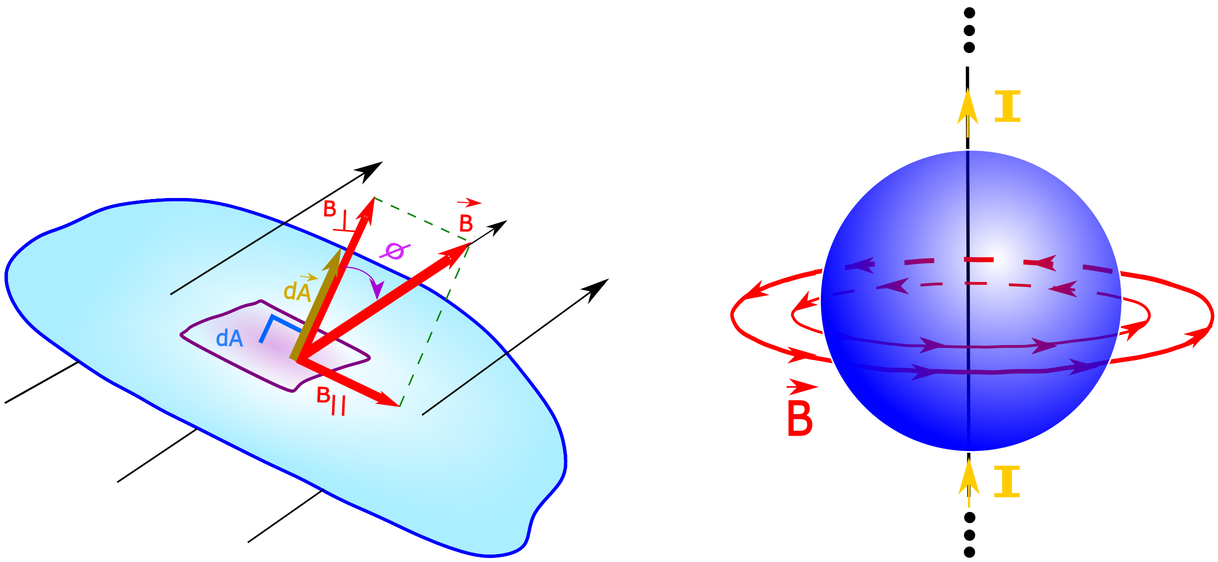
Electrodynamics

Chapter 9

Electromagnetic Induction

I hope that you enjoy this chapter as much as I did writing it. It is a remarkable and beautiful topic the one that we are just about to start discussing. The physics behind the use of electricity at big scales of almost all modern technology lies in the electromagnetic induction. To mention the greatness of electromagnetic induction, let me give you some historical context to obtain a grasp. Before approximately $1880's$ we used to light the darkness during nights with candles. However, brilliant engineers as *Nikola Tesla*, with the use of electromagnetic induction were capable to light entire cities! The kind of electric current that Tesla used to light up us from darkness is called as *alternate current*, and the way to produce such electric current is with electromagnetic induction. Electric batteries are not necessary to create a voltage across a wire! Remarkable! And even more surprising, generating just some volts from a battery, now we can elevate electric systems to huge amounts of volts, elevating to the order of thousands!

9.1 Magnetic Flux



(a) (b)

Figure 9.1

We define the magnetic flux through a flat surface as

$$\Phi_B = \vec{B} \cdot \vec{A} \tag{9.1}$$

The analogy to the electric flux, if the surface is not flat, in general we can split the surface in many rectangles, and approximate the magnetic flux through such surface as

$$\Phi_B \approx \sum_{i=1}^{N} \vec{B}_i \cdot \Delta \vec{A}_i \tag{9.2}$$

If we let the rectangles be extremely small (infinitesimal) the last sum becomes an integral over the surface:

$$\boxed{\Phi_B = \int_S \vec{B} \cdot d\vec{A}} \tag{9.3}$$

And if the surface encloses a volume, we write $\oint_S$ just to denote that we integrate over a closed surface

$$\Phi_B = \oint_S \vec{B} \cdot d\vec{A} \tag{9.4}$$

We haven't said done anything new in terms of calculations, since this is analog to the electric case. However there is something conceptually new in here! Recall that the closed integral of the electric flux gave us Gauss law as:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \tag{9.5}$$

Where q_{enc} is the enclosed electric charge. Also, remember that when the electric field lines go out the flux was positive, making sense because that means that you are enclosing positive electric charge. When the flux was negative, it means that the electric field lines go into the surface and that you enclosed a negative charge. However, in the magnetic case each field line that goes out, also goes into any surface that we use to calculate magnetic flux! To visualize this, think about the case of a current in an infinite wire, the field lines are circular. So, no matter the shape of the surface you make, all field lines that go out the surface, inevitably also go in as shown for the case of a sphere in figure 9.1b. So, for each line that contributes positively in the closed integral in equation 9.4, there is a negative contribution that cancels out the positive contribution. Hence,

$$\boxed{\oint_S \vec{B} \cdot d\vec{A} = 0} \tag{9.6}$$

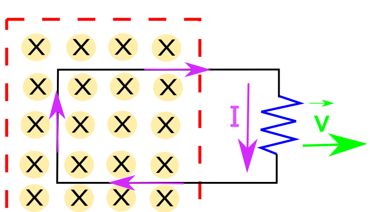
More specifically we say that in Nature, there are not *magnetic mono-poles*! We can think as a mono-pole in the electric case as a positive charge $+q$ where electric field lines have an starting point. This is no the case for the magnetic field, there is not such thing as an isolated magnet with just $North$ pole or $South$ pole. Actually, if you take any magnet, and you cut it in half, the halves once again have a North and South pole. Cut it again, and now you have four smaller magnets with south and north pole. And so on and so forth. We haven't found in Nature magnetic mono-poles, in case of finding one, what is stated in last equation would not hold! Last equation will be one of the four Maxwell equations, with which along with Lorentz force, you can describe all electromagnetic phenomena. So, if you find some day a magnetic mono-pole, it would be a big deal! A huge discovery.

9.2 Faraday's Law

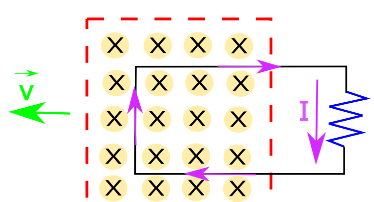
"Faraday was once interviewed by reporters when he came up with this law, and they said to him "... So, What?!... So, you get a little bit of electricity... So, What?! " His answer was "Someday you will tax it"... And he was right... he had vision "

— Walter Lewin

Picture the following three experiments



(a)



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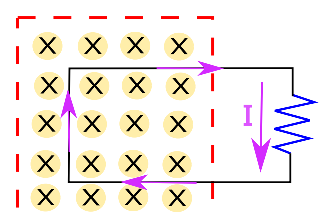


(c)

Figure 9.2

- In figure 9.2a, we have a source of magnetic field which is constant. Then you move the closed wire with certain speed v to the right as shown in figure 9.2a. If you connect such wire to a voltmeter you register a potential difference. Wait! What?! A voltage?! But there is no battery! How possibly is there an electromotive force (emf)? Also, of course given that there is a voltage, a current is produced as shown in figure 9.2a! If you move the closed wire to the opposite direction with certain speed $-v$ to the left, an opposite direction electric current is produced and a voltage with opposite sign is registered in the voltmeter. This is beautiful! Think beyond the results of the experiment. Imagine the possibilities! The experiment tells you that you don't need a battery to create an emf! Not wires connected to a battery, just move the circuit inside the magnetic field.

- In figure 9.2b,the magnetic field source is moved and the closed wire is maintained static. A current and certain emf are registered in the wire if you connect it to a voltmeter. Once again, no batteries connected to the wire,and even so, there is a current and an emf.

- In figure 9.2c, the field field source and the closed wire do not move. However, there is a change of the strength of the magnetic field (its magnitude). A current and efm are generated once again!

These experimental results can be summarized in one equation

$$\boxed{\varepsilon = -\frac{d\Phi_B}{dt}} \tag{9.7}$$

known as **Faraday's Law**, where Φ_B is the magnetic flux through the surface area enclosed by the loop wire. What Faraday's Law establishes is that whenever there is a change in the magnetic flux through the enclosed area of a closed loop, there will be an induced emf and current.

If we substitute in the last equation the magnetic flux in equation 9.3, we have that

$$\varepsilon = -\frac{d}{dt}\left(\int \vec{B} \cdot d\vec{A}\right) = -\int \frac{d}{dt}\left(|\vec{B}| dA \cos\theta_{AB}\right) \tag{9.8}$$

so, notice that an emf can be produced whenever you change either the magnitude of the magnetic field, the area where the magnetic field lines crosses the area, the angle between the magnetic field and the area vector (θ_{AB}) or any combination of them. In general, if the conductor wire which an emf is induced has several loops, equation 9.7 becomes

$$\boxed{\varepsilon = -N\frac{d\Phi_B}{dt}} \tag{9.9}$$

where N is the number of wrapped loops of the wire.

Now, probably you ask yourself "*What is actually the direction of the area vector in the magnetic flux?*" In the previous section, we said that it is perpendicular to loop surface. However, there are two directions which can achieve this. To be consistent with possible results of Faraday's Law, we use a right hand rule again. Stick your thumb finger of your right hand in the direction of the area vector and your fingers curl in the direction of the loop (see diagrams in figure 9.3). If $\varepsilon < 0$, it means that the direction of the induced electric current is opposite to the direction of you curled fingers. If $\varepsilon > 0$, it means that the electric current direction is the direction of your curled fingers. So, see (diagrams in figure 9.3) to practice what would be the direction of the induced current and emf.

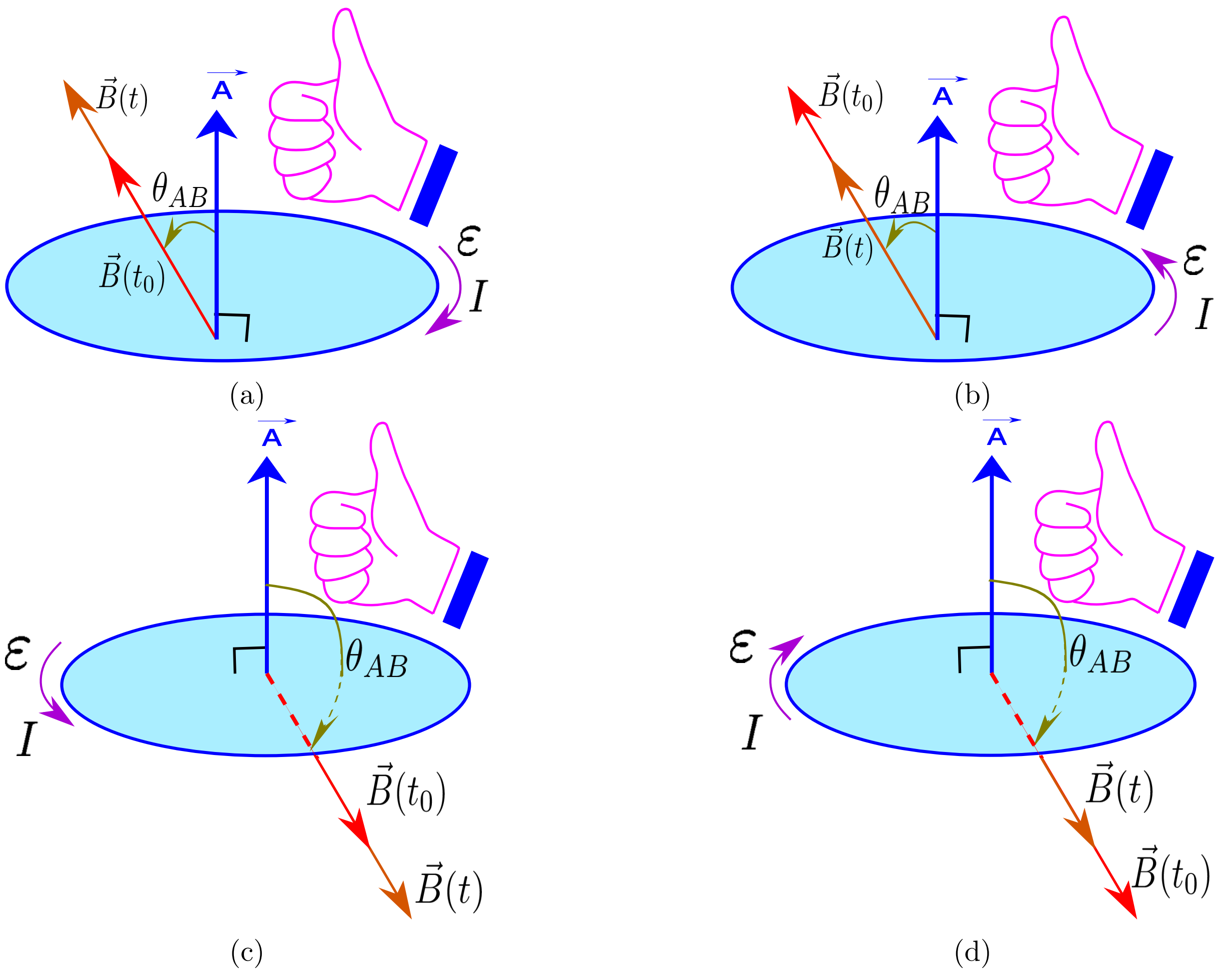


Figure 9.3

The minus sign in equation 9.7 is extremely important. So important that the fact that the minus sign is placed in there we call it as *Lenz Law*. The minus sign in Faraday's Law says that *Nature opposes to the change of magnetic flux*. Whatever is the source of the electric current induced in the conducting wire, this electric current will create a magnetic field in such a way that will try to oppose to the change of magnetic flux. So, in general we state that

Lenz Law: The induced electric current in a loop conducting wire is in such direction that it creates a magnetic field that opposes the change of the magnetic flux.

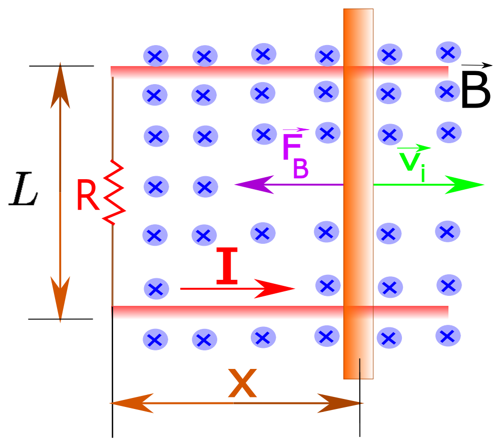


Figure 9.4

In order to get a better grasp, suppose the following scenario. Suppose two parallel conducting bars are connected to a resistance as shown in figure 9.4. A constant magnetic field is applied all along between the two parallel conducting wires as shown in the figure. Now, you place a third conducting bar perpendicular to the other two bars. We establish the direction of the area vector the same as the magnetic field. Now, we give a little push to the bar and suppose that there is no friction between the conducting bars. When the bar starts to move to $+x$ the magnetic flux increases, because the area gets bigger and bigger. Therefore,

$$\frac{d\Phi_B}{dt} > 0 \tag{9.10}$$

but, Lenz law stablishes that the induced current must be so that the magnetic field that it produces opposes to the change of the external magnetic flux.

$$\varepsilon = -\frac{d\Phi_B}{dt} < 0 \tag{9.11}$$

so, given that the magnetic field is going towards the page, the induced magnetic field must be outwards the page. So, the induced electric current must travel anti-clockwise (check this with right hand rule).
Now, if the bar moves from left to right, the magnetic flux decreases, because the area gets smaller and smaller. Therefore,

$$\frac{d\Phi_B}{dt} < 0 \tag{9.12}$$

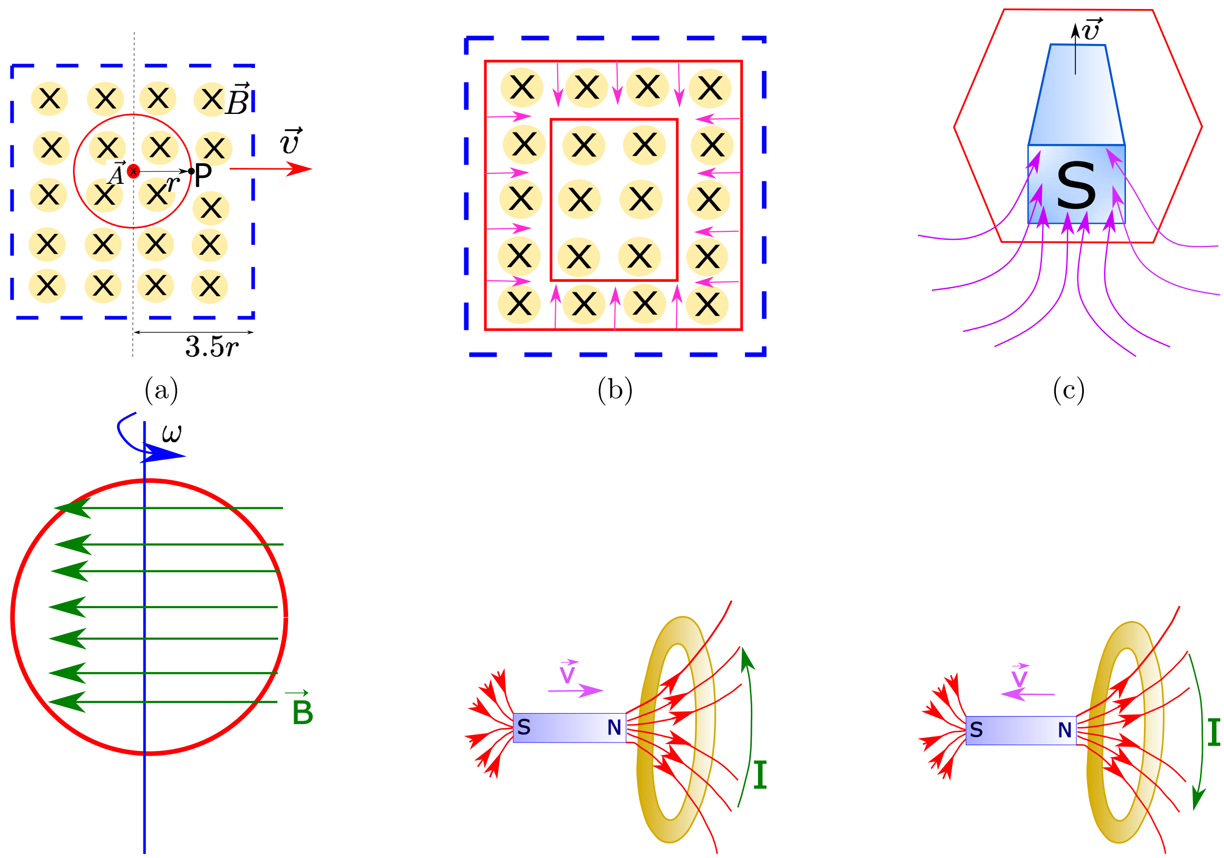
however, by Lenz law the induced current must go clockwise so that the generated magnetic field opposes the change.

$$\varepsilon = -\frac{d\Phi_B}{dt} > 0 \tag{9.13}$$

so, the magnetic field direction is towards the page. In either case, the induced electric current will tend to maintain the original magnetic flux through the enclosed area by the loop of the current.

Example 1: Practicing Lenz Law

For each case shown in figure 9.5, determine the direction of the electric current



(a) (b) (c) (d) (e) (f)

Figure 9.5

Solution:

For each of the cases, we need to calculate the change with respect time of the magnetic flux

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}\left(\int \vec{B}\cdot d\vec{A}\right) = \frac{d}{dt}\left(\int |\vec{B}|dA\cos\theta_{BA}\right) \tag{9.14}$$

where θ_{BA} is the angle between the magnetic field and the area vector. For each case a direction of the area vector must be given to determine the change in the flux. So, we consider the area vector such that we do not have to worry about the sign of $\cos\theta_{BA}$ (except for the case in figure 9.5d). So, we choose the direction the area vector so that the angle between the area vector and the magnetic field is $0 \leq \theta_{BA} < 90$ (except for 9.5d), so $\cos\theta_{BA} > 0$.

a) For figure 9.5a notice that while the circle is inside the magnetic field there is no electric current, because even if the coil is moving, there is no change in the magnetic flux and we assume that the magnetic field is uniform all along the square of length $7.0r$. If we use as the origin of our reference frame in the dotted axis and the area vector towards the page as shown in the figure 9.5a, and call as x the position of point P shown in the figure, when $3.5r < x < 4.5r$ there will be an electric current in the wire. Whenever there is a portion of the enclosed area by the loop (the red circle) outside the region of the magnetic field and a portion of the enclosed area by the loop inside the magnetic field region, there will be a change in the magnetic flux, and therefore a current. Using the convention of the figure 9.5a, the vector area points in the same direction of the magnetic field . So the vector $d\vec{l}$ runs clockwise. When the circle starts to get out of the magnetic field region, the magnetic flux decreases because less magnetic field lines crosses the area A. So

$$\frac{d\Phi_B}{dt} < 0 \qquad \text{when } 3.5r < x < 4.5r \Longrightarrow \varepsilon > 0 \qquad \text{when } 3.5r < x < 4.5 \tag{9.15}$$

So, given that ε is positive, then the current flows in the direction of $d\vec{l}$. So, the electric current flows clockwise when $3.5r < x < 4.5r$. Once the complete circle is outside of the region of the magnetic field, the electric current stops flowing , there is no more change in the magnetic flux.

b) In figure 9.5b the arrows show that the squared circuit decreases its enclosed area. We make that the area vector goes into the page, so that the vector $d\vec{l}$ is clockwise. In such case, then the magnetic flux through the area decreases (less magnetic field lines crosses the area enclosed by the squared circuit). Therefore,

$$\frac{d\Phi_B}{dt} < 0 \Longrightarrow \varepsilon > 0 \tag{9.16}$$

therefore, given that $\varepsilon > 0$ the induced electric current flows clockwise in the same direction of $d\vec{l}$

c) In figure 9.5c, we have that the magnet is going away from the loop. We place the area vector towards the page, such that $d\vec{l}$ is clockwise. Given that the magnetic field strength is proportional to $\frac{1}{r^2}$, we know that further the magnet, lower the magnitude of the magnetic field. So, as the magnet moves away from the loop with velocity $\vec{v}$

$$\frac{d\Phi_B}{dt} < 0 \Longrightarrow \varepsilon > 0 \tag{9.17}$$

Hence, given that $\varepsilon > 0$ the induced electric current flows clockwise in the same direction of $d\vec{l}$

d) For figure 9.5d, let's place the area vector pointing outwards the page. In radians, the angle between the magnetic field and the area vector at any time t is $\theta_{BA} = \omega t + \pi/2$, where we have the $\pi/2$ because at time $t = 0$ the magnetic field and the area vector are perpendicular. So, the magnetic flux is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = |\vec{B}| \cos(\omega t + \pi/2) A \tag{9.18}$$

and we have that the induced emf is

$$\varepsilon - \frac{d\Phi_B}{dt} = |\vec{B}|\omega \sin(\omega t + \pi/2) A \tag{9.19}$$

Therefore, the induced emf ε is positive and negative every some certain time. Also, there will be some moments when the electric current and the induced emf is zero! More specifically the current is zero very time the argument of the sine function is $n\pi$ (where n is any integer number),

$$\omega t' + \pi/2 = n\pi \Longrightarrow t' = \frac{\pi}{\omega}\left(n - \frac{1}{2}\right) \tag{9.20}$$

So, given that ε changes its sign (after being zero any time t' given in 9.20) then the current will be flowing clockwise and anticlockwise. This kind of electric current which changes its direction every certain period of time is known as *alternate current*.

e) For figure 9.5e, we choose the area vector pointing to the right. Now, since the magnet is getting closer to the ring, then the magnetic field magnitude increases, because closer to the source of the magnetic field greater its magnitude. So,

$$\frac{d\Phi_B}{dt} > 0 \Longrightarrow \varepsilon < 0 \tag{9.21}$$

the electric current flows in contrary direction of $d\vec{l}$. In figure 9.5e is shown the direction of the induced electric current.

f) For figure 9.5f, we choose the area vector pointing to the right. Now, since the magnet is getting further from the ring, then the magnetic field magnitude decreases, because further to the source of the magnetic field smaller its magnitude. So,

$$\frac{d\Phi_B}{dt} < 0 \Longrightarrow \varepsilon > 0 \tag{9.22}$$

the electric current flows in the direction of $d\vec{l}$. In figure 9.5f is shown the direction of the induced electric current.

Example 2: Inducing current using a variable current in a toroid

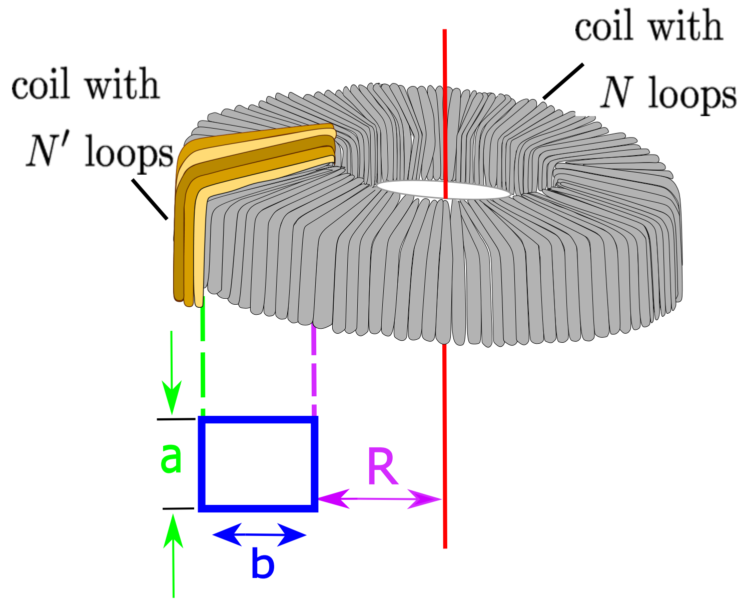


Figure 9.6

A toroid of rectangular cross sectional area $A = a \times b$, where $a = 2.0$cm and $b = 3.0$cm with internal radius $R = 4.0$cm is made up of $N = 500$ loops of conducting wire. The electric current carried by each loop is given by the function $I = I_{max} \sin \omega t$, where $I_{max} = 50.0$A. The frequency $f = 60$Htz. So, another conducting wire is wrapped around certain section of the toroid as shown in figure 9.6. The wrapped conducting wire has $N' = 20$ loops.

- Determine the induced *emf* in the wrapped conducting wire with N' loops, as function of time.
- If the resistance of the conducting wrapped wire with N' loops is $R_\Omega = 20\Omega$. What is the electric current as function of time?

Solution:
So, first of all let's calculate the magnetic flux through one loop of the wrapped conducting wire with N' loops. So,

$$\Phi_{B,1} = \int \vec{B} \cdot d\vec{A} = \int \frac{\mu_0 N I}{2\pi r} dA \tag{9.23}$$

where we used the magnitude of the magnetic field generated by a toroid with N loops (equation 8.84) and r is a radial distance from the axis of the toroid to any point. Also, we have that the differential of area $d\vec{A}$ had exactly the same direction of the magnetic field. Now, using the function of the electric current given by the exercise, we have that the magnetic flux is

$$\Phi_{B,1} = \int \frac{\mu_0 N I_{max} \sin(\omega t)}{2\pi r} dA \tag{9.24}$$

From figure 9.6, we can see that a differential of area in the rectangular loop is

$$dA = adr \tag{9.25}$$

So, the magnetic flux is

$$\Phi_{B,1} = \frac{\mu_0 N I_{max} \sin(\omega t)}{2\pi} a \int_R^{R+b} \frac{dr}{r} = \frac{\mu_0 N I_{max} \sin(\omega t)}{2\pi} a \ln\left(\frac{R+b}{R}\right) \tag{9.26}$$

where we took out from the integral all terms that are independent of r. Also, the integration limits follow because we are interested to calculate the magnetic flux in the squared coil which has internal radius R and outer radius $R+b$ (see figure 9.6).

Once we know the magnetic flux through one loop, we just multiply by the number of loops to obtain the total flux. So,

$$\Phi_B = N' \left(\frac{\mu_0 N I_{max} a}{2\pi} \ln\left(\frac{R+b}{R}\right) \sin(\omega t)\right) \tag{9.27}$$

Now that we know the total magnetic flux, we are in position to calculate the change of magnetic flux, which will just be the time derivative of last equation. Therefore,

$$\frac{d\Phi_B}{dt} = N'\omega \frac{\mu_0 N I_{max} a}{2\pi} \ln\left(\frac{R+b}{R}\right) \cos(\omega t) \tag{9.28}$$

Therefore, the induced emf

$$\varepsilon = -\frac{d\Phi_B}{dt} = -N'\omega \frac{\mu_0 N I_{max} a}{2\pi} \ln\left(\frac{R+b}{R}\right) \cos(\omega t) \tag{9.29}$$

So, let's just plug in now the values

$$\begin{aligned} & -N'\omega \frac{\mu_0 N I_{max} a}{2\pi} \ln\left(\frac{R+b}{R}\right) = \\ & -(20)(2\pi \cdot 60\text{Htz}) \frac{\left(4\pi \times 10^{-7} \frac{\text{T}\cdot\text{m}}{\text{A}}\right)(500)(50\text{A})(2 \times 10^{-2}\text{m})}{2 \cdot \pi} \ln\left(\frac{4 \times 10^{-2}\text{m} + 3 \times 10^{-2}\text{m}}{4 \times 10^{-2}\text{m}}\right) \\ & \approx -0.42\text{V} \end{aligned} \tag{9.30}$$

Therefore, the induced emf is

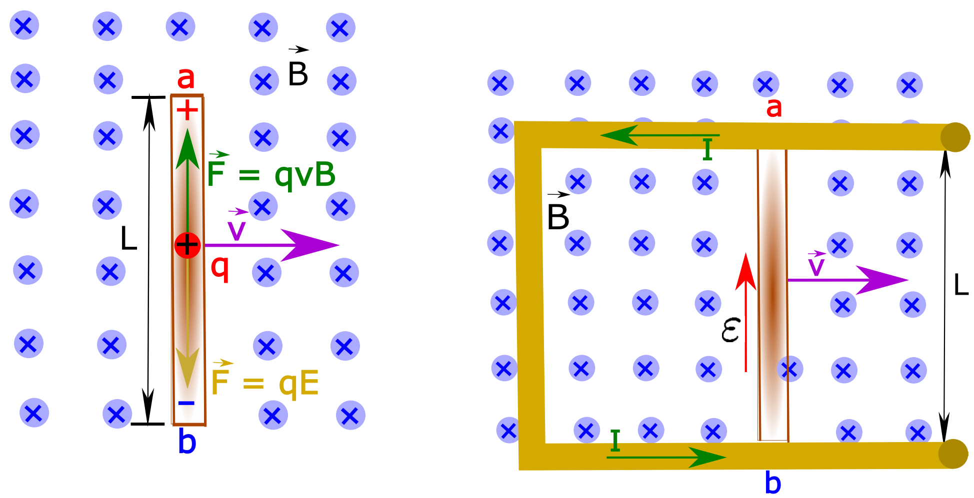
$$\varepsilon = -0.42\text{V} \cos\left(376.99 \frac{\text{rad}}{\text{s}} t\right) \tag{9.31}$$

where we used that $\omega = 2\pi f$. Finally, using ohm's law, the function of the induced electric current is

$$I = \frac{\varepsilon}{R_\Omega} = -\frac{0.42\text{V}}{20\Omega} \cos\left(376.99 \frac{\text{rad}}{\text{s}} t\right) = -0.021\text{A} \cos\left(376.99 \frac{\text{rad}}{\text{s}} t\right) \tag{9.32}$$

9.3 Motional emf

Faraday's Law mentions that any time there is a change in the magnetic flux in the closed loop an emf will be induced. However, why is that such emf is induced? There are two reasons to such induction. The first one has to do with Lorentz Force, which we discuss in this section. The second one has huge implications in physics as we will see, and we will discuss broadly about these implications. However, we wait until we cover both reasons of the induction of the emf, so that our discussion is much richer.



(a) (b)

Figure 9.7

Suppose there is a constant and uniform magnetic field, and an external agent moves a straight bar with constant velocity through the magnetic field as shown in figure 9.7a. Since, the bar is moving with certain velocity $\vec{v}$ then a force is exerted on the electric charges of the bar. Depending on their sign, the exerted force is either upwards for positive charges or downwards for negative charges due to the magnetic force (given by equation 7.4). Therefore, positive charges will move upwards to the end of the bar, and negative charges to the opposite end of the bar as shown in figure 9.7a. So, as the electric charges start to accumulate at the ends of the bar, a potential difference is established. A moment comes when the potential difference is big enough to create an electric field such that the force exerted on the charges is balanced out with the electric force exerted on them, i.e.

$$|\vec{F}_B| = |\vec{F}_E| \Rightarrow q|\vec{B}||\vec{v}| = q|\vec{E}| \tag{9.33}$$

Taking into account that for the bar of lenght L, the potential difference is $\Delta V = |\vec{E}|L$ (assuming that the electric field is constant). Then, we have that

$$\Delta V = |\vec{B}||\vec{v}||L \tag{9.34}$$

Now, suppose this same bar, now sliding above two rails with a U shaped wire with total resistance R as shown in figure 9.7b. For simplicity let's assume that there is no friction between the rails and the sliding bar. As the bar moves, the area gets bigger and bigger so the magnetic flux through the enclosed area grows. Explicitly

$$\Phi_B = |\vec{B}|Lx \tag{9.35}$$

where x is the position of the bar at certain time t and the area is Lx. Also, we used the direction of the area vector in the same direction of the magnetic field. Therefore,

$$\varepsilon = -\frac{dI_B}{dt} = -\frac{d}{dt}\left(|\vec{B}|Lx\right) = -|\vec{B}|L|\vec{v}| \tag{9.36}$$

Notice that the magnitude of the induced emf is the same of the voltage we obtained analyzing the bar just with Lorentz force. The minus sign of the last equation just means that the current has to travel other way around to what we established. And make sense! We used the area vector to the same direction of the magnetic field so your curled fingers direction is clock-wise. So the electric current instead of travelling clockwise , it travels anti-clock wise. So, our discussion of the emf induced with Faraday's Law coincides exactly with our physical analysis with the magnetic force exerted on the charges. In general, for the particular case when the closed loop wire moves, the induced emf is called as *motional emf*.

If we want to calculate the electric current, we can easily obtain it just using Ohm's Law

$$\varepsilon = RI \Longrightarrow I = \frac{\varepsilon}{R} = \frac{|\vec{B}|L|\vec{v}|}{R} \tag{9.37}$$

where we ignored the minus sign, because you can switch the direction of the current to the correct direction and leave it as positive.

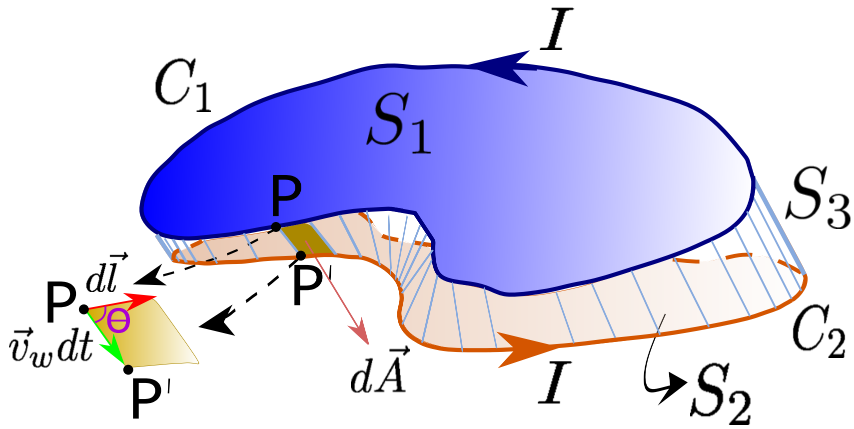


Figure 9.8: C_1 represents the closed path of the circuit at time t. Such circuit carries certain current I. Then, the circuit moves, and finishes with an arbitrary shape. When the closed circuit has moved, it arrives in the configuration of path C_2. The path C_2 represents the exact same circuit after it has moved in certain infinitesimal time dt. The surfaces S_1 and S_2 are the enclosed areas by the circuit in its initial configuration and final configuration respectively. Surfaces S_1, S_2 and the side gridded surface S_3 enclose certain volume.

Now, what is the relation between Faraday's Law and the motional emf for any general case? We will study when the circuit starts with any arbitrary shape, moves to any arbitrary direction and finishes with any arbitrary shape. All these happening in a differential of time dt. So, we build an arbitrary shape of a closed loop at time t and a final arbitrary shape of the same closed loop which has moved in an infinitesimal time dt to an arbitrary direction (see figure 9.8). So, we have that the change of magnetic flux, the difference of magnetic flux between the initial configuration of the wire and the final configuration of the wire is

$$d\Phi = \Phi(t+dt) - \Phi(t) = \int_{S_2} \vec{B} \cdot d\vec{A} - \int_{S_1} \vec{B} \cdot d\vec{A} \tag{9.38}$$

i.e. we are calculating the difference of magnetic flux through surface S_2 and S_1. Now, we will do a smart move. We create a surface S_3 that joins surfaces S_1 and S_2, in such way that surfaces S_1, S_2 and S_3 make a closed surface (a surface that encloses a volume). So, we know from equation 9.6 that

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{9.39}$$

so, if we use as the closed surface the union of S_1, S_2 and S_3, we have that

$$\oint \vec{B} \cdot d\vec{A} = \int_{S_2} \vec{B} \cdot d\vec{A} + \int_{S_1} \vec{B} \cdot d\vec{A} + \int_{S_3} \vec{B} \cdot d\vec{A} = 0 \tag{9.40}$$

Now, we do not know what is the direction of the magnetic field, because we are dealing with a very general case, the magnetic field lines could be pointing any direction. However, we know this "*If the magnetic field lines goes into surface S_1, then the magnetic field lines* *must go outwards at least one (could be both) of the surfaces S_2 and S_3. Because magnetic field lines make closed loops. And this holds for any other arbitrary surface S_1,S_2 or S_3. If the magnetic field lines go into surface S_2, then the magnetic field lines must go outwards at least one (could be both) of the surfaces S_1 and S_3. If the magnetic field lines go into surface S_3,then the magnetic field lines must go outwards at least one (could be both) of the surfaces S_1 and S_2."* What is important about this observation, is that two surfaces magnetic flux have the opposite sign of the third surface. Therefore, we have that

$$\int_{S_2} \vec{B} \cdot d\vec{A} - \int_{S_1} \vec{B} \cdot d\vec{A} - \int_{S_3} \vec{B} \cdot d\vec{A} = 0 \tag{9.41}$$

Hence, isolating the difference of the magnetic flux through surface S_2 and S_1, we have that

$$d\Phi = \Phi_{S_3} = \int_{S_3} \vec{B} \cdot d\vec{A} \tag{9.42}$$

i.e. the change of the magnetic flux is the magnetic flux through surface S_3. Now, let's call $\vec{v}_w$ the velocity of the wire and $\vec{v}_c$ the velocity of the electric charges down the wire. Therefore, the total velocity of the electric charges is

$$\vec{v} = \vec{v}_w + \vec{v}_c \tag{9.43}$$

Now, notice that we can write $d\vec{A}$ (the differentials of areas of the surface S_3, see figure 9.8) as

$$d\vec{A} = \left(\vec{v_w} \times d\vec{l}\right) dt \tag{9.44}$$

Therefore ,

$$\frac{d\Phi}{dt} = \oint \vec{B} \cdot \left(\vec{v_w} \times d\vec{l}\right) \tag{9.45}$$

where notice that the integral is now a closed integral. The reason is simple, now you are integrating with respect $d\vec{l}$. We need that the differential area $d\vec{A} = \left(\vec{v_w} \times d\vec{l}\right) dt$ sweeps out all the surface S_3. We achieve this by making the vector $d\vec{l}$ travel all the closed path C_1 that encloses the area S_1. Now, since $\vec{v}_c$ and $d\vec{l}$ have the same direction, we have that

$$\vec{v} \times d\vec{l} = (\vec{v}_w + \vec{v}_c) \times d\vec{l} = \vec{v}_w \times d\vec{l} \tag{9.46}$$

So, equation 9.45 becomes

$$\frac{d\Phi}{dt} = \oint \vec{B} \cdot \left(\vec{v} \times d\vec{l}\right) \tag{9.47}$$

Now, we have that

$$\vec{B} \cdot \left(\vec{v} \times d\vec{l}\right) = -\left(\vec{v} \times \vec{B}\right) \cdot d\vec{l} \tag{9.48}$$

(you can check this by yourself). Hence, equation 9.47 becomes

$$\frac{d\Phi}{dt} = -\oint \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l} = -\oint \frac{1}{q}\left(q\vec{v} \times \vec{B}\right) \cdot d\vec{l} = -\oint \frac{1}{q}\vec{F}_B \cdot d\vec{l} \Rightarrow \varepsilon = \oint \frac{1}{q}\vec{F}_B \cdot d\vec{l} \tag{9.49}$$

where notice once again we obtain that the responsible of the induced emf is the magnetic force! Of course the repeated $q's$ are unnecessary. They were written to show that the magnetic force is indirectly present in last equation. We could actually say that the force per unit charge $\vec{F}_B/q$ is present in the last equation.

So, we have that in general for motional emf in closed loop

$$\boxed{\varepsilon = -\frac{d\Phi}{dt} = \oint \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l}} \tag{9.50}$$

Important to remark, is that a motional emf is induced not necessarily in closed loops. If an open wire with any shape moves with velocity $\vec{v}$ in any non-uniform magnetic field (so the magnetic field can vary from point to point), the induced emf is given by

$$\boxed{\varepsilon = \int \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l}} \tag{9.51}$$

The first example that we have already discussed of this induced emf is the straight wire moving with velocity $\vec{v}$.

Example 3: Force acting on a conductor sliding bar

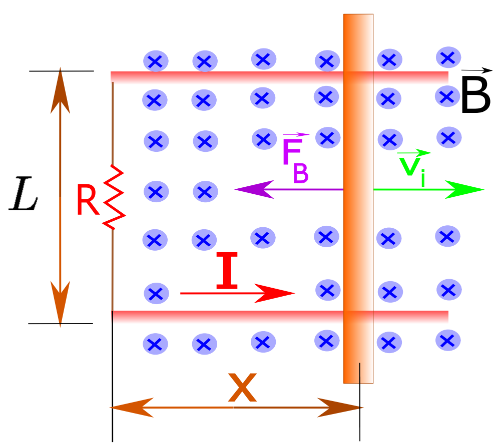


Figure 9.9

The conductor bar shown in figure 9.9 slides without friction over two parallel conductor rails. There is a magnetic field towards the page as shown in the figure. The sliding bar has certain mass m and length L. The bar is pushed and has certain initial velocity $\vec{v}_i$ to the right.

- Find the velocity of the bar by using Newton's Laws
- Show that the same result is found by using conservation of energy

Solution:

If there is no friction, then the only force applied to the bar is the magnetic force. Therefore, by second Newton's Law

$$ma_x = -\vec{F}_B = -IL|\vec{B}| \tag{9.52}$$

where the minus sign follows because the force is applied to $-x$, and we substituted the force exerted on a straight wire where the magnetic field is perpendicular to the electric current (equation 7.78). Now , recall that $a_x = \frac{dv_x}{dt}$, therefore

$$\frac{dv_x}{dt} = -\frac{ILB}{m} \Rightarrow \frac{dv_x}{dt} = -\left(\frac{BLv_x}{R}\right)\frac{LB}{m} \tag{9.53}$$

where we used equation 9.37 for the electric current. So, separating variables and integrating we have

$$\int_{v_{0x}}^{v_x} \frac{dv_x}{v_x} = -\frac{B^2L^2}{mR}\int_0^t dt \Rightarrow \ln\left(\frac{v_x}{v_{0x}}\right) = -\frac{B^2L^2}{mR}t \tag{9.54}$$

Hence the velocity is

$$v_x = v_{0x}e^{-B^2L^2t/mR} \tag{9.55}$$

Now, if we apply conservation of energy, we have the following

$$-\Delta K = E_R \tag{9.56}$$

where E_R is the energy stored in the resistance of the circuit. Probably there is no friction between the rails and the bar, however there is an electric current which the collision of the electrons with the material release heat. So, what last equation is saying is that as the bar looses kinetic energy, the resistor gains energy as heat. If we divide the last equation by time Δt, we obtain units of power (J/s). Therefore, last equation becomes

$$-\frac{\Delta K}{\Delta t} = P_R \tag{9.57}$$

where the label R is just to let know that is the power due to Joule effect (heat transferred to the resistance). Now, if we let the time $\Delta t \to 0$, the value will be exact not just an approximation. So,

$$\frac{dK}{dt} = I^2R \tag{9.58}$$

where used equation 5.32 for the Joule heat, which was substituted in the right-hand side of the equation and the left-hand side became a differentiation because we took the limit of $\Delta t \to 0$. So, substituting the kinetic energy

$$-\frac{d}{dt}\left(\frac{1}{2}mv_x^2\right) = I^2R \Rightarrow mv_x\frac{dv_x}{dt} = \left(\frac{BLv_x}{R}\right)^2 R \tag{9.59}$$

where we used equation 9.37 for the electric current and derived the velocity with respect time using the chain rule. Finally, separating variables

$$\int_{v_{ox}}^{v_x} \frac{dv_x}{v_x} = -\frac{B^2L^2}{mR}\int_0^t dt \tag{9.60}$$

which is exactly the same integration we obtained when we used Newton's Laws, so the final result will be exactly the same

$$v_x = v_{0x}e^{-B^2L^2t/mR} \tag{9.61}$$

as it should! It could not be that energy conservation result were different to the result applying Newton's laws.

Example 4: Emf induced in a rotating bar

A conducting bar of length L rotates with constant angular speed ω as shown in figure 9.10. A constant and uniform magnetic field is perpendicular to the plane of rotation of the bar. Find the induced emf between the ends of the bar.

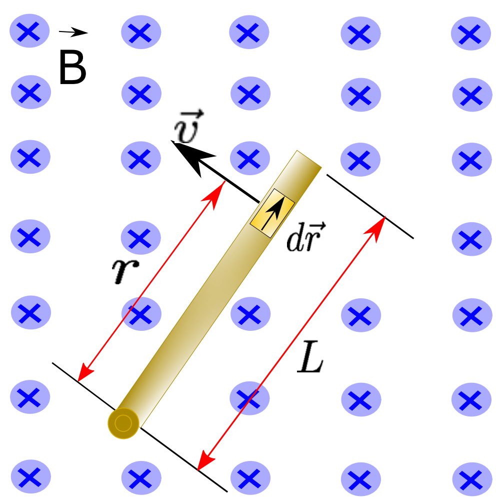


Figure 9.10

Solution:

First of all, can we use *Faraday's Law*? No! Because Faraday's Law applies for closed loops, in this case there is no closed circuit, just the bar. However, even though there is an emf induced. Why? Because the bar is moving with angular velocity and a magnetic force will be exerted to all the electric charges of the bar, making positive charges go to one of the ends of the bar, while the negative while move to the other end of the bar, making a potential difference between the ends. It is similar to what we saw in the moving bar horizontally. Now, if we cut the bar in infinitesimal chunks, you can visualize that each infinitesimal chunk has a certain tangent speed $v = \omega r$ where r is the radial distance from the axis of rotation to the infinitesimal chunk. Now every single small chunk generates an infinitesimal emf $d\varepsilon = Bvdr$ and we can think as the complete bar as made of tons of small batteries in parallel with magnitude emf $d\varepsilon$. Given that when batteries are in parallel the efm sum, we have that the magnitude of the emf is

$$\varepsilon = \int d\varepsilon = \int_0^L |\vec{B}|\omega r dr = |\vec{B}|\omega \frac{L^2}{2} \tag{9.62}$$

where the limits of integration follow because we cover the whole bar. Now, before we are done, notice that when the bar moves, the exerted magnetic force on the electric charges will make that the positive charges move to the axis of rotation, while the negative charges will move to the top end of the bar (use the right hand rule to see the direction of the exerted force on the electric charges). Since we started measuring the emf from the axis of rotation to the end of the bar, then the emf is

$$\varepsilon = -|\vec{B}|\omega \frac{L^2}{2} \tag{9.63}$$

where the negative sign means that the electric current will flow from the end of the bar towards the axis of rotation. In this exercise we used lots of physical intuition to solve the problem. However, what if we use equation 9.51? We have that

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{9.64}$$

where $\vec{v}$ is the tangent velocity of each electric charge , and $\vec{r}$ is the vector starting from the axis of rotation to the position of the electric charges. So, we have that

$$\varepsilon = \int \left((\vec{\omega} \times \vec{r}) \times \vec{B} \right) \cdot d\vec{r} \tag{9.65}$$

where we used that for this case $d\vec{l} = d\vec{r}$. Now, the direction of the vector $\vec{v} = \vec{\omega} \times \vec{r}$ is already known due to the direction of movement of the bar. And the direction of $\vec{v} \times \vec{B}$, using the right hand rule is towards the axis of rotation as shown in the figure. So, $(\vec{\omega} \times \vec{r}) \times \vec{B}$ and $d\vec{r}$ are anti-parallel (see the direction of $d\vec{r}$ in figure 9.10). So, equation 9.65 becomes

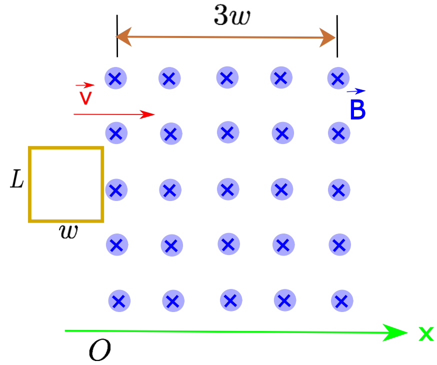
$$\varepsilon = -\int_0^L |\vec{B}|\omega r dr = -|\vec{B}|\omega \frac{L^2}{2} \tag{9.66}$$

where the limits of integration have been already included. Notice we have obtained the same magnitude, and the advantage is that we already know the sign of ε, so we already know the direction of the current, and much less intuition needed to solve the complete problem. Just straightforward use of cross products.

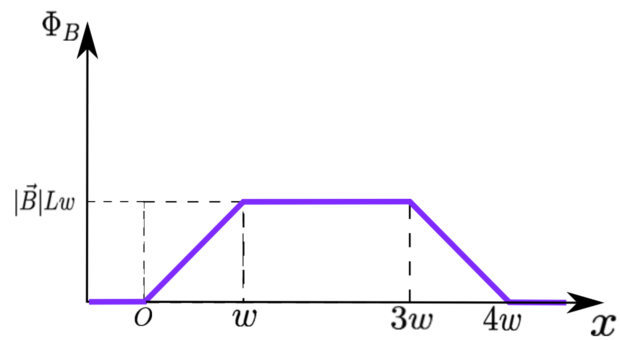
Example 5: Induced emf in a Moving Loop

A squared conducting wire, with dimensions L, w and resistance R, moves with constant speed v to the right as shown in the figure 9.11a. There is a uniform and constant magnetic field towards the page as shown in figure 9.11a. If x is the position of the right-hand side of the squared loop along the $x - axis$

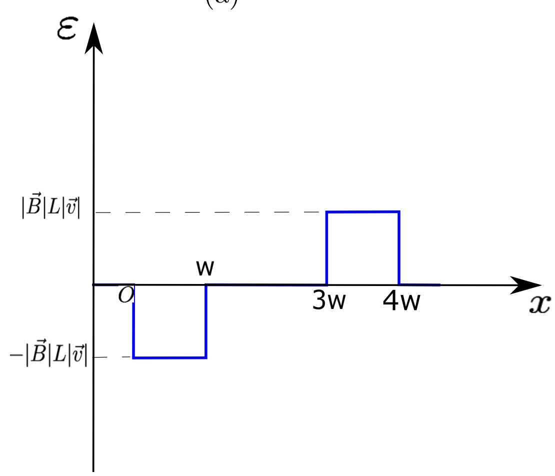
- Make a plot of the magnetic flux through the enclosed area of the squared loop as function of position x
- Make a plot of the induced emf in the squared loop as function of position x
- Make a plot of the force an external agent needs to apply to the squared loop to counter the magnetic force exerted on the loop so that its speed is constant



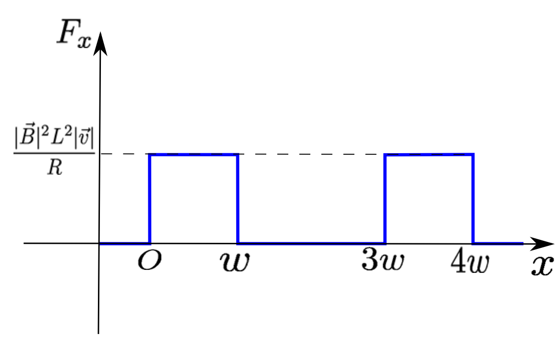
(a)



(b) Plot of the electric flux through the enclosed area of squared loop with respect to the position of the right-hand side wire of the squared loop in figure 9.11a. The plot is based on equations 9.67, 9.71, 9.74 and 9.78.



(c) Plot of the induced emf in the squared loop with respect to the position of the right-hand side wire of the squared loop in figure 9.11a. The plot is based on equations 9.68, 9.72, 9.75and 9.79.



(d) Plot of the external force exerted on the squared loop with respect to the position of the right-hand side wire of the squared loop in figure 9.11a. The plot is based on equations 9.70, 9.73, 9.77and 9.80.

Figure 9.11

Solution:

First of all, we establish the direction of the vector $d\vec{l}$ in the closed loop of the square circuit clockwise, so that the area vector points towards the page (exactly the same direction of the magnetic field in figure 9.11a. Notice in figure 9.11a that when the right hand side of the squared loop is to the right of position O, i.e. when $x > 0$, there is necessarily a flux through the loop. Given that the magnetic field is constant, we have that

$$\Phi_B = |\vec{B}|A = |\vec{B}|Lx \qquad \text{for } 0 < x < w \tag{9.67}$$

where A is the area where there is magnetic flux in the loop. So ,the change of the magnetic flux in the range $0 < x < w$ is

$$\frac{d\Phi_B}{dt} = |\vec{B}|L|\vec{v}| \qquad \text{for } 0 < x < w \Longrightarrow \varepsilon = -|\vec{B}|L|\vec{v}| \tag{9.68}$$

where we used that the speed is $\frac{dx}{dt}$. Given that the induced emf is negative, then the electric current travels in the opposite direction of the established vector $d\vec{l}$. So, the current flows anti-clockwise. If it is the case, then, by the right hand rule the magnetic field force exerted on the right-hand side wire of the squared circuit is

$$\vec{F}_B = I\vec{L} \times \vec{B} = -IL|\vec{B}|\hat{x} = -\left(\frac{|\vec{B}|L|\vec{v}|}{R}\right)L|\vec{B}| = -\frac{|\vec{B}|^2L^2|\vec{v}|}{R}\hat{x} \tag{9.69}$$

where notice that the only magnetic force exerted is on the right-hand side wire of the squared loop, because the left-hand side wire of the squared loop still is outside of the range of the magnetic field, and the top and bottom parts of the squared loop, their forces cancel out (same magnitude opposite directions). Also, we applied ohm's law (as we did to obtain equation 9.37). However, given that the velocity is constant, the force exerted by an external agent must be of the same magnitude as the magnetic force but with opposite direction. Hence

$$F_x = \frac{|\vec{B}|^2L^2|\vec{v}|}{R} \qquad \text{for } 0 < x < w \tag{9.70}$$

where F_x is the external force exerted on the squared loop by an external agent to keep a constant speed.

Once the squared loop is completely immersed in the magnetic field, the magnetic flux reaches a maximum value (the area where there is flux does not increase any more, it is now the total area of the squared loop) and remains constant until the squared loop right-hand side wire reaches $x = 3w$ (see figure 9.11a). So,

$$\Phi_B = |\vec{B}|Lw \qquad \text{for } w < x < 3w \tag{9.71}$$

where Lw is the area of the squared loop. So, given that the magnetic flux is constant, then the emf is zero

$$\varepsilon = 0 \qquad \text{for } w < x < 3w \tag{9.72}$$

hence, there is no induced current in the squared loop. In such case there is no magnetic force exerted on the squared loop. Therefore, there is no need to apply an external force to maintain a constant speed (if we apply a force actually we accelerate the squared loop). Thus,

$$F_x = 0 \qquad \text{for } w < x < 3w \tag{9.73}$$

Once the right-hand side wire of the squared loop is in the range $3w < x < 4w$, some area of the loop is not any more immersed in the magnetic field. So the magnetic flux changes, and once again an induced current flows in the loop.

$$\Phi_B = |\vec{B}|A = |\vec{B}|L\,(4w - x) \qquad \text{for } 3w < x < 4w \tag{9.74}$$

where $A = L\,(x - 3w)$ is the area where there is magnetic flux in the squared loop. So ,the induced emf

$$\frac{d\Phi_B}{dt} = -|\vec{B}|L|\vec{v}| \qquad \text{for } 3w < x < 4w \Longrightarrow \varepsilon = |\vec{B}|L|\vec{v}| \tag{9.75}$$

given that $\varepsilon > 0$ then the current flows in the direction of vector $d\vec{l}$, so current flows clockwise and the magnetic force exerted on the left-hand side wire of the loop, by using the right-hand rule, is

$$\vec{F_B} = I\vec{L} \times \vec{B} = -IL|\vec{B}|\hat{x} = -\left(\frac{|\vec{B}|L|\vec{v}|}{R}\right)L|\vec{B}| = -\frac{|\vec{B}|^2L^2|\vec{v}|}{R}\hat{x} \tag{9.76}$$

where notice that only a magnetic force exerted is on the left-hand side wire of the squared loop, because the right-hand side wire of the squared loop now is outside of the range of the magnetic field, and the top and bottom parts of the loop, the exerted forces cancel out (same magnitude opposite directions). Once again, given that the velocity is constant, the force exerted by an external agent must be of the same magnitude as the magnetic force but with opposite direction. Hence

$$F_x = \frac{|\vec{B}|^2L^2|\vec{v}|}{R} \qquad \text{for } 3w < x < 4w \tag{9.77}$$

Finally, once the squared loop is completely outside the magnetic field range (for $x > 4w$), there is no more magnetic flux

$$\Phi_B = 0 \qquad \text{for } x > 4w \tag{9.78}$$

therefore no induced emf

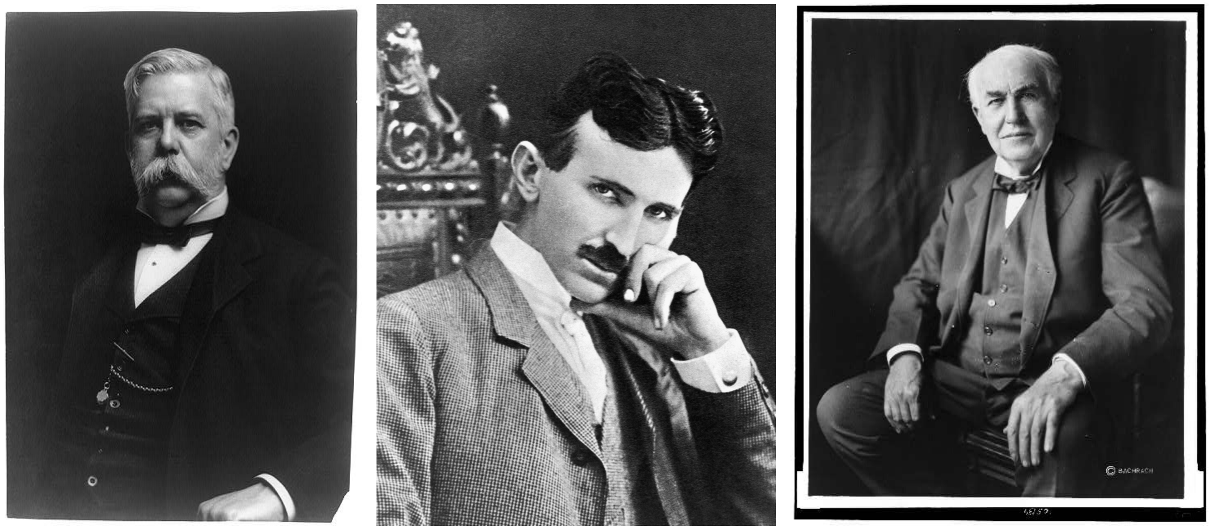
$$\varepsilon = 0 \qquad \text{for } x > 4w \tag{9.79}$$

consequently no induced electric current, so no magnetic force exerted on the squared loop, so no necessity of applying an external force to maintain the squared loop moving with constant speed

$$F_x = 0 \qquad \text{for } x > 4w \tag{9.80}$$

So, the plots of the magnetic flux, the induced emf and the needed external force to maintain the squared loop to move with constant speed are in figures 9.11b. 9.11c and 9.11d.

9.4 The war of the currents



(a) George Westinghouse (b) Nikola Tesla (c) Thomas Alva Edison

Figure 9.12: Original pictures taken from references [19],[20] and [21] respectively.

The war of the currents was a period of time that took place as an intellectual (and monetary interests) revelry between those engineers and scientists that supported the direct current and the alternate current. More specifically this took place in the United States during the late $1880's$ and early $1890's$. After the war of the currents took place, the world would never be the same. The war ended in 1893 when the alternate current showed its power in the Chicago's World Fair. In this war, we have two giants as leaders, *George Westinghouse* who wanted to supply electric energy to all U.S.A with alternate current, and *Thomas Alva Edison* who was the counterpart trying to supply direct current. By the time the War started, Thomas Alva Edison was already like a rock star of science and engineering in those days. He had tremendous influence and the support of one of the most powerful bankers *J.P. Morgan*. So, whatever he mentioned about the alternate current to general public, they would believe him. So, when Westinghouse was trying to expand in the USA using alternate current, Thomas Alva Edison made public demonstrations electrifying and killing animals, saying that he used alternate current in such demonstrations and spreading the idea that alternate current was dangerous. When Westinghouse was almost overthrown, a genius appeared into the action, the giant **Nikola Tesla** (years before was hired by Thomas Alva Edison, however Edison did not support Nikola's ideas). He invented a poly-phase motor that could make a completely integrated alternate current (AC) system. In May 1892, George Westinghouse won the contract to light the famous Chicago's Fair. One year later, on May 1rst 1983, when the night came, the world witnessed the most incredible display of lights that had ever been seen before! This was the ultimate proof that alternate current was more powerful and cheaper than direct current. Then, Westinghouse won the contract to make a electric power plant at the Niagara Falls, an idea that came from the brilliant Nikola Tesla.

Figure 9.13: The Chicago World's Fair in 1893. Before that night, the human beings had never seen before such an amazing spectacle. Complete blocks were illuminated by thousands of lights. It was that night that the Alternate Current showed its power. Original picture taken from reference [22].

In this section, we discuss three electric devices. Two of them (the alternator and the transformer) are cornerstones for alternate current, while Faraday's Dynamo is a device that can produce direct current. For all of these devices their physical functioning basis lies in Faraday's Law. Finally, we make a quantitative comparison between alternate and direct current, showing the greatness of the alternate current.

Example 6: Faraday's Dynamo

A conductor disk of radius R rotates with a constant angular speed ω about its central axis as shown in figure 9.14. If the the disk is immersed in a constant magnetic field $\vec{B}$ as shown in figure 9.14, determine the emf induced between the center of the disk and the border of the disk. ***Solution:***

We cannot apply the flux rule for this case. The calculation of an induced emf must be along a closed loop, and in this case the electric current will flow radially in all directions in the disk. There is a motional emf because the electric charges in the disk are moving respect to the magnetic field. The tangent velocity of each charge is

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{9.81}$$

So, using equation 9.51, we have that the induced emf is

$$\varepsilon = \int \left((\vec{\omega} \times \vec{r}) \times \vec{B} \right) \cdot d\vec{r} \tag{9.82}$$

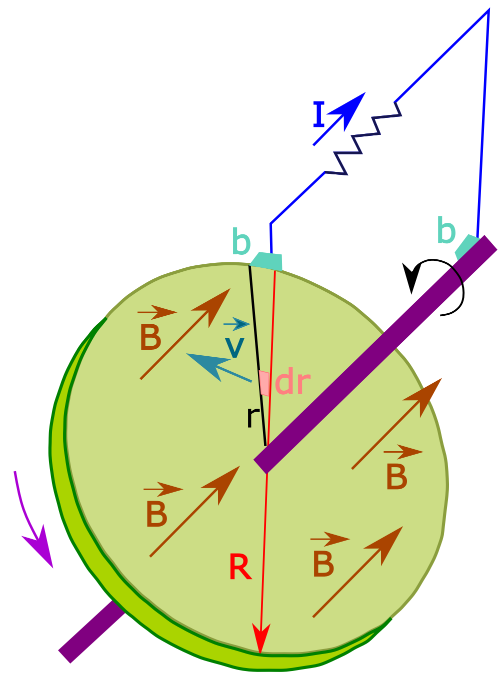


Figure 9.14

where we used $d\vec{l} = d\vec{r}$. Now, the direction of the vector $\vec{v} = \vec{\omega} \times \vec{r}$ is already known due to the direction of movement of the disk. And the direction of $\vec{v} \times \vec{B}$, using the right hand rule is radially away the axis of rotation. For example as shown in figure 9.14, the current would be flowing through the sliding conductor contact in b. So, $(\vec{\omega} \times \vec{r}) \times \vec{B}$ and $d\vec{r}$ are parallel. Therefore, the emf is

$$\varepsilon = \int_0^R |\vec{B}|\omega r dr = |\vec{B}|\omega \frac{R^2}{2} \tag{9.83}$$

Notice, that the induced emf is exactly the same magnitude as the bar in example 4! In this case we obtained an opposite sign, because the magnetic field has opposite direction to the magnetic field in the example 4. We can think that the disk is composed of several bars of infinitesimal width, each generating an emf of magnitude $|\vec{B}|\omega \frac{R^2}{2}$. Using a bar to generate an emf is not very practical as using the disk. If you place an sliding conductor contact at the edge of the disc, and another at the center, then you have continuously a constant flowing of electric current and emf as shown in figure 9.14. However, for the bar this is not possible. Since the sign of the emf does not change with time (unless suddenly the disc rotates to the opposite direction or the magnetic field changes its direction), you create a direct current out of this device. Thomas Alva Edison used Dynamos to generate direct current and supply electricity to neighborhoods in New York.

Example 7: The Alternator

The figure 9.15 shows a simple version of an alternator which loop moves with constant angular speed. The figure 9.15 shows different stages of the loop which moves. Given that the induced emf in the alternator is not constant and changes its sign, the electric current induced in the wire changes its direction. The kind of electric current induced is called *alternate current*. The loop rotates with a constant angular speed ω as shown in the figure. The magnetic field $\vec{B}$ is constant and uniform. At time $t = 0$, the angle between the magnetic field and the area vector is $\phi = 0$ as shown in figure 9.15a.

- Find the induced efm in the squared loop as function of time
- Find the induced electric current in the squared loop as function of time

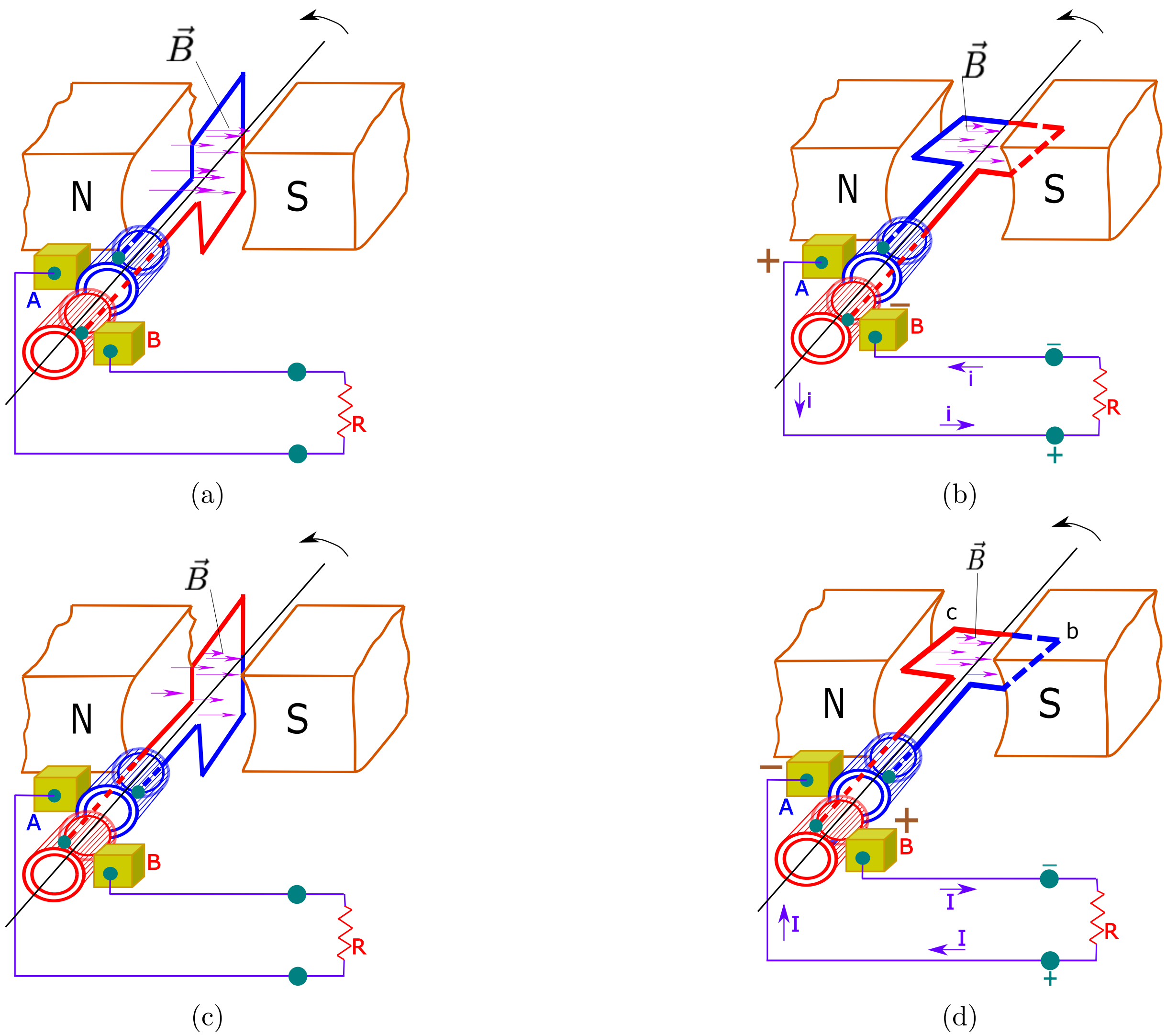


Figure 9.15

Solution:
The angle between the area vector and the magnetic field can be written as

$$\theta_{BA} = \theta_0 + \omega t \tag{9.84}$$

where the label BA just means that is the angle between the magnetic field and the area vector, ω is the angular speed of rotation of the loop and θ_0 is the initial angle. However, at $t = 0$ the angle is zero , so we can just say that

$$\theta_{BA} = \omega t \tag{9.85}$$

So, the induced emf in the loop is

$$\varepsilon = -\frac{d}{dt}\left(\vec{B}\cdot\vec{A}\right) = -\frac{d}{dt}\left(|\vec{B}||\vec{A}|\cos\omega t\right) = \omega|\vec{B}||\vec{A}|\sin\omega t \tag{9.86}$$

So, notice that this simple device has given us an alternating current! After every $t = \frac{2\pi n}{\omega}$ for $n = 1, 2, 3, ...$, the sign of the emf changes, so the direction of the induced electric current changes. The electric current is given simply by Ohm's law $\varepsilon = RI$, so

$$I = \frac{\varepsilon}{R} = \frac{\omega|\vec{B}||\vec{A}|}{R}\sin\omega t \tag{9.87}$$

where R is the resistance of the conducting wires. Now think beyond the simple generator of electricity you see in figure 9.15 . Suppose you could make something huge! To give some numbers, for example $A = 3\mathrm{m}^2$, and $|\vec{B}| = 1.0\mathrm{T}$, and you can make the loop rotate fast as $\omega = 3600\mathrm{r.p.m} \approx 377\frac{\mathrm{rad}}{\mathrm{s}}$. Then using equation 9.86, we have that the induced emf is

$$\varepsilon = 1131\mathrm{V}\sin\left(377\frac{\mathrm{rad}}{\mathrm{s}}t\right) \tag{9.88}$$

No, you make the winding such that it has 350 loops. So, the induced emf is

$$395850\mathrm{V}\sin\left(377\frac{\mathrm{rad}}{\mathrm{s}}t\right) \tag{9.89}$$

Oh my God! You have a device that can generate an exorbitant voltage! With this amount of voltage, you are capable to transport electricity to places kilometers away from the alternator! Is this huge amount of voltage used by domestic electrical devices? No! Before delivering the voltage to houses, the so called *transformer* is used to decrease the voltage to approximately 120V (in México, this depends on the country). We talk in the next example about the transformer.

Now, there are sometimes that the emf is zero, and sometimes the current goes to one direction and after a while the current will switch its direction. So, if a light bulb is connected to the the alternator there will be times that the light bulb turns off (you can think that the resistance in figure 9.15 is the light bulb). You will see a blinking light! This would be terribly annoying. Or even worst! Imagine you close the circuit by connecting a T.V. to the alternator. It would be turning on and turning off! However, notice the following, the frequency is

$$f = \omega/2\pi = 17999.99\frac{1}{\mathrm{s}} \tag{9.90}$$

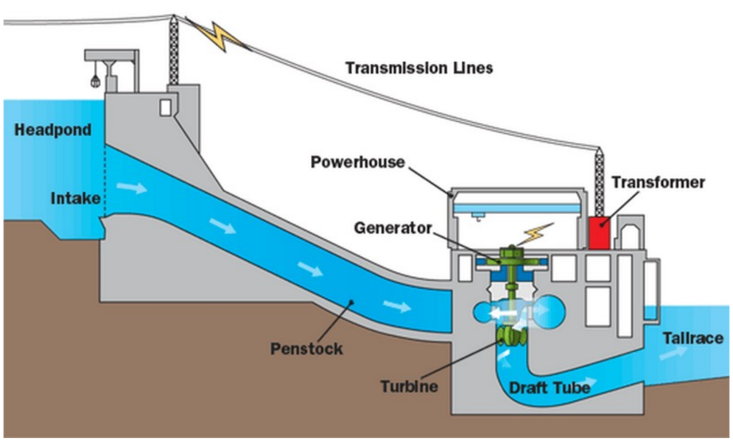
So the period is,

$$T = \frac{1}{f} = \frac{1}{17999.99\frac{1}{\text{s}}} = 5.55 \times 10^{-5}\text{s} \tag{9.91}$$

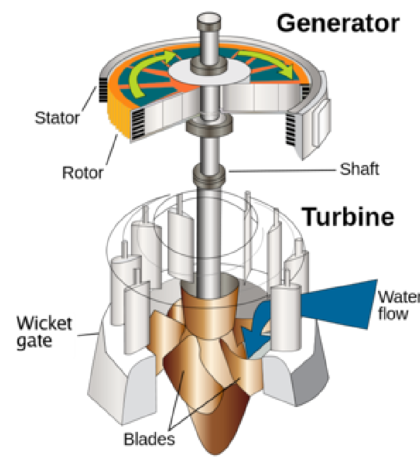
And in one period, the emf is zero twice. So,

$$t_z = \frac{T}{2} = 2.77 \times 10^{-5}\text{s} \tag{9.92}$$

where the label z is just to denote that is the time it takes to be zero the emf. So, every 2.77×10^{-5}s the emf is zero, so there is no current, and therefore no lights! However, the time is so tiny that you would not even notice! Unless you have super powers and you can blink so fast to notice that lights turned off! In the case of the T.V. they have a so called *power supply* that changes the alternate current to direct current and also transforms the voltage to the accurate one for the circuit inside the T.V.



(a) The potential energy of water is transformed to kinetic energy. The water makes the blades of the turbine spin. Once the turbine spins, the generator (the alternator) generates electricity by electromagnetic induction.



(b) How turbines are used to make the rotor to spin. The rotor produces a magnetic field with electromagnets. The stator is made of coils of conducting wires (generally copper) . So, an emf is induced in the coils of the stator. Electric current is generated and transported by the transmission lines to a transformer. The wicket gates control how water flows into the turbine.

Figure 9.16: Original pictures taken from references [17] and [18].

Now, how is the alternator principle used in the real world? There are several designs for a power plant. However, we mention how a hydroelectric power plant operates. These power plants use water to spin the blades of a turbine. So, the turbine can rotate with a huge angular speed. Such turbine, moves loops of coil in circular motion where there is a magnetic field. So, an emf is induced in the coil and an alternating current! That current is transported with transmission lines to a substation. In that substation the voltage is increased with transformers and then that is transported kilometers away to costumers. How beautiful is to think that all our commodity of electricity is due to Faraday's Law. In one way or another, we can say that Faraday's Law moves our economy nowadays. If it were not by Faraday's Law, we would not have all the electrical network we have today. Our economies run due to Faraday's Law!

Now, you know why *Nikola Tesla* proposed to use the Niagara Falls to generate alternate current.

Example 8: The Transformer

The device shown in figure 9.17 is called *transformer*. The transformer consists of two independent windings, one connected to a source of alternate current, and the other connected to devices with certain resistance R. Both windings wound around a core of iron as shown in the figure. We can think also, as the first winding is the input and the second as the output. It is common to call the winding connected to the source of alternate current as the *primary* and the winding connected to the resistance R as the *secondary*.

Suppose a transformer which the primary winding has 20 loops and the secondary has 3500 loops. What would be the voltage in the secondary winding if the voltage in the primary is $\Delta V = 1500\text{V}$?

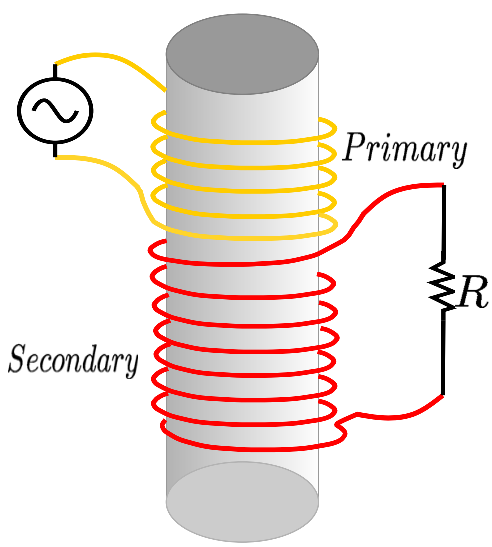


Figure 9.17

Solution:

Probably when you the figure in 9.17, you think *Ok, a core made of iron, and two windings wound to it... So, what?! Not big deal...Actually kind of disappointing and boring.* But my dear reader, if you thought that, you couldn't be more wrong! It is the simplicity of certain powerful systems that give them much more beauty! So, to get a grasp what the transformer does and how it works, we will make a simplification of the transformer, making certain assumptions. However, even though we are making certain assumptions, this encloses the idea of what a transformer does and how it works.

Let's start analyzing the primary winding. This coil is connected to an alternate current source as could be an alternator. So, the electric current is changing its magnitude and direction every time . So, in the area enclosed by the loops in the primary there is a changing magnetic flux! Therefore, there is an induced emf

$$\varepsilon_1 = -N_1 \frac{d\Phi}{dt} \tag{9.93}$$

where N_1 is the number of loops wound in the primary, and $\frac{d\Phi}{dt}$ is the change of magnetic flux in the enclosed area of just one loop in the primary. Notice that ε_1 is a counter emf in the primary winding. The electric current induced in that winding will be such that will try to stop the magnetic flux change. Now, since the core is made of iron, the magnetic dipoles of the core will align to the magnetic field generated by the primary. So, when the magnitude and direction of the magnetic field of the primary changes, also the direction of the magnetic dipoles and the magnetic field they generate! So, the dipoles in the core, generate a changing magnetic flux also in the secondary! For an ideal transformer, the magnetic flux for one loop of the secondary is the same for one loop of the primary (this is not true in a real life transformer, there are effects in the material of the core as hysteresis that makes that the changing magnetic flux is not exactly the same. However, we are making a simplification). Therefore,

$$\varepsilon_2 = -N_2 \frac{d\Phi}{dt} \tag{9.94}$$

Therefore,dividing equation 9.93 over equation 9.94, we have that

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{N_1}{N_2} \tag{9.95}$$

and if we assume that the resistance of the windings is negligible (once again making a simplification), then

$$\boxed{\frac{\Delta V_1}{\Delta V_2} = \frac{N_1}{N_2}} \tag{9.96}$$

where the voltage ΔV_1 is the voltage supplied to the primary winding and the voltage ΔV_2 is the one induced in the secondary winding. So, we have two cases

- if $N_1 > N_2$ the output voltage or the voltage in the secondary **decreases** respect to the primary voltage.

- if $N_2 > N_1$ the output voltage or the voltage in the secondary **increases** respect to the primary voltage.

This principle is used in the transformers in real life. At the power plant, the alternator supplies certain maximum voltage ΔV_{max} and this is then highly increased up with transformers to the order of approximately around $250,000\text{V} - 400,000\text{V}$ (sometimes even higher). Then, the voltage is too high to be the voltage source to domestic circuits, so another transformer now reduces the voltage to order of approximately around $2000\text{V} - 4000\text{V}$ in a substation. However, the voltage still is too high! Probably you have seen near your house or near the University a pole mounted device as the one shown in the figure 9.18b (figure taken from [24]). That is a transformer! And typically that reduces the voltage to around 120V (in México). After all those steps, the connections of our houses will be supplying 120V. We connect then all our electrical devices to that voltage supply. However, when you are sitting comfortably at your home watching T.V. or using your computer, many transformers are working to supply you with the correct voltage, and also working to elevate the voltage to huge amounts of voltage to transport the electric energy up to your house kilometers away from the power plant. This is beautiful, and *Nikola Tesla* and *George Westinghouse* were capable to visualize such power in the alternate current. In figure 9.18a there is a simplified cartoon of the steps needed to trasnport electricity all up to our homes.

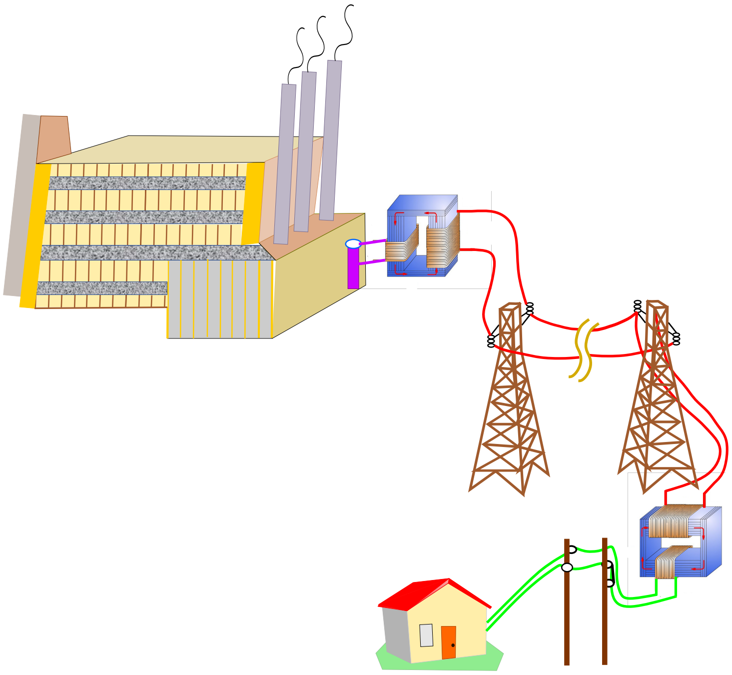
Now, could you create a transformer with direct current? NO! Absolutely no! If in the primary you had direct current, then there would be a magnetic field in the core of iron, but this would not be changing! And what matters is a changing magnetic flux to induce an emf, so the beautiful transformers can only exist due to alternate current.

Finally, answering the question of this exercise, isolating ΔV_2 in equation 9.96, and plugging in values, we have that

$$\Delta V_2 = \frac{N_2}{N_1}\Delta V_1 = \frac{3500}{20}1500\text{V} = 262500\text{V} \tag{9.97}$$

Wow! An amazing increment of voltage! Now that voltage is enough to transport electricity to many kilometers from the power plant!

Now, after this discussion. What do you think about that iron core? If you thought it was boring and not interesting, what about now? It is beautiful the power of the physics isn't it?



(a) A simplified cartoon of how electricity is transported to our homes.

(b) An example of a real transformer

Figure 9.18

Example 9: Alternate Current vs Direct Current

An average of 120kW of power is supplied to a small city by a power plant 80km away from it. The transmission lines use copper (resistivity $\rho = 1.71 \times 10^{-8}\Omega \cdot \text{m}$) and the radius of the cables are 3.5cm. Calculate

- the lost of power due to Joule effect if the energy is transmitted with direct current with 240V.
- the lost of power due to Joule effect if the energy is transmitted with alternate current with $24,000$V

Solution:

We have glorified alternate current. We have mentioned that it has revolutionized the way we can transfer electricity to far away from the power plant cities. But, now comes the time to test how good really is the alternate current. So, let's do a simple calculation to show that indeed the alternate current is the winner.

First of all, let's calculate the resistance of the transmission wires. Using equation

5.20, we have that

$$R = 1.71 \times 10^{-8}\Omega \cdot \mathrm{m} \cdot \frac{80000\mathrm{m}}{\left(\pi \cdot (3.5 \times 10^{-2}\mathrm{m})^2\right)} \approx 0.35\Omega \tag{9.98}$$

where we kept just the first two digits after the decimal point. Now, we firstly analyze when we use direct current. So, the electric current is given by

$$I = \frac{P}{\Delta V} = \frac{120 \times 10^3\mathrm{W}}{240\mathrm{V}} = 500\mathrm{A} \tag{9.99}$$

where P is the power with which is supplied the electric current and we used the voltage when we use direct current. Now, the power lost by heat in the wires (Joule effect) is given by

$$P_J = I^2 R = (500\mathrm{A})^2\, 0.35\Omega = 87500\mathrm{W} \tag{9.100}$$

Huge amount of power is lost! Most of the energy is lost due to heat when we use direct current! If we calculate the percent, we can get a better grasp

$$\frac{P_J}{P} \cdot 100 = \frac{87500\mathrm{W}}{120 \times 10^3\mathrm{W}} \approx 73\% \tag{9.101}$$

What?! 73% of the energy is lost by heat in the transmission cables! Direct current is not a good option. But, let's see first what is the story if we use alternate current. The electric current is

$$I = \frac{P}{\Delta V} = \frac{120 \times 10^3\mathrm{W}}{24000\mathrm{V}} = 5\mathrm{A} \tag{9.102}$$

So, the lost power due to joule effect is

$$P_J = (5\mathrm{A})^2\, 0.35\Omega = 8.75\mathrm{W} \tag{9.103}$$

Incredible! When we use alternate current practically the lost of energy is negligible. If we calculate the percent of the power lost when we use alternate current, we can notice that it is practically negligible

$$\frac{P_J}{P} \cdot 100 = \frac{8.75\mathrm{W}}{120 \times 10^3\mathrm{W}} \approx 0\% \tag{9.104}$$

So, practically all the energy is delivered to costumers. Now, we have a better grasp why *Nikola Tesla* and *George Westinghouse* firmly said that alternate current was much better. Thinking as businessman, if we lose much less energy with alternate current, we can use much less thicker wires, which implies less money to build transmission lines to transfer electricity. Less lost in heat, means also less money for maintenance. Cheaper to transmit thousands of power, so much more profit because more costumers have access to use electricity as energy supply. Thinking as engineer, we can build much more impressive and powerful machines. Thinking as scientist, is incredibly beautiful how science revolutionizes our world.

9.5 Induced emf and non-conservative electric fields

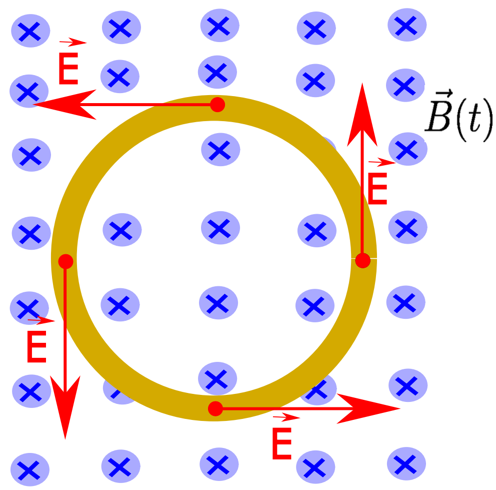


Figure 9.19

We have mentioned that there are two different ways that the induced emf takes place when there is a changing magnetic flux in a closed loop. We have mentioned the first way that this induced emf takes place is due to magnetic force exerted on the electric charges in the wire. However, suppose the following scenario, a fixed (not moving) closed loop immersed in a changing magnetic field as shown in figure 9.19. Due to Faraday's Law there will be an induced emf in the closed loop, and a current will start to flow in the wire. However, what force is moving the charges? There is no magnetic force because the wire is not moving! What is the source of such force that moves the electric charges? So, we come to the conclusion that the changing magnetic field must induce an electric field in the conductor. This electric field would be the responsible of moving the electric charges in the wire, exerting a force on them. We have that the work that the electric field does to move an electric charge in the closed loop is

$$W = \oint \vec{F}_E \cdot d\vec{l} = \oint q\vec{E} \cdot d\vec{l} \tag{9.105}$$

where we just substituted the electric force from the second to third equality. However, we also say that the work is

$$W = q\varepsilon \tag{9.106}$$

Therefore, equating last two equations we obtain that the induced emf is

$$\boxed{\varepsilon = -\frac{d\Phi}{dt} = \oint \vec{E} \cdot d\vec{l}} \tag{9.107}$$

where we have already included Faraday's Law in the equality, and notice that this induced electric field is *non-conservative*. If it were conservative as in the case when a static electric charges generates an electric field, the closed integral is zero.

Now, what is the direction of the electric field? To solve last equation first of all we need to know the direction of the electric field to know what is the dot product inside the closed integral. We can obtain the electric field direction noticing something qualitatively. The following equations correspond to Ampère's Law for magnetic fields and no magnetic mono-poles existence; and the equations when there is an electric field solely due to induced emf by changing in time magnetic field.

$$\begin{aligned} \oint \vec{B} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{A} \qquad &, \qquad \oint \vec{B} \cdot d\vec{A} = 0 \\ \oint \vec{E} \cdot d\vec{l} = \int \left(-\frac{\partial \vec{B}}{\partial t} \right) \cdot d\vec{A} \qquad &, \qquad \oint \vec{E} \cdot d\vec{A} = 0 \end{aligned} \tag{9.108}$$

where in the last equation we used Gauss Law applied to a *non-conservative* induced electric field (there is no electric charge enclosed). Notice the similarity of both set of equations. Any solution that solves the first set of equations for the magnetic field, the solution for the second set of equations for the electric field must be of the same form. The solution that satisfies both first equations is the *Bio-Savart Law* (equation 8.1). So, qualitatively we can use the same set of tricks and tools we have applied to obtain the magnetic field direction with Bio-Savart Law to non-conservative electric fields. The trick is to interchange when applying the right hand rule the direction of the electric current with the direction of the magnetic field and the result of applying the right hand rule interchange it with the direction of the electric field.

$$\begin{aligned} \vec{I} &\Longleftrightarrow \vec{B} \\ \vec{B} &\Longleftrightarrow \vec{E} \end{aligned} \tag{9.109}$$

Example 10: Non conservative electric field in a solenoid due to alternate current

A very long solenoid (you can model it as infinite) of radius R has n loops for each unit of length. It carries an electric current I that varies as

$$I = I_{max} \cos \omega t \tag{9.110}$$

where I_{max} is the maximum electric current and ω is the angular frequency of the source of the alternate current. Determine

- The magnitude of the induced electric field outside the solenoid a distance $r > R$ along its central axis.
- What is the magnitude of the induced electric field inside the solenoid a distance r from its central axis.

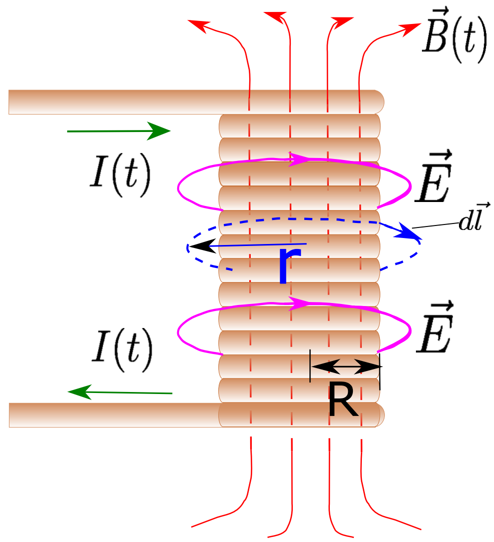


Figure 9.20

Solution:

First of all we need to know what is the direction of the induced electric field. Remember that we can find the non-conservative electric field direction by applying the tricks we use when applying Bio-Savart Law. The trick is as follows

- The changing magnetic field lines assume them as electric currents
- Apply the right hand rule or solve as you were using Bio-Savart. Then whatever direction you obtain with the right hand rule or by using Bio-Savart, instead of being the direction of a magnetic field, that will be direction of the induced electric field!

So, in other words we must use what we mentioned in equation 9.109. If the magnetic field lines were instead electric currents, the magnetic field would be circles around the electric current lines as we have seen when we studied an infinite wire carrying electric current. Therefore, we have that the induced electric field lines will be circles around the magnetic field lines, with constant magnitude at different distances from the central axis (see figure 9.20).

Therefore, using a closed loop as circle of arbitrary radius r, we have that

$$\oint \vec{E} \cdot d\vec{l} = |\vec{E}| \oint dl = |\vec{E}| 2\pi r \tag{9.111}$$

where the electric field could get out of the integral because is constant at the circle of radius r, also we have that $d\vec{l}$ and $\vec{E}$ are parallel (see figure 9.20). The last result holds no matter if the circular closed loop we choose is inside ($r < R$) the solenoid or at the solenoid or outside the solenoid ($r \geq R$).

So, first let's see what is the induced electric field inside the solenoid. The negative of the change of magnetic flux through the area that encloses the closed loop of radius $r < R$ is

$$-\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\int \vec{B} \cdot d\vec{A}\right) = -\frac{d}{dt}\left(\mu_0 n I \pi r^2\right) = \mu_0 n I_{max} \omega \sin\left(\omega t\right) \pi r^2 \tag{9.112}$$

where we used the area of a circle of area r, the magnetic field generated by an infinite solenoid 8.92, and used the function of the electric current given by the exercise (equation 9.110). Finally, by equation 9.107, we equate equation 9.111 and equation 9.112, so

$$|\vec{E}| 2\pi r = \mu_0 n I_{max} \omega \sin\left(\omega t\right) \pi r^2 \Rightarrow |\vec{E}| = \frac{\mu_0 n I_{max} \omega r}{2} \sin\left(\omega t\right) \tag{9.113}$$

At the radius of the solenoid, the induced electric field can be obtained by just substituting $r = R$ in last equation

$$|\vec{E}| = \frac{\mu_0 n I_{max} \omega R}{2} \sin\left(\omega t\right) \qquad \textit{for } r = R \tag{9.114}$$

Now, outside the solenoid, the negative of the change of magnetic flux through the area that encloses the closed loop of radius $r > R$ is

$$-\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\int \vec{B} \cdot d\vec{A}\right) = -\frac{d}{dt}\left(\int_0^R \vec{B} \cdot d\vec{A} + \int_R^r \vec{B} \cdot d\vec{A}\right) \tag{9.115}$$

where the second integral in the third equality will be zero, because there is no magnetic field outside the solenoid ($r > R$). Hence, we have that

$$-\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\mu_0 n I \pi R^2\right) = \mu_0 n I_{max} \omega \sin\left(\omega t\right) \pi R^2 \tag{9.116}$$

where we used the area of a circle of radius R, and the function of the electric current given by the exercise (equation 9.110). So, by equation 9.107, we equate equation 9.111 and equation 9.116, therefore

$$|\vec{E}| 2\pi r = \mu_0 n I_{max} \omega \sin\left(\omega t\right) \pi R^2 \Rightarrow |\vec{E}| = \frac{\mu_0 n I_{max} \omega R^2}{2r} \sin\left(\omega t\right) \tag{9.117}$$

Therefore , we have that the electric field is given by

$$\vec{E}(r,t) = \begin{cases} \frac{1}{2}\mu_0 n I_{max} \omega r \sin\left(\omega t\right) & \text{for } r < R \\ \frac{1}{2r}\mu_0 n I_{max} \omega R^2 \sin\left(\omega t\right) & \text{for } r \geq R \end{cases} \tag{9.118}$$

9.6 Self-Inductance and Energy in Magnetic Fields

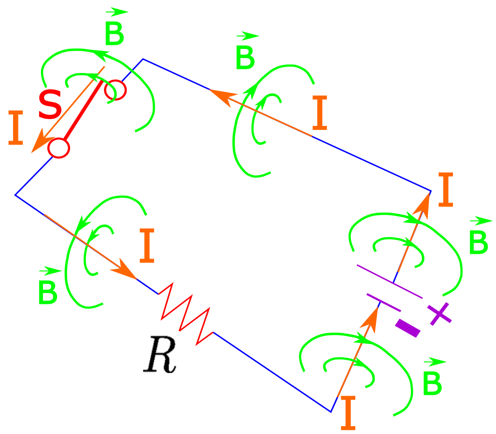


Figure 9.21

Let's analyze the circuit shown in the figure 9.21. When the switch is closed, a current starts to flow. The electric current creates a magnetic field, and since it takes a while until the current becomes constant, there is a varying magnetic flux through the area enclosed by the circuit. Hence, by Faraday's Law, we know that there will be an induced emf trying to oppose the change of magnetic flux, consequently trying to oppose the electric current to vary. The induced emf by the current in the circuit, that is trying to stop itself the change is also called as *back emf*. Now, the magnetic field generated by the electric current is proportional to the electric current in the wire. We can notice this from steady currents by the Bio-Savart Law (equation 8.20)

$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{s} \times \hat{r}}{r^2} \tag{9.119}$$

and given that the magnetic flux through the enclosed area by the loop wire is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \tag{9.120}$$

we can say that the magnetic flux is also proportional to the electric current

$$\Phi_B = LI \tag{9.121}$$

where the constant of proportionality is L, which we call as **self-inductance** or simply **inductance**. Now, the *back emf* is generated when the electric current is changing. In such case, we have that the changing magnetic flux is

$$\frac{d\Phi_B}{dt} = L\frac{dI}{dt} \tag{9.122}$$

Therefore, the back emf in the closed loop is given by

$$\boxed{\varepsilon_L = -L\frac{dI}{dt}} \tag{9.123}$$

where we keep the label L in the back emf to recall for any future calculations that this is the induced emf due to the self inductance.

Now, by Faraday's Law in equation 9.107, we also have that

$$\oint \vec{E} \cdot d\vec{l} = -L\frac{dI}{dt} \tag{9.124}$$

where the closed path will be the circuit itself! So, in figure 9.22 we show the path we will use to integrate $\vec{E}$. Probably you think "*What electric field must be taken into account?*". Well, in figure 9.22 are shown the two electric fields that must be taken into account. One is the electric field that is generated inside the conducting wires where the electric current is flowing. Such current is flowing because there is an electric field inside exerting a force on the electrons. The second electric field is the electric field in the battery. There is an electric field between the positive and negative terminal. For convenience we establish the direction of the $d\vec{l}$ vector as the direction of the current (but the direction of $d\vec{l}$ is irrelevant for the final result, if you take the vector $d\vec{l}$ the other way around we obtain the same result). So, we have that the closed integral is

$$\oint \vec{E} \cdot d\vec{l} = \int \vec{E_R} \cdot d\vec{l} + \int \vec{E_B} \cdot d\vec{l} \tag{9.125}$$

where we labelled the electric field in the resistance as $\vec{E_R}$ and the electric field in the battery as $\vec{E_B}$. Now, the diagram in figure 9.22 shows an electric field in the resistance, however the electric field is all along the wire. The resistance R in the circuit is the total resistance of all the circuit. Only due to drawing, we see an electric field in that section, but actually the resistance R is all along the wire , and the electric field also is all along the wire. Now, using equation 3.10, we have that

$$\int \vec{E_R} \cdot d\vec{l} + \int \vec{E_B} \cdot d\vec{l} = -\Delta V_R + \Delta V = -IR + \Delta V \tag{9.126}$$

where ΔV_R is the potential difference across the resistance, and ΔV is the potential difference of the battery. Therefore, equation 9.124 becomes

$$-IR + \Delta V = -L\frac{dI}{dt} \Longrightarrow \Delta V = IR + L\frac{dI}{dt} \tag{9.127}$$

A quite nice differential equation, that when it is solved it gives the behaviour of the electric current with respect time. In many circuit applications, a device called as **inductor** is widely used. The inductor is just a solenoid, which creates a changing magnetic field inside it, and therefore a back emf. In such applications, the self inductance of the loop is considered negligible and just the back emf generated by the inductor is considered. However, we must be aware that always, does not matter if there is an inductor or not in the circuit, there is always a back emf and a self inductance L. In analogy with resistance, when analyzing circuits generally we do not consider the resistance of the whole conducting wires (as we did now) because they are made of conductors (generally copper) which resistance is practically negligible. However, even though for the calculations is considered as negligible the resistance is there! All materials in the circuits have at least some resistance. Now, the mentioned circuits with inductors and resistors are called *L-R circuits* and the differential equation that models their behaviour is exactly equation 9.127.

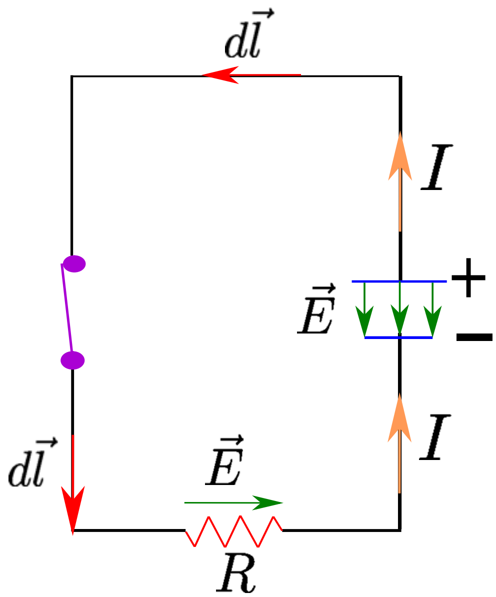


Figure 9.22

Now, the infinitesimal work done by the battery to move an infinitesimal amount of electric charge is

$$dW = I\Delta V dt = I\left(IR + L\frac{dI}{dt}\right)dt = \left(I^2R + LI\frac{dI}{dt}\right)dt \tag{9.128}$$

where we used that $I = \frac{dq}{dt}$ and $W = q\Delta V$. So, we have that

$$P = \frac{dW}{dt} = I^2R + LI\frac{dI}{dt} \tag{9.129}$$

where notice that the first term is the power by joule effect. What is the second term? Something interesting I can assure you. Since we want to analyze the term with the inductance, we do not pay attention to the joule heat term and neglect it. You can think that the total work is the energy dissipated by joule effect plus the other term that we are analyzing, so we are just analyzing the term that contains other information rather than the energy lost due to the resistance in the conductor. So, we have that

$$dW = \left(LI\frac{dI}{dt}\right)dt = \frac{L}{2}\left(\frac{d\left(I^2\right)}{dt}\right)dt = \frac{L}{2}d\left(I^2\right) \tag{9.130}$$

Therefore, integrating both sides of last equation, starting from certain current 0 up to a final current I, we have that the work required to obtain such current

$$W = \frac{L}{2} I^2 = \frac{1}{2} I \Phi_B \tag{9.131}$$

where we used equation 9.121. Notice that the work is related to the magnetic flux. So, this energy must be the one stored in the magnetic field! Associating the energy it was needed to obtain the current I that generates certain magnetic field, then we have that the potential energy associated to the magnetic field is given by

$$\boxed{U = \frac{L}{2} I^2 = \frac{1}{2} I \Phi_B} \tag{9.132}$$

There is a better way to see directly the magnetic fields in the last equations as it was in the case of the electrostatic potential energy in equation 3.67. However, we will have to wait until next chapter to derive the general case. Nevertheless, using a particular case, with an infinite solenoid which magnetic field inside points to $\hat{y}$ let's obtain the same expression meanwhile. Suppose an infinite solenoid with alternating current carried by the loops. We obtained in equation 8.93, that the magnetic field inside an infinite solenoid is given by

$$\vec{B} = \mu_0 n I \hat{y} \tag{9.133}$$

Now, the magnetic flux through one loop of wire is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \mu_0 n I A \tag{9.134}$$

where A is the area enclosed by the solenoid. Now, given that $N = nl$ where N is the number of loops in certain length, and that the flux magnetic flux is trough N loops, we have that the total magnetic flux is

$$\Phi_B = nl \left(\mu_0 n I A \right) = \mu_0 n^2 I V \tag{9.135}$$

where V is volume ($V = lA$). So using equation 9.121, the inductance is

$$L = \frac{\Phi_B}{I} = \mu_0 n^2 V \tag{9.136}$$

And by using 9.131, we have that

$$U = \frac{L}{2} I^2 = \frac{\mu_0 n^2 V}{2} \left(\frac{|\vec{B}|}{\mu_0 n} \right)^2 = \frac{|\vec{B}|^2}{2\mu_0} V \tag{9.137}$$

So, we can say that the magnetic field potential energy density (potential energy per unit volume) is given by

$$\boxed{u = \frac{\vec{B} \cdot \vec{B}}{2\mu_0}} \tag{9.138}$$

and even we found this with an infinite solenoid, the last equation is general. A more formal deduction of the energy stored in fields is given in next chapter.

Chapter 10

Maxwell Equations and introduction to electromagnetic waves

In this chapter we obtain the famous Maxwell Equations, which along with Lorentz force en-globe all the electric and magnetic phenomena. We begin talking about the continuity equation. After we establish Maxwell Equations, and finally we talk about electromagnetic waves, a beautiful implication of Maxwell equations. All along this chapter is assumed that you already manage vector calculus. In previous chapters we have been as careful as possible to not introduce much vector calculus, now is inevitable.

10.1 Continuity Equation

The continuity equation states local conservation of charge and is written as

$$\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}} \tag{10.1}$$

i.e. if certain region of space has certain electric charge Q, and it varies in time, the electric charge that region lost or gain had to go or come from somewhere respectively. To notice why last equation states that, let's rewrite it in integral form . So, integrating both sides of equation 10.1 over the volume that contains the electric charge Q

$$\int_V \left(\nabla \cdot \vec{J}\right) dV = -\int_V \frac{\partial \rho}{\partial t} dV = -\frac{\partial}{\partial t}\left(\int_V \rho dV\right) \tag{10.2}$$

where the label V just means that we are integrating over the volume V that contains the electric charge in all our integrals. And we have that the electric charge in volume V is

$$Q = \int_V \rho dV \tag{10.3}$$

therefore equation 10.2 becomes

$$\int_V \left(\nabla \cdot \vec{J}\right) dV = -\frac{\partial Q}{\partial t} \tag{10.4}$$

If we use the divergence theorem, we can express the left hand side of the last equation in a surface integral, so

$$\boxed{\oint_S \vec{J}\cdot d\vec{A} = -\frac{\partial Q}{\partial t}} \tag{10.5}$$

where surface S encloses the volume V. Something to remark is that equation 10.1 and equation 10.5 are equivalent.

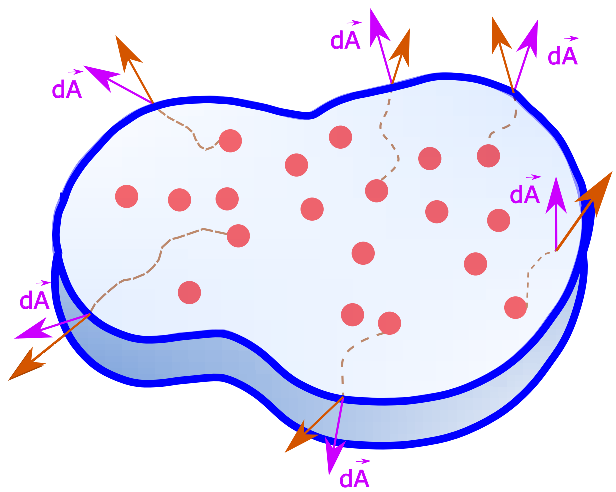


Figure 10.1: Electric charges are in the volume V shown. A surface S, encloses such volume and electric charges. Some electric charges go out the volume through the surface. The differential of area in each point of the surface points outwards the volume by convention.

Probably is easier to notice in the form of equation 10.5 why it establishes the conservation of electric charge. Suppose that in the volume V shown in figure 10.1, electric charges leave the volume. Then, electric currents are generated going outwards the volume. So, we can say that the change of the electric charge in the volume V is due to the currents that go out of the volume. And that is exactly what is established in the left hand side of equation 10.5. Recall that $I = \vec{J}\cdot\vec{A}$. So, in the left hand side of equation 10.5, we are summing all the currents in each differential area of surface S. Now, notice the extreme importance of the minus sign in the right hand side of equation 10.5. If charge leaves the volume V, then

$$\frac{\partial Q}{\partial t} < 0 \Rightarrow -\frac{\partial Q}{\partial t} > 0 \tag{10.6}$$

where the change of electric charge is negative because the electric charge in the volume V decreases. And notice that if the electric charges go out the volume, then for every current

$$dI = \vec{J}\cdot d\vec{A} > 0 \tag{10.7}$$

because the differential area vector by convention always points out of the surface that encloses the volume V, so the dot product must be positive because the vector $\vec{J}$ also points out the volume (of course not necessarily exactly the same direction of each $d\vec{A}$ but both going out the volume V). Hence, the left hand side of equation 10.5 is also positive. So, the negative sign in equation 10.5 gives consistency to the continuity equation.

Now, suppose that electric charges go into the volume V. So, the change of the electric charge in the volume V is

$$\frac{\partial Q}{\partial t} > 0 \tag{10.8}$$

because the electric charge in the volume V increases. Now, if the electric charges go into the volume then

$$dI = \vec{J} \cdot d\vec{A} < 0 \tag{10.9}$$

which means that the electric charges are flowing opposite to the differential vectors $d\vec{A}$. Which makes sense because all the charges are going into the volume! So once again, we have that the change of the electric charge in volume V is due to the currents, this time they are going into the volume. And the minus sign in the right hand side of equation 10.5 gives mathematical consistency.

Finally, of course, there could exist the case when electric charges go out the volume V , and some others go into the volume V. In such case, if globally (i.e. taking into account all electric charges) the electric charge decreases then

$$\oint_S \vec{J} \cdot d\vec{A} > 0 \tag{10.10}$$

i.e. some electric currents will be going out the volume contributing positively to the last integral, some electric charges going into the volume contributing negatively, but at the end taking into account all the contributions in the surface S, the infinitesimal sum (the integral) ends up being positive because there has to be more electric current going out than the electric current going into the volume.

So, in conclusion the continuity equation in 10.1 and equation 10.5 tells us that if electric charge in certain region in space changes, is because it went somewhere, travelling either in or out of the volume in form of electric currents.

10.2 Rewriting the laws

The following four equations

$$\begin{aligned}
&\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \qquad \text{Gauss Law}\\
&\oint_S \vec{B}\cdot d\vec{A} = 0 \qquad \text{No Magnetic Monopoles}\\
&\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \text{Faraday's Law}\\
&\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \qquad \text{Ampère's Law}
\end{aligned} \tag{10.11}$$

along with the Lorentz force

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{10.12}$$

can model all what we have learned so far in the first nine chapters! Amazing isn't it? However, even we have made a tremendous advance, the four last equations have a hidden snake. A profound theoretical inconsistency exists. The brilliant Professor of Cambridge University *James Clerk Maxwell* fixed this inconsistency in a beautiful way and lead him to discover the existence of electromagnetic waves. And even more amazingly that light itself is an electromagnetic wave!

So, before we handle the theoretical problem solved by Maxwell; let's rewrite in differential form the four last equations. Beginning with the Gauss Law, if we apply the divergence theorem to the left-hand side of the equation and write the enclosed charge in the right-hand side of the equation in terms of its charge density, Gauss Law can be written as

$$\int_V \left(\nabla\cdot\vec{E}\right) dV = \frac{1}{\epsilon_0}\int_V \rho dV \Longrightarrow \int_V \left(\nabla\cdot\vec{E} - \frac{\rho}{\epsilon_0}\right) dV = 0 \tag{10.13}$$

where in the second equality we just moved everything to the left-hand side of the equation. Now, since V is any arbitrary volume, then for any volume V the integration always has to be zero, therefore we can state with confidence that

$$\boxed{\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}} \tag{10.14}$$

this is Gauss Law written in differential form and is equivalent to the integral form. It's just two different mathematical forms of expressing the same physics. Depending of the problem to tackle, it is convenient to use one or the other form.

Now, the *no magnetic mono-poles* existence in differential form is written as

$$\boxed{\nabla\cdot\vec{B} = 0} \tag{10.15}$$

where the procedure to obtain it is practically the same we applied to obtain Gauss Law in differential form (apply the divergence theorem to the left-hand side of the equation of no mono-poles in integral form and since is equal to zero the integral for an arbitrary volume, what is integrated must be zero). From the last equation, we can express in another way the magnetic field. We can say that

$$\boxed{\vec{B} = \nabla \times \vec{A}} \tag{10.16}$$

where $\vec{A}$ is called as the **vector potential**. The reason why we can express the magnetic field in this form is that the divergence of a curl of any vector $\vec{C}$ is always zero,

$$\nabla \cdot \left(\nabla \times \vec{C}\right) = 0 \tag{10.17}$$

So, given that equation 10.15 always holds for any magnetic field, nothing stops us to express the magnetic field in terms of certain vector potential. It is analogous to the case of the $potential function$ when we defined it. Recall that in equation 3.3 we said that given that the electrostatic force is conservative (not dependant of the path), the closed integral in any closed path of the electric field is zero, so we could express the electric field in terms of a potential function as

$$\vec{E} = -\nabla V \tag{10.18}$$

Something similar happens in the case of the magnetic field. However, instead of expressing it in terms of a potential function, it is a potential vector. The potential vector $\vec{A}$ is not as useful in practical applications in real life as the potential function V. In practical applications differences of potential function ΔV are widely used to calculate as many things as possible as we have widely seen all along the 10 chapters, due to the scalar nature of the potential function. Nevertheless, the vector potential theoretically by using vector calculus can be highly useful.

Now , for writing Faraday's Law in differential form, we apply Stokes's Theorem to the left-hand side of the equation, and write explicitly the flux Φ_B

$$\int_S \left(\nabla \times \vec{E}\right) \cdot d\vec{A} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right) \Longrightarrow \int_S \left(\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right) = 0 \tag{10.19}$$

where we introduced the derivative into the integration because only the magnetic field is time dependant. Probably, you will be uncomfortable with this because, we discussed that Faraday's Law also included a change in area. Indeed, but recall that in that case the emf is induced due a magnetic force exerted on the charges in a certain wire. So, we keep that in equation 10.12 and the Faraday's Law that we are writing down keeps just the induced emf due to non conservative electric fields. Finally, since the surface S is arbitrary, then the integration must be zero for any surface, therefore the Faraday's Law written in differential form is

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \tag{10.20}$$

Finally, but not least, for writing Ampère's Law in differential form, we apply Stokes's Theorem to the left hand side of the equation , and writing the enclosed current in terms of the current density, we have that

$$\int_S \left(\nabla \times \vec{B}\right) \cdot d\vec{A} = \mu_0 \int_S \vec{J} \cdot d\vec{A} \Longrightarrow \int_S \left(\nabla \times \vec{B} - \mu_0 \vec{J}\right) \cdot d\vec{A} = 0 \tag{10.21}$$

and since the integration is over any arbitrary surface S, then it must hold that for any surface S, the Ampère's Law written in differential form becomes

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{10.22}$$

So, summarizing what we have obtained

$$\begin{aligned} \nabla \cdot \vec{E} =& \frac{\rho}{\epsilon_0} \qquad \text{Gauss Law} \\ \nabla \cdot \vec{B} =& 0 \qquad \text{No Magnetic Monopoles} \\ \nabla \times \vec{E} =& -\frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday's Law} \\ \nabla \times \vec{B} =& \mu_0 \vec{J} \qquad \text{Ampère's Law} \end{aligned} \tag{10.23}$$

Example 1: Energy in Magnetic Fields Revisited

Find the potential energy stored in magnetic fields.
Solution:

In last chapter we found the energy density stored in magnetic fields by using a particular case, the infinite solenoid. However, we aim to derive the potential energy in general in this case. Now we have the enough machinery to achieve the task. We start from equation 9.132, so we have that

$$U = \frac{1}{2} I \Phi_B = \frac{1}{2} I \int \vec{B} \cdot d\vec{S} \tag{10.24}$$

where $d\vec{S}$ is differential of area. I know! We have used all along the 10 chapters $d\vec{A}$ for differential of area. However, unfortunately we have as $\vec{A}$ the vector potential, and I do not want you to get confused. So, just for this case $d\vec{S}$ is used for differential of area. So, expressing the magnetic field in terms of the vector potential (equation 10.16), we have

$$U = \frac{1}{2} I \int \left(\nabla \times \vec{A}\right) \cdot d\vec{S} = \frac{1}{2} I \oint \vec{A} \cdot d\vec{l} \tag{10.25}$$

where in the last equality we used Stokes Theorem. Now, since $d\vec{l}$ has exactly the same direction as the electric current, we can express last equation as

$$U = \frac{1}{2} \oint \left(\vec{A} \cdot \vec{I}\right) dl \tag{10.26}$$

and we can think of the last equation as the particular case for currents, but if there are current densities, in general we can say that

$$U = \frac{1}{2} \int_V \left(\vec{A} \cdot \vec{J} \right) dV \tag{10.27}$$

where volume V must contain the electric current densities. However, if V is greater than that is fine, because in such case for any region in space where there is no electric current then $\vec{J} = 0$ and there is no contribution in the last integral. Now, using Ampère's Law in differential form $\nabla \times \vec{B} = \mu_0 \vec{J}$, we have that last equation becomes

$$U = \frac{1}{2\mu_0} \int_V \vec{A} \cdot \left(\nabla \times \vec{B} \right) dV \tag{10.28}$$

Now, using the relation (not proved)

$$\nabla \cdot \left(\vec{A} \times \vec{B} \right) = \vec{B} \cdot \left(\nabla \times \vec{A} \right) - \vec{A} \cdot \left(\nabla \times \vec{B} \right) = \vec{B} \cdot \vec{B} - \vec{A} \cdot \left(\nabla \times \vec{B} \right) \tag{10.29}$$

where we in the second equality we used equation 10.16. Therefore, we can say that

$$\vec{A} \cdot \left(\nabla \times \vec{B} \right) = \vec{B} \cdot \vec{B} - \nabla \cdot \left(\vec{A} \times \vec{B} \right) \tag{10.30}$$

so substituting last equation into equation 10.28, we have that

$$U = \frac{1}{2\mu_0} \int_V \left(\vec{B} \cdot \vec{B} \right) dV - \int_V \nabla \cdot \left(\vec{A} \times \vec{B} \right) dV \tag{10.31}$$

and if we apply the divergence theorem in the second term of the right-hand side of the last equation, we obtain

$$U = \frac{1}{2\mu_0} \int_V \left(\vec{B} \cdot \vec{B} \right) dV - \oint_S \left(\vec{A} \times \vec{B} \right) \cdot d\vec{A} \tag{10.32}$$

where the surface S bounds the volume V. If we let S go up to infinity covering all space, we have that the second term vanishes because at infinity $\vec{B} \to 0$ and also must be the case for $\vec{A} \to 0$, the contributions in the surface integral are negligible as the surface grows more and more up to infinity. It is not the case for the volume integral, as more volume is covered, the contribution is greater. Of course there comes to certain point when the volume is so large that there is no more contribution in the integral, but as we get bigger the volume greater the contribution of the first integral. Notice that, taking the volume to cover all space is completely fine! Because last equation is just equation 10.27 (we used just some properties and made some algebra to rewrite equation 10.27 to get last equation). So, we have that the energy stored in magnetic fields is

$$U = \frac{1}{2\mu_0} \int_{allspace} \left(\vec{B} \cdot \vec{B} \right) dV \tag{10.33}$$

10.3 A problem with Ampère's Law

In last section we rewrote four equations in a differential format. However, we did nothing than just writing in an equivalent way the four equations in 10.11. In this differential form is easy to identify a problem with the set of equations we have, specifically one of them. If we, apply a divergence operation to both sides of Ampère's Law, we have that

$$\nabla \cdot \left(\nabla \times \vec{B}\right) = \mu_0 \nabla \cdot \vec{J} \tag{10.34}$$

but, the divergence of any rotational is always zero. Therefore, we have that

$$\nabla \cdot \vec{J} = 0 \tag{10.35}$$

however this is not always true! The general conservation of charge equation is the continuity equation (equation 10.1)! So, we have a big problem! Ampère's Law drives to an inconsistency with electric charge conservation! And this shouldn't surprise us. Every time we used Ampère's Law we said that the electric current was constant and stable. For example, suppose a conducting wire transports a stable and constant current. The wire lies in the x axis. So, for such current, it's current density is like

$$\vec{J} = J_x \hat{x} \tag{10.36}$$

where J_x is a constant. Therefore, if we calculate

$$\nabla \cdot \vec{J} = \frac{\partial J_x}{\partial x} = 0 \tag{10.37}$$

of course is zero. So, Ampère's Law was consistent to the cases we used it because the currents were constant. Also, you can think of it as charges where not accumulating in any place of space. So, pick any volume where the current passes as shown in figure 10.2, so given that the current is constant everywhere, the charges that go into the volume, the same amount of charges also go out of the volume! Therefore,

$$\frac{\partial Q}{\partial t} = 0 \tag{10.38}$$

i.e. the charge in the volume does not change. And if the charge does not change, of course also the charge density does not change. So, is not surprise that for stable and constant currents we obtain 10.35.

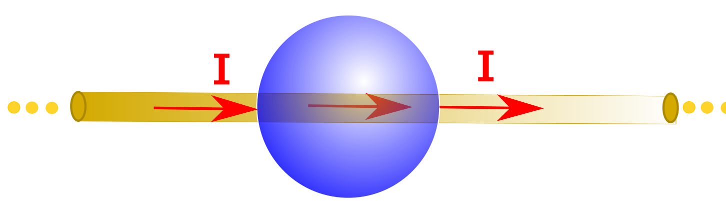


Figure 10.2: An steady electric current. No charges are accumulated in the spherical volume. The same amount of electric charge that goes into the volume, goes out of the volume

The problem with Ampère's Law can also be shown with a simple but beautiful thought experiment (A thought experiment by Maxwell to adresss the problem with Ampère's Law). Suppose a capacitor is charging as shown in figure 10.3. So, if we create an Amperian loop C as shown in figure 10.3, the surface S_1 and S_2 are bounded by the same loop C. And there should not be a problem with using either S_1 and S_2 to determine the enclosed current I_{enc} because we are using the same Amperian loop. For both surfaces S_1 and S_2 we should obtain the same result! However, if we apply Ampère's Law with surface S_1

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 I \tag{10.39}$$

but with surface S_2, we have that

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = 0 \tag{10.40}$$

because there is no enclosed current with such surface!(no current penetrates surface S_2). So, we have an inconsistency! Two different results we obtain using the same Amperian loop. And think about it, of course Ampère's Law shouldn't be able to model this kind of cases, the current is not constant! In the wire conductor is constant, and when the current arrives to one plate of the capacitor, the charge starts to pile up, and in the gap between the plates there is no current! These kind of cases is not possible to model with just Ampère's Law So, Ampère's Law must be modified.

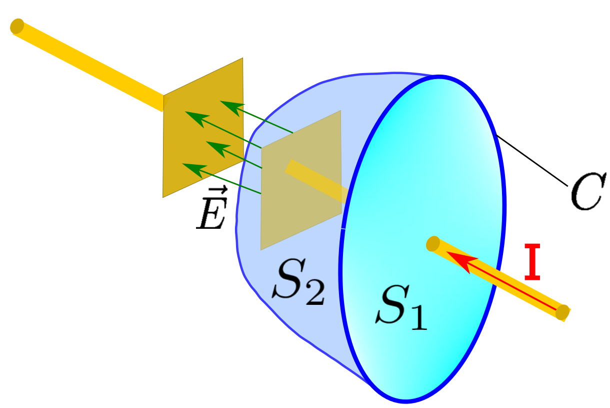


Figure 10.3

10.4 Maxwell Equations

The problem just mentioned in the previous section was solved by Maxwell by adding a term to Ampère's Law; and the new equation became

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}} \tag{10.41}$$

where the second term is called as *displacement current* and the equation as **Ampère-Maxwell Law**. The name of *displacement current* could be quite misleading. It is not a current. The units match as the same of a current (Amperes(A)). But it is a changing electric field flux through the same Amperian loop we use to calculate the enclosed current. The second term relates a varying electric field with a magnetic field! A way to interpret the last equation is ***the source of magnetic fields are electric currents and varying electric fields!*** If we want to write the Ampère's-Maxwell equation in differential form, we apply stokes and the divergence theorem, so we have

$$\int_S \left(\nabla \times \vec{B}\right) \cdot d\vec{A} = \mu_0 \int_S \vec{J} \cdot d\vec{A} + \mu_0 \epsilon_0 \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A} \tag{10.42}$$

and since all the integrals apply for any arbitrary surface S, then, it must be true that

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}} \tag{10.43}$$

which is equivalent to equation 10.41.

So, finally the **Maxwell Equations** are

$$\boxed{\begin{array}{lll} \oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} & \text{Gauss Law} & \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\ \oint_S \vec{B} \cdot d\vec{A} = 0 & \text{No Magnetic Monopoles} & \nabla \cdot \vec{B} = 0 \\ \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} & \text{Faraday's Law} & \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} & \text{Ampère-Maxwell Law} & \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{array}} \tag{10.44}$$

where the integral and differential form are shown. These four equations along with Lorentz force can model all the electromagnetic phenomena in vacuum(absence of dielectric and magnetic materials)! Amazing, how powerful a set of four equations can be.

Now, let's show that the theoretical issues mentioned in last section are now completely solved. If we apply the divergence operator to both sides of the Ampère-Maxwell Law, we have

$$\nabla \cdot \left(\nabla \times \vec{B}\right) = \nabla \cdot \left(\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \Longrightarrow 0 = \mu_0 \nabla \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\nabla \cdot \vec{E}\right) \tag{10.45}$$

where we used that the divergence of any curl of a vector is always zero. Also, we used that the divergence operator and the partial derivative with respect time commute. Now, if we use Gauss Law in differential form, we obtain

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{10.46}$$

which is exactly the continuity equation! So, we have recovered the charge conservation law! Nice! What about the thought experiment with the capacitor in figure 10.3 ? Is it also solved?

So, we will use the Ampère-Maxwell Law in integral form. We use the Amperian Loop C shown in figure 10.3, and we will prove that either surface S_1 or S_2 lead to the same result.

For the surface S_1 in figure 10.3, we have that

$$I_{enc} = I \tag{10.47}$$

and the electric flux in such surface is

$$\Phi_E = 0 \tag{10.48}$$

because there is no electric field lines crossing surface S_1 (no electric flux through that surface). Hence, using surface S_1, the Ampère-Maxwell Law reads

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \tag{10.49}$$

Now, with the surface S_2 in figure 10.3, we have that

$$I_{enc} = 0 \tag{10.50}$$

because no electric current penetrates such surface. However, there is an electric flux through surface S_2. There is an electric field between the plates of the capacitor. So, recalling the magnitude of the electric field generated between the plates of a capacitor

$$|\vec{E}| = \frac{\sigma}{\epsilon_0} = \frac{Q(t)}{A\epsilon_0} \tag{10.51}$$

where we wrote the electric charge as function of time, because as time passes more and more electric charge is pilled up in the plates of the capacitor. Therefore, we have that the electric flux through the surface S_2 in figure 10.3 is

$$\Phi_E = \int \vec{E} \cdot d\vec{A} = \frac{Q(t)}{\varepsilon} \tag{10.52}$$

where we used that the area where there is electric flux is the area of the capacitor. We can do this if the surface S_2 is as close as possible to the plate of the capacitor (this is just to simplify the calculations, you could take any other surface S_2 that is between the plates and you would obtain exactly the same result). Therefore, the change with respect time of the electric flux is

$$\frac{d\Phi_E}{dt} = \frac{I}{\epsilon_0} \tag{10.53}$$

where we used that $I = \frac{dQ}{dt}$. So, using surface S_2, the Ampère-Maxwell Law reads as

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \tag{10.54}$$

exactly the same as when we used surface S_1! So, the experimental thought paradox is also solved!

10.5 Electromagnetic Waves

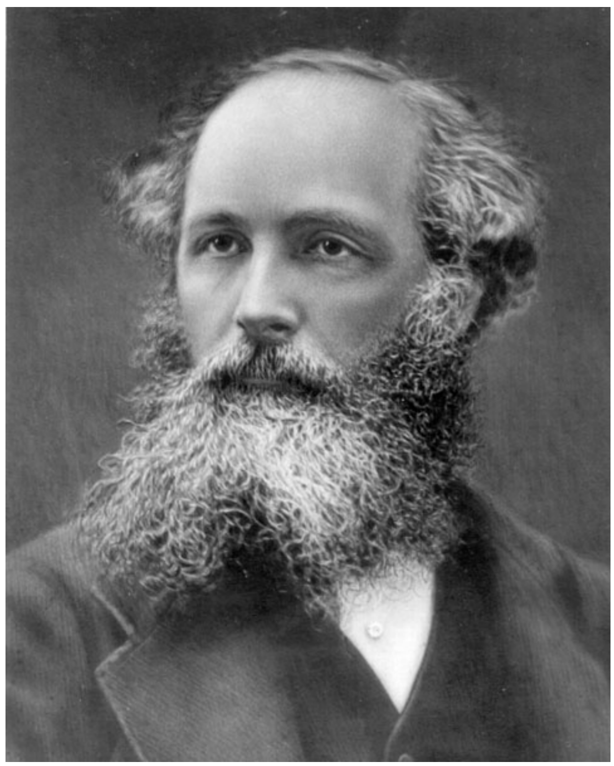


Figure 10.4: James Clerk Maxwell. Original picture from reference [25]

"The velocity of transverse undulations in our hypothetical medium, calculated from the electro-magnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena"

— James Clerk Maxwell

Now that we have Maxwell Equations, a beautiful result rises from them. Suppose we are in complete vacuum, so there are no electric charges, therefore $\vec{J} = 0$ and $\rho = 0$. Using the differential form, let's apply a curl to both sides of *Ampère-Maxwell* equation, so we have that

$$\nabla \times \left(\nabla \times \vec{B}\right) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \tag{10.55}$$

for the left-hand side of equation 10.55, we have that

$$\nabla \times \left(\nabla \times \vec{B}\right) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = -\nabla^2 \vec{B} \tag{10.56}$$

where we used the no mono-poles existence from Maxwell Equations. Now, we have that

$$\nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\nabla \times \vec{E}\right) = -\mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial \vec{B}}{\partial t}\right) \tag{10.57}$$

where in the second equality we used the fact that the curl operator ∇ and the partial derivative $\frac{\partial}{\partial t}$ can commute. The reason is that crossed derivatives with respect time and the coordinates x, y and z for any component of the electric field commute

$$\frac{\partial^2 E_x}{\partial t \partial x} = \frac{\partial^2 E_x}{\partial x \partial t}, \qquad \frac{\partial^2 E_x}{\partial t \partial y} = \frac{\partial^2 E_x}{\partial y \partial t}, \qquad \frac{\partial^2 E_x}{\partial t \partial z} = \frac{\partial^2 E_x}{\partial z \partial t} \tag{10.58}$$

where explicitly was written the x component of the electric field, but this also applies to the y and z component. So, it follows that if we derive firstly with respect time and then apply the curl operator(which has the position derivatives), we obtain exactly the same if we firstly calculate the curl and then apply the partial derivative with respect time. In the last step in equation 10.57 we used Faraday's Law. Therefore, equation 10.55 becomes

$$\boxed{\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{B}}{\partial t^2}} \tag{10.59}$$

a wave equation for $\vec{B}$! Recall from your waves course, the three dimensional wave equation is given by

$$\nabla^2 \vec{A} = \frac{1}{v^2} \frac{\partial \vec{A}}{\partial t^2} \tag{10.60}$$

for any vector field $\vec{A}$ where v is the speed of propagation of such wave. And, something similar happens with the electric field when we apply a curl to Faraday's Law,

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla \times \left(\frac{\partial \vec{B}}{\partial t}\right) \Rightarrow \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{B}\right) \tag{10.61}$$

where we used for the left-hand side

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E} \tag{10.62}$$

where also we used $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ according to Gauss Law in differential form. However, we are in completely empty space (no charges) so $\rho = 0$. Finally, if we substitute *Ampère-Maxwell* equation in 10.61 (taking into account that we are in completely empty space so $\vec{J} = 0$), we have that

$$\boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}} \tag{10.63}$$

so we have another wave equation for the electric field with the same speed! We call to these as *electromagnetic waves*! And they propagate with a very specific speed. Comparing the general wave equation 10.60 with what we have found in equation 10.59 and equation 10.63, we have that the speed of propagation of electromagnetic waves is

$$c = \sqrt{\frac{1}{\mu_0 \epsilon_0}} \tag{10.64}$$

and if we plug in the values,

$$c = \sqrt{\frac{1}{(4\pi \times 10^{-7}\mathrm{N/A}^2)(8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N \cdot m}^2})}} = 299863380.5 \frac{\mathrm{m}}{\mathrm{s}} \approx 3 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}} \tag{10.65}$$

which is the speed of light! We generally round the number to $3 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}$. It was the speed of electromagnetic waves so identical to the experimental evidence of the speed of light so far in the times of Maxwell that he mentioned the quote stated at the beginning of this section. So, is light an electromagnetic wave? Years later, **Heinrich Rudolf Hertz** produced with a smart experiment electromagnetic waves! And, it was in 1887, when **Hertz** found experimentally that the speed of propagation of electromagnetic waves was exactly the speed of light! Also, he was able to show that the electromagnetic waves can be reflected, refracted and diffracted! He showed experimentally that Maxwell Equations prediction that light is an electromagnetic wave was indeed correct! This is amazing! Light is an electromagnetic wave! However, many theoretical questions rise. The speed of light is respect to what? i.e. when you move respective to certain reference frame is that we say that you have certain speed. For example, when you are inside your car, and the car is moving, respective to the car you are at rest so your speed is $0\frac{\mathrm{m}}{\mathrm{s}}$. However, if an observer standing at the ground sees you inside the car will say that your speed is the speed of the car! So, the speed we have obtained is respect to what? Many physicists supported the idea that something covered all the universe, called as the *ether*. Also, notice that when we study waves as waves in a rod, or sound waves there is a medium

Figure 10.5: Heinrich Rudolf Hertz. Original picture taken from reference [26].

where the wave propagates. So, also many physicists said that the electromagnetic waves medium of propagation was the ether. However, many experiments were done to try to prove the existence of ether, but all of them failed. Years later, another giant of physics came to the rescue, Albert Einstein. He postulated his famous theory of *special relativity* where there was no necessity of the ether to explain the speed of light. The speed of light is the maximum speed that any object in the Universe can be travelling, and is the same in all inertial reference frames! The implications of special relativity are tremendous! Space and time are not absolute! The time an observer experiences is not the same as another observer if the move one relative to the other with speeds near the speed of light!(Actually with any speed, but higher the speed approaching the speed of light the relativistic effects are more and more notable).

Example 2: A monochromatic plane wave

Show that

$$\vec{E} = E_{max}\cos\left(kz - \omega t\right)\hat{x} \tag{10.66}$$

solves the electric field wave equation. Then find the direction of propagation of the magnetic field and the direction of oscillation.

Solution:

We need to show that the given function in equation 10.66 satisfies equation 10.63. So, we have that

$$\nabla^2\vec{E} = \left(\frac{\partial^2\left(E_{max}\cos\left(kz - \omega t\right)\right)}{\partial z^2}\right)\hat{x} = -E_{max}k^2\cos\left(kz - \omega t\right)\hat{x} \tag{10.67}$$

where the partial derivatives respect to x and y were not written because the electric field does not depend on those component variables. On the other hand, we have that

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E_{max}\cos\left(kz - \omega t\right)\right)\hat{x} = -\frac{\omega^2}{c^2}E_{max}\cos\left(kz - \omega t\right)\hat{x} \tag{10.68}$$

And we have that,

$$\frac{\omega^2}{c^2} = \left(\frac{2\pi/T}{\lambda/T}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2 = k^2 \tag{10.69}$$

where we just used the definition of the angular frequency ω, the speed of propagation (in this case the speed of light) and the wave number k. Therefore, we have that indeed

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} \tag{10.70}$$

so the wave equation for the electric field is satisfied. Now, if we are interested to know the direction and function of the magnetic field, we can apply Faraday's Law in differential form. The curl of the electric field in general is

$$\nabla\times\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z} \tag{10.71}$$

where E_x, E_y and E_z are the x,y and z components of the electric field. In this particular case, the electric field just has x component. It propagates in the z direction, but the electric field oscillates in the x component. The only component of the electric field is just dependant of z, so all other derivatives are zero. Therefore, we have that the curl is

$$\nabla \times \vec{E} = \frac{\partial E_x}{\partial z}\hat{y} = \frac{\partial \left(E_{max}\cos\left(kz - \omega t\right)\right)}{\partial z}\hat{y} = -E_{max}k\sin\left(kz - \omega t\right)\hat{y} \tag{10.72}$$

Now, using Faraday's Law

$$-E_{max}k\sin\left(kz - \omega t\right)\hat{y} = -\frac{\partial \vec{B}}{\partial t} \tag{10.73}$$

and integrating then with respect time we have that

$$\vec{B} = E_{max}\frac{k}{\omega}\cos\left(kz - \omega t\right)\hat{y} = \frac{E_{max}}{c}\cos\left(kz - \omega t\right)\hat{y} \tag{10.74}$$

where we used $k/\omega = 1/c$, which can be deduced from equation 10.69. So, notice that the electric field and magnetic field are in phase (when the electric field is maximum also is the magnetic field, when the electric field is zero also is the magnetic field) but oscillate in perpendicular directions. If we plot the electromagnetic wave at certain instant (because the wave is moving!), we obtain what is shown in figure 10.6.

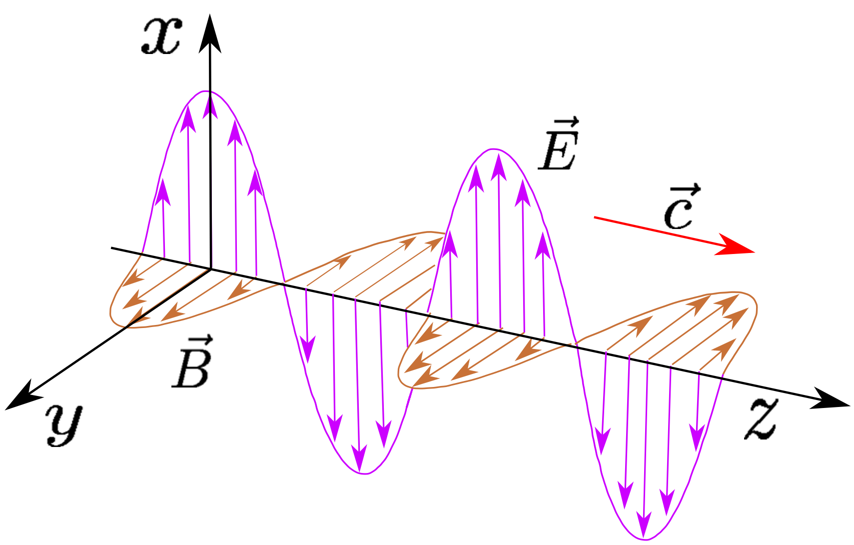


Figure 10.6: Electromagnetic wave. Be careful about this representation. This represents the electromagnetic wave at certain time because electromagnetic waves move at the speed of light! Also, this representation along the z is just a subset representation of all the electromagnetic wave. There are electric and magnetic fields in all space oscillating equally at different planes as shown in figure 10.8

Now, why the name of *monochromatic* and also *plane* wave? The name monochromatic is due to the fact that the electromagnetic wave that we analyzed has only one frequency.

Electromagnetic waves with different frequencies correspond to different colors in the visible spectrum. So, we call them as monochromatic due to their only one value frequency dependence. In table 10.1 are shown the different approximate ranges of wavelength and frequencies of different colors. In figure 10.7 are shown the different ranges of wavelength and frequencies of electromagnetic waves.

Visible Light Spectrum		
Wave Length (m)	Color	Frequency (Hz)
400nm to 440nm	Violet	6.8×10^{14} to 7.5×10^{14}
440nm to 480nm	Blue	6.25×10^{14} to 6.8×10^{14}
480nm to 560nm	Green	5.36×10^{14} to 6.25×10^{14}
560nm to 590nm	Yellow	5.08×10^{14} to 5.36×10^{14}
590nm to 630nm	Orange	4.76×10^{14} to 5.08×10^{14}
630nm to 700nm	Red	4.29×10^{14} to 4.76×10^{14}

Table 10.1: Approximate ranges of frequencies and wavelengths of visible light.

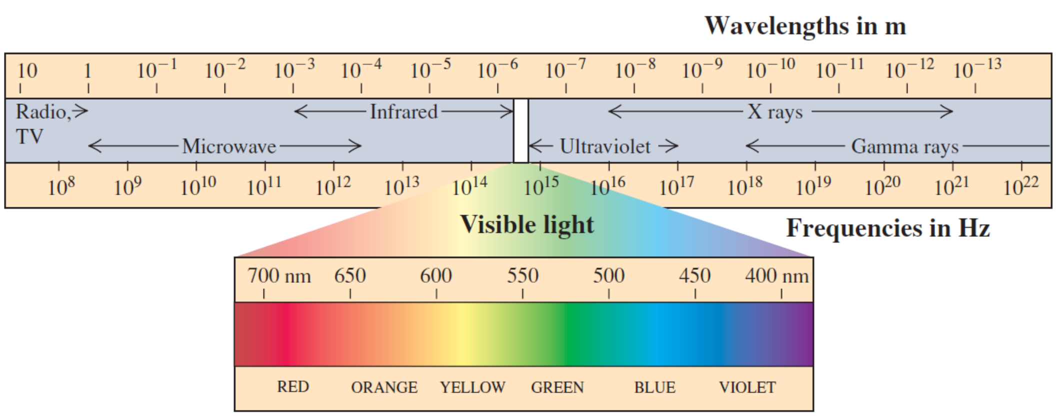


Figure 10.7: Original figure taken from [7]

Now, why plane wave? Notice that the electric and magnetic field only depend on the z component and time t. So, for any z fixed value, all points in such plane have exactly the same magnitude and direction of the electric field. Probably you think, I see explicitly just the dependence on time t, z and $\hat{x}$ in the electric field. The magic word is *field*. Recall a field is a function that designates to every single point in space and time a vector. In this case which vector? The one given at equation 10.66. So, if you pick the points with coordinates $(1, 1, z_0)$ and $(0, 0, z_0)$ for any fixed value z_0 at certain time t', at both points there is a vector, which direction is towards $\hat{x}$ with magnitude

$$E_{max} \cos (kz_0 - \omega t') \tag{10.75}$$

and for any random point with coordinates (x, y, z_0) will have the same vector direction with magnitude 10.75. So, for each value of z, there is a perpendicular plane with a set of vectors all oscillating with the same pattern. Each of those planes are called as *wave-fronts*. Finally, we say that the wave is *linearly polarized* because the electric field and magnetic field vary strictly in one direction. It is a convention to mention the direction of the electric field polarization. So, in this case the electromagnetic wave is in the x direction *polarized*.

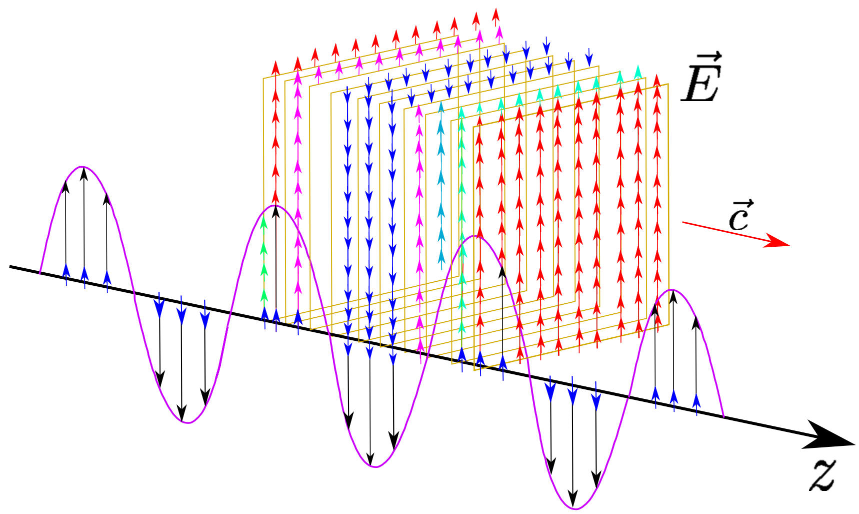


Figure 10.8: Wave-fronts of a plane electromagnetic wave. In every plane, all points have exactly the same direction and magnitude of electric field. Also this happens to the magnetic field, but is not shown in this picture (only the electric field is plotted). This is a more realistic representation of a plane wave than figure 10.6 because there are electric and magnetic fields in each point in space. The figure 10.6 represents the plane wave only along the z axis. The wave fronts move with the speed of light. So, as in figure 10.6 this representation shows certain instant of time, and we show explicitly the direction of the velocity of the electromagnetic wave.

10.6 Poynting Theorem

The total potential energy in electric and magnetic fields is

$$U = \int_{allspace} \left(\frac{\epsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{1}{2\mu_0} \vec{B} \cdot \vec{B} \right) dV \tag{10.76}$$

where we summed the potential energy stored in electric fields (equation 3.67) and magnetic fields (equation 10.33) respectively. Now, we are interested to know how the energy due to magnetic and electric fields in certain volume in space changes and if it is the case, what is the energy that escapes from the volume. Therefore, instead of integrating over all space, we take the integration over a volume V and calculate how it changes with respect time,

$$\frac{dU}{dt} = \frac{d}{dt}\left(\int_V \left(\frac{\epsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}\right)dV\right) = \int_V \left(\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \frac{1}{\mu_0}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right)dV \quad (10.77)$$

Now, from Ampère-Maxwell equation we have that

$$\nabla\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \Longrightarrow \frac{\partial\vec{E}}{\partial t} = \frac{1}{\epsilon_0\mu_0}\nabla\times\vec{B} - \frac{1}{\epsilon_0}\vec{J} \quad (10.78)$$

and from Faraday's Law,

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E} \quad (10.79)$$

Therefore, equation 10.77 becomes

$$\frac{dU}{dt} = \int_V \left(\epsilon_0\vec{E}\cdot\left(\frac{1}{\epsilon_0\mu_0}\nabla\times\vec{B} - \frac{1}{\epsilon_0}\vec{J}\right) - \frac{1}{\mu_0}\vec{B}\cdot\left(\nabla\times\vec{E}\right)\right)dV \quad (10.80)$$

Now, we have the following relation of the divergence of a cross product of two vectors (this holds for any two vectors, but we already use the electric and magnetic field),

$$-\nabla\cdot\left(\vec{E}\times\vec{B}\right) = \vec{E}\cdot\left(\nabla\times\vec{B}\right) - \vec{B}\cdot\left(\nabla\times\vec{E}\right) \quad (10.81)$$

Therefore, using the last relation into 10.80, we have that

$$\frac{dU}{dt} = -\frac{1}{\mu_0}\int_V \nabla\cdot\left(\vec{E}\times\vec{B}\right)dV - \int_V \left(\vec{E}\cdot\vec{J}\right)dV \quad (10.82)$$

and if we apply the divergence theorem on the first term of the right-hand side of last equation, we have that

$$\boxed{\frac{dU}{dt} = -\frac{1}{\mu_0}\oint_S \left(\vec{E}\times\vec{B}\right)\cdot d\vec{A} - \int_V \left(\vec{E}\cdot\vec{J}\right)dV} \quad (10.83)$$

where the last equation is called as the *Poynting Theorem*. The first term in the right-hand side of last equation is telling us how energy is flowing out across the boundary (surface S) of volume V, while the second term is telling us the energy lost by the fields due to the work done on electric charges in volume V. The vector that is integrated in the first term in the right-hand side of last equation is called as *Poynting vector*, defined as

$$\boxed{\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)} \quad (10.84)$$

which has units $\frac{\mathrm{J}}{\mathrm{sm}^2} = \frac{\mathrm{W}}{\mathrm{m}^2}$ (amount of energy per unit time per unit area). The direction of the poynting vector $\vec{S}$ is the direction of propagation of an electromagnetic wave. We have that

$$\vec{S}\cdot d\vec{A} \quad (10.85)$$

is the energy per unit time crossing the infinitesimal surface $d\vec{A}$ (the power passing through the infinitesimal surface of area $d\vec{A}$). So, the poynting vector $\vec{S}$ is the energy flux density.

Therefore , the equation 10.83 makes sense, any change of potential energy associated to the electric and magnetic fields in volume V is either because the fields have done work on the charges in volume V or because energy flowed out of the volume though the surface S. How can this energy have flowed out? Well, one way is electromagnetic waves. The electromagnetic waves carry energy, and if they cross out the boundary S then that energy is not any more in the fields in volume V. Also, probably, you think how do we know that the first term in the equation 10.83 actually means energy that flowed out of volume V through the boundary surface S? By just arranging the equation 10.83, we can easily see the interpretation,

$$\frac{dU}{dt} + \int_V \left(\vec{E} \cdot \vec{J}\right) dV = -\frac{1}{\mu_0} \oint_S \left(\vec{E} \times \vec{B}\right) \cdot d\vec{A} \tag{10.86}$$

so the left-hand side of last equation is the total change in energy in volume V, due to both fields and electric charges. Given that energy must be conserved, the right-hand side of last equation must describe the energy that escapes through the the boundary of volume V (through surface S).

Finally, how do we know that the term

$$-\int_V \left(\vec{E} \cdot \vec{J}\right) dV \tag{10.87}$$

is the energy lost by the fields due to the work done on the electric charges? Let's prove it. In general, inside volume V can be any distribution of electric charges. The infinitesimal force exerted on an infinitesimal amount of electric charge is given by

$$d\vec{F} = dq \left(\vec{E} + \vec{v} \times B\right) \tag{10.88}$$

where we used Lorentz force but using the fact that the force is applied in an infinitesimal amount of electric charge. If we divide by an infinitesimal amount of volume

$$\vec{f} = \rho \left(\vec{E} + \vec{v} \times B\right) \tag{10.89}$$

where $\vec{f}$ is the force density, the exerted force on electric charges per unit of volume and the charge density $\rho = \frac{dq}{dV}$. Now, if we multiply by the velocity of the each electric charge in every infinitesimal amount of volume

$$\vec{f} \cdot \vec{v} = \rho \left(\vec{E} + \vec{v} \times B\right) \cdot \vec{v} = \rho \vec{E} \cdot \vec{v} = \vec{E} \cdot \vec{J} \tag{10.90}$$

where the term $(\vec{v} \times B) \cdot \vec{v} = 0$ because the cross product of the velocity and the magnetic field is perpendicular to both $\vec{B}$ and $\vec{v}$, so the dot product $(\vec{v} \times B) \cdot \vec{v}$ vanishes. Also, we used that $\rho\vec{v} = \vec{J}$ (equation 5.10). So, if we integrate over volume V, we have that

$$\int_V \left(\vec{f} \cdot \vec{v}\right) dV = \int_V \left(\vec{E} \cdot \vec{J}\right) dV \tag{10.91}$$

Given that $\vec{f}dV = d\vec{F}$, then $\left(\vec{f}\cdot\vec{v}\right)dV = d\vec{F}\cdot\vec{v} = dP$ where dP is a differential of power. Recall from your mechanics courses that power is $P = \vec{F}\cdot\vec{v}$. So, the integral in the left-hand side of the last equation is the total power. And, also recall that power is the amount of energy transferred per time $P = \frac{dW}{dt}$. Therefore we have obtained,

$$\frac{dW}{dt} = \int_V \left(\vec{E}\cdot\vec{J}\right)dV \tag{10.92}$$

but the last equation is the amount of energy that is transferred to the electric charges given that a force is exerted on them. So, the amount of energy lost by the fields is just the same magnitude, but opposite sign. So we have shown that indeed the term 10.87 is the amount of energy lost by the fields due to the work done on the electric charges in volume V.

Example 3: Energy carried by a monochromatic electromagnetic wave

The magnetic and electric fields of an electromagnetic wave are given by

$$\vec{E} = E_{max}\cos\left(kz-\omega t\right)\hat{x} \qquad\qquad \vec{B} = \frac{E_{max}}{c}\cos\left(kz-\omega t\right)\hat{y} \tag{10.93}$$

- Calculate the poynting vector
- The average of the poynting vector (the intensity of the electromagnetic wave)

Solution:

The poynting vector is

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{E_{max}^2}{c\mu_0}\cos^2\left(kz-\omega t\right)\hat{z} \tag{10.94}$$

Now, since the frequency of oscillation could be considerably high for an electromagnetic wave, instead of being interested in the amount of energy per unit of area per unit of time, we want its average over a complete cycle or period. Therefore, the magnitude of the average of the poynting vector over one period is

$$\left\langle\vec{S}\right\rangle = \frac{1}{T}\int_0^T \frac{E_{max}^2}{c\mu_0}\cos^2\left(kz-\omega t\right)dt\hat{z} = \frac{E_{max}^2}{c\mu_0}\left(\frac{1}{2}+\frac{1}{T}\int_0^T\cos\left(2kz-2\omega t\right)dt\right)\hat{z} \tag{10.95}$$

where we used a trigonometric identity $\cos^2(z) = \frac{1+\cos(2z)}{2}$. Now, let's do the simple change of variables $u = 2\left(kz-\omega t\right) \Longrightarrow du = -2\omega dt$. Therefore, the integral in equation 10.95 becomes

$$-\frac{1}{2\omega}\int_{2kz}^{2kz-2\omega T} cos\left(u\right)du = -\frac{1}{2\omega}\left(\sin\left(2kz-2\omega T\right)-\sin\left(2kz\right)\right) = 0 \tag{10.96}$$

and it vanished given that the sine function is cyclic; due to $2\omega T = 2\left(\frac{2\pi}{T}\right)T = 4\pi$, and $\sin(2kz) = \sin(2kz - 4\pi)$. Therefore, the average of the poynting vector is

$$\left\langle \vec{S} \right\rangle = \frac{E_{max}^2}{2c\mu_0}\hat{z} \tag{10.97}$$

and it must be thought as the average of energy carried by the electromagnetic waves per unit time per unit of area. The magnitude of the average is also called as **intensity**. Unfortunately, mostly represented with the letter I. A disadvantage given that we already had I for electric currents. Be aware of the context, units and you will know if a parameter with I is either intensity or electric current.

Example 4: Analyzing an electromagnetic wave

A sinusoidal electromagnetic wave with frequency of 40MHz travels in empty space towards $+z$ direction as shown in the figure.

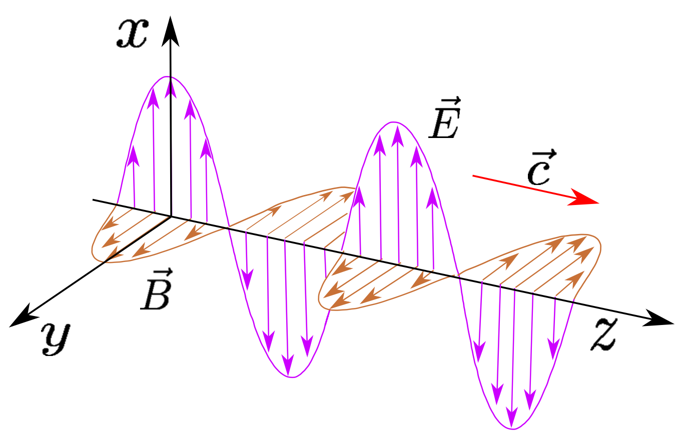


Figure 10.9

- Determine the wavelength λ and the period of the wave T.
- Determine the magnitude and direction of the magnetic field when the maximum value of the electric field is 750N/C
- What is the average energy for unit of time that crosses (power) an area of $1\text{m} \times \text{m}$ perpendicular to the propagation direction of the electromagnetic wave.
- What would be the direction of the magnetic field if the electric field were towards $-y$ in certain instant of time instead and the direction of propagation of the electromagnetic wave is exactly the same.

Solution:

We can easily calculate the period

$$T = \frac{1}{f} = \frac{1}{40 \times 10^6 \text{Htz}} = 2.5 \times 10^{-8}\text{s} \tag{10.98}$$

In order to calculate the wavelength

$$\lambda = \left(3 \times 10^8 \text{m/s}\right)\left(2.5 \times 10^{-8}\text{s}\right) = 7.5\text{m} \tag{10.99}$$

Now, from equation 10.74 we can obtain the magnitude of the magnetic field. Given that the electric field and the magnetic field are in phase, then when the electric field is maximum also is the magnetic field. So, we can just take the maximum value of the function in equation 10.74, therefore

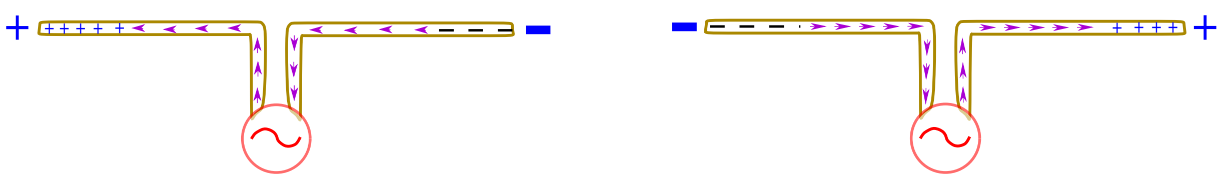
$$|\vec{B}|_{max} = \frac{E_{max}}{c} = \frac{750\text{N/C}}{3 \times 10^8 \text{m/s}} = 2.5 \times 10^{-6}\text{T} \tag{10.100}$$

Now, we calculate the average energy for unit of time that crosses (power) an area of $1\text{m} \times \text{m}$ perpendicular to the propagation direction of the electromagnetic wave

$$\langle P \rangle = \int \left\langle \vec{S} \right\rangle \cdot d\vec{A} = \frac{E_{max}^2}{c\mu_0} A = \frac{(750\text{N/C})^2}{(3 \times 10^8 \text{m/s})\,(4\pi \times 10^{-7}\text{N/A}^2)} \left(1\text{m}^2\right) = 1492.07\text{W} \tag{10.101}$$

Finally, notice that the propagation direction is the same as the pointing vector, because the electric and magnetic fields are transverse waves (both fields are perpendicular to the propagation of the electromagnetic wave). So, using the right hand rule the only way that the poynting vector direction is towards $+z$ given that the electric field points to $-y$ is that the magnetic field direction is towards $+x$.

Example 5: Dipole Antenna Radiation



(a) (b)

Figure 10.10: An alternating current supply is connected to two rods. Given that the polarity of the voltage changes, charges are accelerated towards one of the rods in different instants, creating electromagnetic waves as shown in figure 10.11

Every time an electric charge accelerates, it emits energy by electromagnetic radiation. So, if we have electric charges in a wire changing constantly their direction, they are are constantly accelerated, therefore they generate electromagnetic waves! This beautiful fact is used for antennas! An antenna can either generate or receive electromagnetic waves to transport information! A simple antenna is as the one shown in figures 10.10a and 10.10b. Two conductor wires are connected to an alternating current source. As shown in the figures, at certain instants of time one wire is positively charged and one is negatively charged. The electrons pile up in one of the rods, while the other is left positively charged. So, notice that given that we have two opposed electrically charged wires, we can model them as a dipole. For that reason we call these antennas as *dipole antennas*. The electromagnetic radiation pattern due to the dipole antenna is as the one shown in figure 10.11. The electric and magnetic field are always perpendicular one to each other. For long distances from the dipole antenna (also called as the *radiation zone*) the electric field generated by a dipole antenna (modeled as an electric dipole) is given by

$$\vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left(\omega\left(t - r/c\right)\right)\hat{\theta} \tag{10.102}$$

where p_0 is the dipole moment and the angle θ is measured from the axis of the dipole.

- Find the magnetic field
- Calculate the Poynting vector
- Calculate the intensity of the electromagnetic wave
- Calculate the mean power emitted by the antenna

Solution:

In order to find the magnetic field, we can use *Faraday's Law*. However, we have been using Cartesian coordinates, and in this particular case we are given the electric field in spherical coordinates. So, the curl of any electric field in general written in spherical coordinates is

$$\nabla\times\vec{E} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right) - \frac{\partial E_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rE_\phi\right)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\theta\right) - \frac{\partial E_r}{\partial\theta}\right]\hat{\phi} \tag{10.103}$$

where E_r, E_θ, E_ϕ are the components of the electric field in spherical coordinates. However, the electric field generated in the radiation zone by the antenna has only θ component and has no dependence on ϕ as can easily be seen in equation 10.102. Therefore, we have that

$$E_r = 0 \qquad , E_\phi = 0 \qquad , \frac{\partial E_\theta}{\partial\phi} = 0 \tag{10.104}$$

Hence, the only term in the curl of the electric field that survives is

$$\nabla\times\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}\left(rE_\theta\right)\hat{\phi} \tag{10.105}$$

So, by plugging the θ component of the electric field in equation 10.102 into the last equation we have that

$$\nabla\times\vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\sin\theta\frac{\partial}{\partial r}\left(\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right)\hat{\phi} = -\frac{\mu_0 p_0 \omega^3}{4\pi rc}\sin\theta\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\phi} \tag{10.106}$$

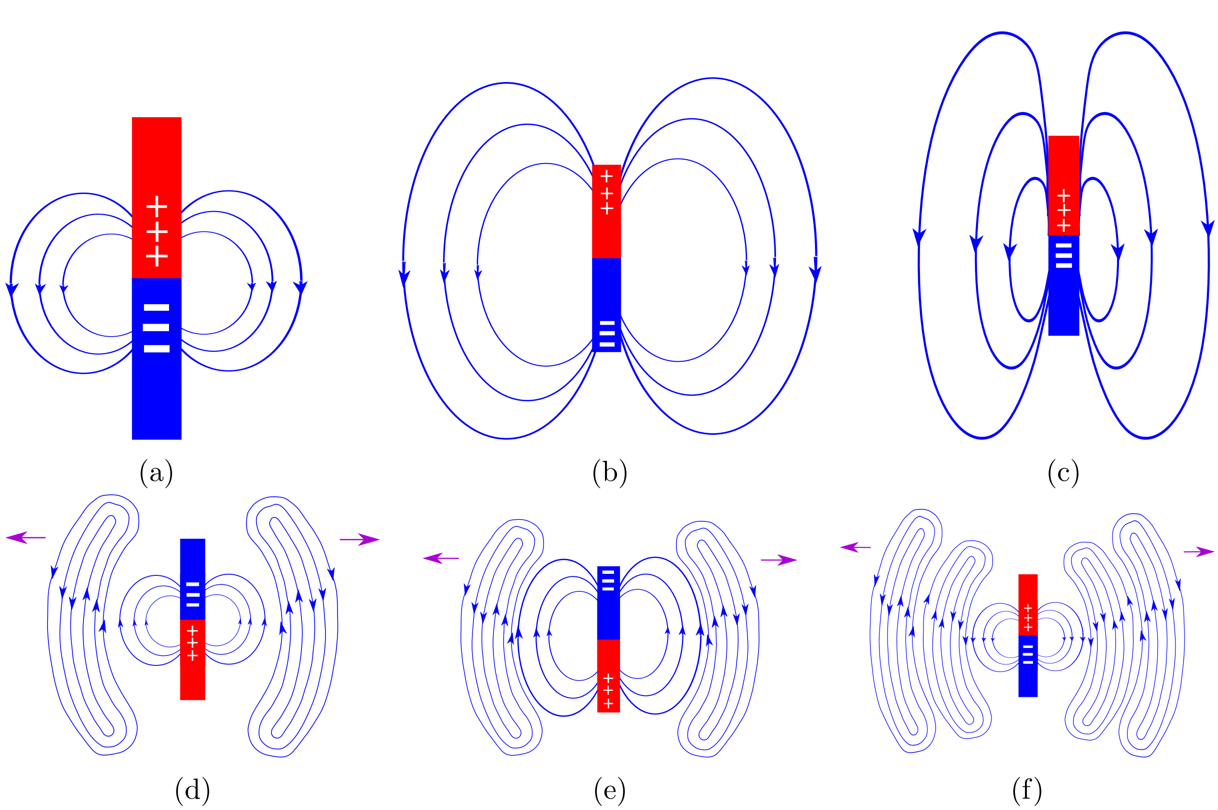


Figure 10.11: Dipole Antenna radiation pattern at different instants. In these figures you see the dipole antenna in figure 10.10 from the top. One wire gets positively electrically charged while the other negatively charged. The accumulation of positive and negative charges tend to be in certain regions of the wires in different times, giving the beautiful patterns shown. Once the electromagnetic waves are emitted, they travel at the speed of light! These patterns plotted in $3D$ would look like figure 10.12.

.

Now, we use Faraday's Law,

$$-\frac{\mu_0 p_0 \omega^3}{4\pi r c}\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\phi}=-\frac{\partial\vec{B}}{\partial t} \tag{10.107}$$

So, integrating with respect time, we obtain the magnetic field

$$\vec{B}=-\frac{\mu_0 p_0 \omega^2}{4\pi r c}\sin\theta\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\hat{\phi} \tag{10.108}$$

Now, we are going to calculate the poynting vector. We have that the cross product in spherical coordinates, in general for any electric field and any magnetic field is the following determinant

$$\vec{E}\times\vec{B}=\begin{vmatrix}\hat{r} & \hat{\theta} & \hat{\phi}\\ E_r & E_\theta & E_\phi\\ B_r & B_\theta & B_\phi\end{vmatrix}$$

However, for the particular case of the electric and magnetic fields generated by the dipole antenna, we have that

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{1}{\mu_0}\begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 0 & E_\theta & 0 \\ 0 & 0 & B_\phi \end{vmatrix} = \frac{1}{\mu_0} E_\theta B_\phi \hat{r} = \frac{\mu_0}{c}\left[\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left(\omega\left(t - \frac{r}{c}\right)\right)\right]^2 \hat{r}$$

Now, we proceed to calculate the intensity. So, the mean poynting vector is

$$< \vec{S} >= \frac{\mu_0}{c}\left[\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\right]^2 \frac{1}{T}\int_0^T \cos^2\left(\omega\left(t - \frac{r}{c}\right)\right) dt\hat{r} \tag{10.109}$$

However, the integral times $\frac{1}{T}$ in last equation ends up to be $\frac{1}{2}$ as in our calculation of example 4 (Energy carried by a monochromatic electromagnetic wave). Therefore, the intensity which is the average of the poynting vector (average of energy per unit area per unit time) is

$$< \vec{S} >= \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{r} \tag{10.110}$$

There is no radiation along the axis of the dipole, given that there $\sin(\theta = 0) = 0$. Taking in consideration the propagation of the emitted electromagnetic waves in 3 dimensions, we obtain the pattern of the radiation by the dipole antenna as shown in figure 10.12. So , if we want to know the total power (energy per unit time), we integrate over a sphere of radius r. Even though the shape of the radiation pattern is toroidal (a donut shape), in the regions where there is no radiation the contribution to the integral will be zero, so we can use the sphere without worrying about those regions where there is no radiation. So, we have that

$$< P >= \int < \vec{S} > \cdot d\vec{A} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int \frac{\sin^2\theta}{r^2}\left(r^2 \sin\theta d\theta d\phi\right) \tag{10.111}$$

where we wrote the differential of area in spherical coordinates. So, if we want to integrate over a complete sphere, we have that

$$< P >= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\phi = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}(2\pi)\int_0^\pi \sin^3\theta d\theta \tag{10.112}$$

and we have that the remaining integral is

$$\int_0^\pi \sin^3\theta d\theta = \int_0^\pi \sin\theta\left(1 - \cos^2\theta\right) d\theta = -\cos\theta\Big|_0^\pi + \int_1^{-1} u^2 du = 2 + \left(-\frac{1}{3} - \frac{1}{3}\right) = \frac{4}{3} \tag{10.113}$$

where we used in the first step the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, then we used the change of variables $u = \cos\theta$, $du = -\sin\theta d\theta$ and the fact that the limits of integration now become $\theta = 0 \to u = \cos 0 = 1$ and $\theta = \pi \to u = \cos\pi = -1$. Therefore, we have that equation 10.112 becomes

$$< P >= \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \tag{10.114}$$

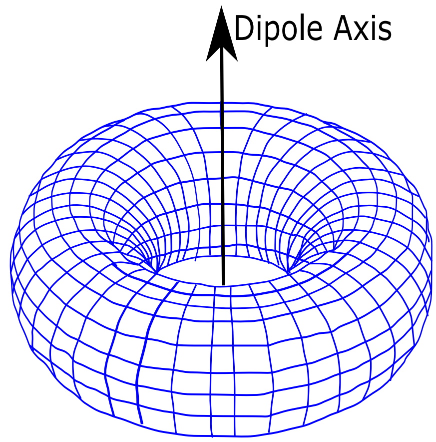


Figure 10.12: Radiation pattern of a dipole antenna. The dipole axis is the oscillation axis of the electric charges. In that axis there is no radiation. This picture shows only one emitted electromagnetic wave. All emitted electromagnetic waves would look like concentric toroids traveling at the speed of light

Example 6: Why is the sky blue?

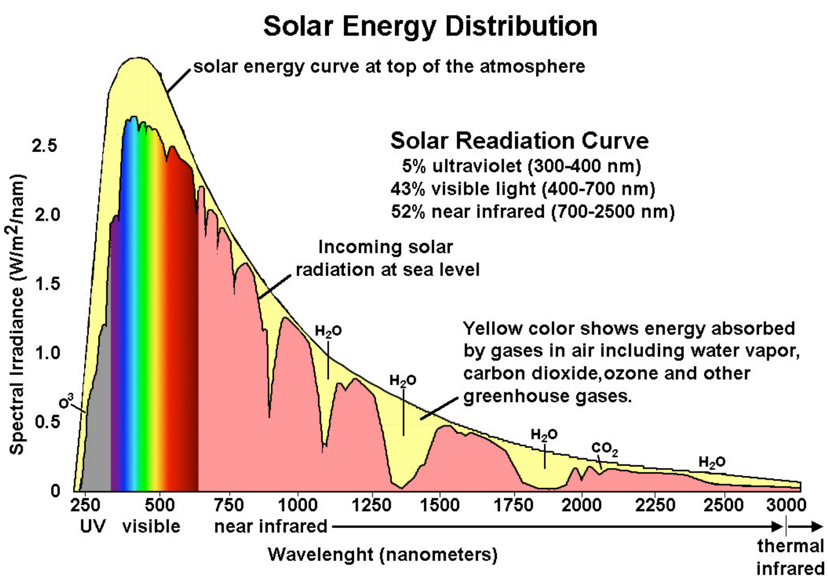


Figure 10.13: The plot shows amount of incident energy to our atmosphere due to different wavelengths of electromagnetic waves coming from the Sun. Original figure taken from [28]

Probably you have heard little kids ask ”Why is the sky blue?”. Sometimes, kids can surprise us how curious they are, and as adults many times we just take things as they are, stop questioning. Now, we are in position to answer one question that probably you have had since you were a kid, “Why is the sky blue?”

Solution:
The sun generates electromagnetic waves of all the visible spectrum. When such radiation arrives to our atmosphere, the varying electric field stimulate the atoms of the atmosphere to oscillate as dipoles! The oscillation of such dipoles is of exactly the same frequency as the incident sun radiation. So, the incident radiation makes the molecules of the atmosphere oscillate as dipoles, and such dipoles re-radiate electromagnetic waves with exactly the same frequency as the incident radiation from the sun! However, this radiation now goes practically to all directions (not along the axis of oscillation of the dipole as shown in figure 10.12), so we say that the electromagnetic radiation is *scattered*.

The process when the incident light *scatters* due to the re-radiated light of the dipoles with the exact same wavelength is called as *Rayleigh scattering*. This kind of scattering takes place only when the molecules or particles, which light influence, are smaller than the wavelength of the incoming electromagnetic wave. In the case of air, the main molecules are oxygen and nytrogen (mostly the 99% of all air). And their sizes are approximately 0.29nm and 0.31nm respectively, which are smaller than the wavelengths of visible light (see table 10.1). So, *Rayleigh scattering* takes place!

Now, the ***probability*** of scattering certain wavelength is given by the so called cross section σ. We do not deduce neither get into more details of the following formula, but think about what the quantity is telling us (***probability*** of certain wavelength of light to be scattered). So, the cross section of *Rayleigh scattering* is

$$\sigma = \frac{<P>}{|<\vec{S}>|} \tag{10.115}$$

where $<P>$ is the power of the re-radiated electromagnetic waves due to the oscillation of the molecules of air and $|<\vec{S}>|$ is the intensity of the incident electromagnetic waves coming from the sun. The term $|<\vec{S}>|$ is calculated by equation 10.97 and $<P>$ from equation 10.114 (because the molecules of the atmosphere oscillate as dipoles when the radiation of the Sun reaches them). Plugging in both formulas, we notice that the cross section of scattering is proportional to w^4. So, the ***probability*** of scattering certain wavelength of light is highly dependant of the frequency of the electromagnetic waves coming from the sun.

Therefore, the electromagnetic waves with higher frequencies (lower wavelength) will scatter much more and we see such light frequencies coming from all directions! Hence, we see all the sky full of such wavelength electromagnetic waves! But wait a minute! If that is true, then from table 10.1 we have that the the power of the re-radiated waves of the excited dipoles with violet light coming from the sun should be much higher! Hence, much more scattering of violet light should be seen. Why is it that the sky is not violet then? Two main reasons. Even though the sun radiates all the visible spectrum, the amount of violet light radiated from the sun is much less than the blue light as shown in figure 10.13. Secondly, our eyes are more sensitive to the blue light than the violet light! So, combining all these facts at the end what we see is a blueish beautiful sky! The amount of blue light is much more than the violet light, and blue light scatters much more than the other longer wavelengths (A simplified cartoon of what we have discussed is shown in figure 10.14). So, from all directions of the Earth atmosphere we receive blueish light. Amazing isn't it?

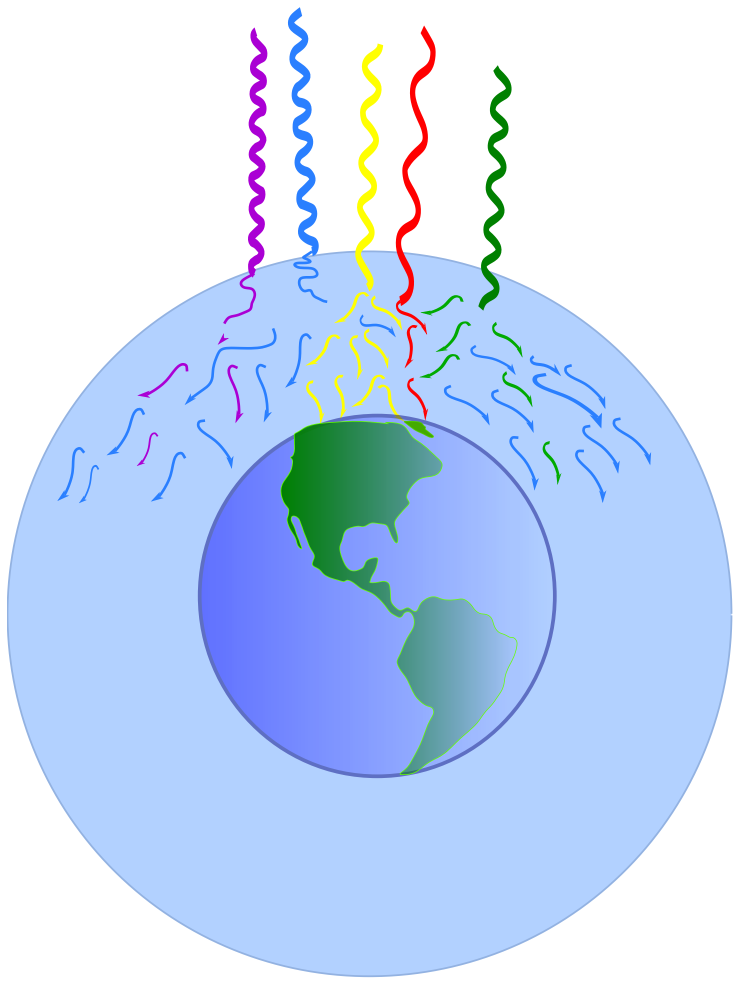


Figure 10.14: Radiation from the sun is scattered when it is incident to the molecules of the atmosphere. Depending of the wavelength of the incoming light, more or less is the light scattered. Given that we receive more blue light than violet light from the Sun, blueish color predominates over violet. Given that blue light scatters much more than other light colors with higher wavelength, we receive blue light color from all directions of the atmosphere.

Now, why are sunsets reddish? Well, when the sunsets take place, the light coming from the Sun has to travel a much bigger layer of atmosphere, scattering much more. So, all the high frequencies of light has been scattered, leaving only the lowest frequencies (greatest wavelength, see table 10.1). So, we see the beautiful reddish sunsets. But then, why are clouds white? The droplets of water that constitute clouds are much bigger than the wavelength of the incident radiation. The scattering of light is not *Rayleigh scattering* anymore in this case. In this case another kind of scattering takes places called as *Mie scattering*. In such process, all wavelengths scatter equally. So, we see a combination of all colours of the spectrum, which produce the colour white! Actually, in outer space the sun looks white. In the earth, if you look directly to the sun (please, do not do it!) the wavelengths red and yellow of light gets scattered muchless. But, look once again at figure 10.13, the amount of yellow light is greater than the red light that arrives to our atmosphere. Therefore, the Sun appears to be yellowish if you look at it directly, because yellow and red light scatters much less than the other light colors. However, more yellow light arrives to our eyes.

10.7 Momentum carried by Electromagnetic Waves

Electric and Magnetic fields store energy as we have studied. Also, they carry momentum! And these fields can exert a force in any given area!

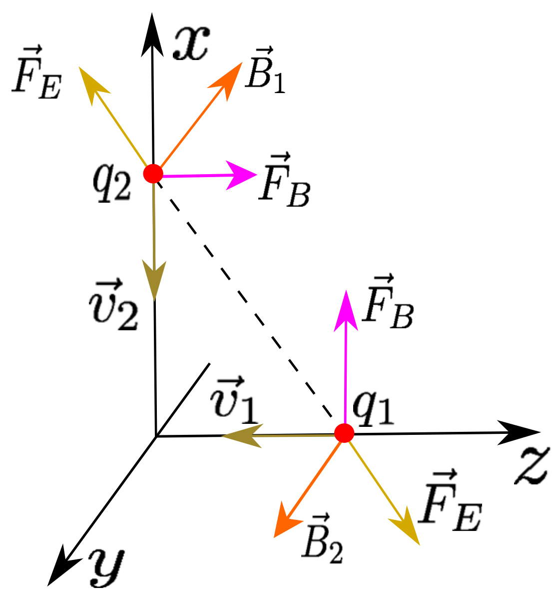


Figure 10.15

So, to start our discussion let's analyze the following system. Suppose two electric charges, q_1 moving in the z axis towards $-z$ with certain velocity $\vec{v}_1$ and q_2 electric charge moving with another certain velocity $\vec{v}_2$ in the x axis towards $-x$ as shown in figure 10.15. Now that the charges move, we cannot say any more that the electric field is given by Coulomb's Law (that was for static cases). However, lets analyze what happens instantaneously when the electric charges have certain position as shown in figure 10.15. At that certain instant, the electric field still points radially, and the magnetic field circles around the corresponding axis (even though we can not use Bio Savart anymore because is not an stable current, nevertheless the form of the field still is circular around the axis where the charges move). Hence, the direction of the magnetic field generated by charge

q_1 at the position of charge q_2 is towards $-y$ and the magnetic field generated by charge q_2 at the position of charge q_1 is towards $+y$ as shown in figure 10.15. So, notice the following, if we use the right hand rule, the magnetic field force exerted on charge q_2 points to the right (towards $+z$) as shown in figure 10.15. However, the magnetic field force exerted on the electric charge q_1 is upwards (towards $+x$) as shown in figure 10.15. So, the forces are of same magnitude but not opposite directions. Therefore, we have that

$$\vec{F}_{12} \neq -\vec{F}_{21} \tag{10.116}$$

i.e. the exerted force on the charge 2 due to charge 1 is not of the same magnitude but opposite direction as the exerted force on the charge 1 due to charge 2. We are violating third Newton's Law! What is going on?! Newton's Third Law is actually momentum conservation, taking into account that $\vec{F} = \frac{d\vec{p}}{dt}$, last equation is telling us that

$$\frac{d\left(\vec{p}_2 + \vec{p}_1\right)}{dt} \neq 0 \tag{10.117}$$

so that momentum is not conserved, it changes with time! Where did the missing momentum go? To the fields! The fields can carry momentum! And the last system alarms us that it must be the case, if not we would be in great trouble, all our electromagnetic theory violates one of the cornerstones of physics. When we add the momentum carried by fields in the total change of momentum of the system, we recover momentum conservation.

Now, there is a quite large derivation of the momentum carried by fields, but we need to introduce the so called *electromagnetic tensor* and that is beyond the scope of this textbook. However, we will take an approach as the famous nobel Prize *Richard Feynman* does in his famous *lectures on Physics* (reference [32]).

From mechanics, we have the following Theorem (and stated as Feynman to study the momentum carried by fields)

Theorem 10.7.1 *Whenever there is a flow of energy in any circumstance at all (field energy or any other kind of energy), the energy flowing through a unit area per unit time, when multiplied by $1/c^2$, is equal to the momentum per unit volume in the space.*

We already know that the energy of electric and magnetic fields flowing through a unit area per unit time is measured by the pointing vector. Therefore, by using the theorem just mentioned, we have that the momentum density carried by electric and magnetic fields (momentum per unit volume in the space) is given by

$$\boxed{\vec{g} = \frac{\vec{S}}{c^2}} \tag{10.118}$$

where we label as $\vec{g}$ as the momentum density carried by electromagnetic fields.

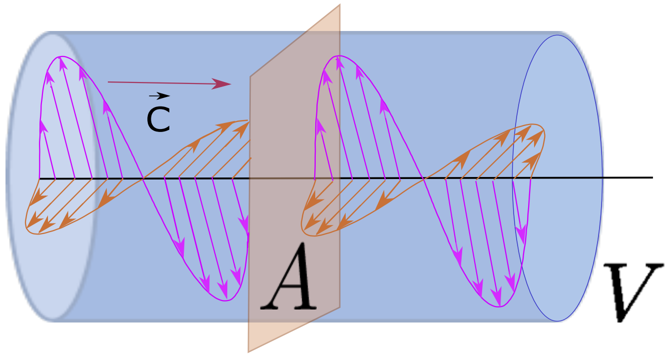
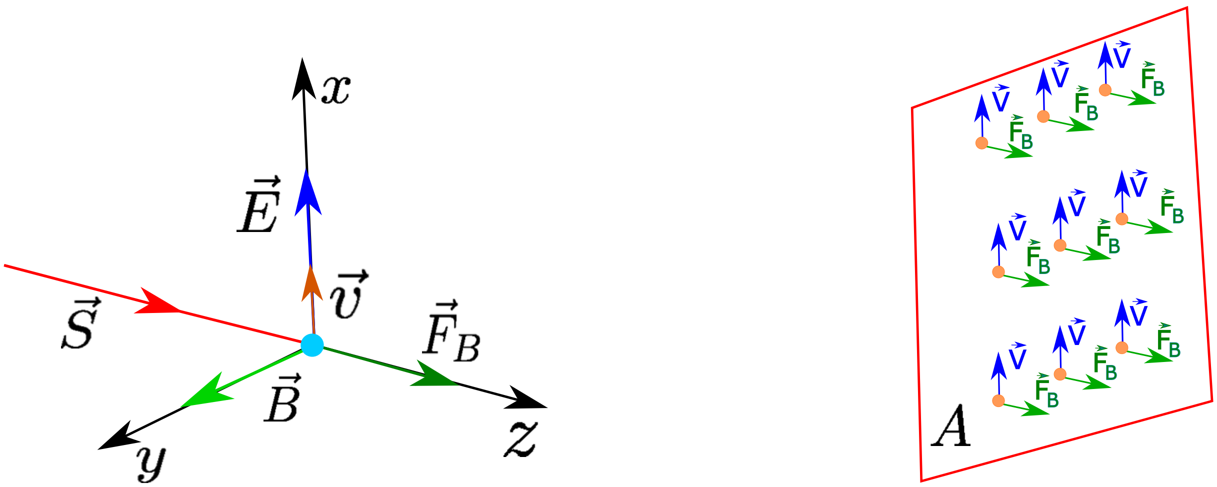


Figure 10.16

Suppose now the following scenario, a monochromatic electromagnetic wave normal (perpendicular) incident to a surface A as shown in figure 10.16, where there are electric charges. Now, as the electromagnetic wave passes through, the momentum of the electric and magnetic fields in the volume V shown in figure 10.16 change! Where did the momentum go? You could probably think, well momentum flew out of the volume as the electromagnetic wave moves. However, if that were the case, notice that the amount of momentum getting into the volume would be the same as the momentum that flew out of the volume because the electromagnetic wave moves with constant speed c. So, that is not causing the change of momentum! Therefore, the momentum must have been transferred to the electric charges. If that is the case, then the momentum lost by the fields is the same amount of momentum gained by the electric charges. Now, how do we know that the momentum in fields changed and there was a momentum change of the electric charges in the surface. As the electromagnetic wave passes trough the surface A where there are electrons, the electric field exerts a force on them! And while the electrons move due to the electric force, a magnetic force also is exerted on them. A force is nothing else than a change of momentum of the charges $\frac{d\vec{p}}{dt}$. And why did it changed? Because the electromagnetic wave transferred it to the charge.

Now, something beautifully remarkable, is that electromagnetic waves exert pressure on any surface A where they are incident! How can this be? Probably from thermodynamics, you recall that the pressure in a gas is due to the collisions of the molecules of the gas with the molecules of the container of such gas. But, in this case there is a pressure exerted on the surface due to light?! How could that be if light is massless? Not intuitive nor trivial at all. But indeed this happens. We can get a better grasp how this works with the picture shown in figure 10.17a. Suppose light goes to $+z$ direction as shown in the picture (the direction of propagation is given by the poynting vector $\vec{S}$) and an electric charge is at the origin. The electromagnetic wave has an electric and magnetic field which are perpendicular one to each other as discussed before. So, the electric field of the wave will move the charge up and down in the x axis. But, when the electric charge moves with certain velocity $\vec{v}$ a magnetic force is exerted. Here comes the beauty, the direction of the magnetic field force will be towards the direction of the propagation of light! So, now suppose not just one charge, but a surface of charges and all of them constrained to certain layer of a metal for example. Since, the plane wave has electric and magnetic fields oscillating at each point of the surface, then all the charges of the metal layer will feel a push towards the direction of the light! You can visualize this in figure 10.17b. The magnitude of the force exerted per unit area $\frac{|\vec{F}|}{A}$ is the pressure that will be exerted on the surface! This kind of pressure due to electromagnetic waves is called as *radiation pressure*.



(a) (b)

Figure 10.17

Now, what is the magnitude of that pressure? Let's analyze an idealized case. Suppose that all the energy and momentum of an normal incident electromagnetic wave is transferred to the electric charges to a surface A, as shown in figure 10.18. So, all the momentum of the electric and magnetic fields in the volume V is transferred to the electric charges in the surface. Given that the electromagnetic waves travel at the speed of light, and oscillate many times in a brief period of time, we are interested more in the average of the momentum transferred. So, all the average momentum transferred to the charges in the surface A is

$$\Delta p =< \vec{g} > Ac\Delta t \Longrightarrow \frac{\Delta p}{\Delta t} =< \vec{g} > Ac \tag{10.119}$$

where p is the momentum of the electric charges, $< \vec{g} >$ is the average momentum density in the volume V shown in figure 10.18. The momentum in the volume V is all transferred to the surface as the electromagnetic wave moves towards A. We calculated the volume as $V = Ac\Delta t$, where $c\Delta t$ is the length travelled by the electromagnetic wave in the interval of time Δt.

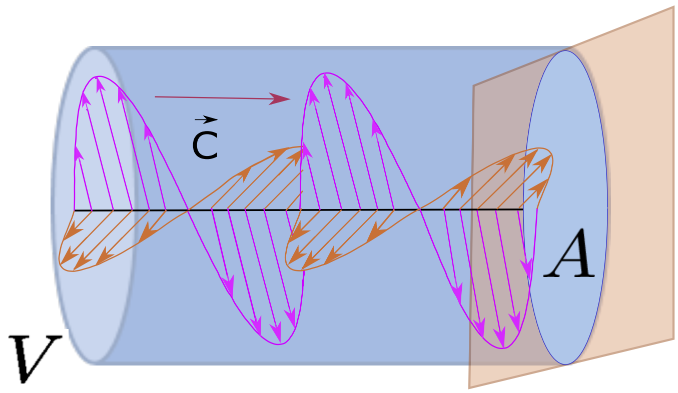


Figure 10.18

Now, we know that the change with respect time of momentum of the electric charges is the force exerted on them, so the mean force exerted on all of them is $<\vec{F}>=\frac{\Delta\vec{p}}{\Delta t}$. Therefore, the pressure on surface A is

$$\boxed{P=\frac{|<\vec{S}>|}{c}} \tag{10.120}$$

where we used the average of $\vec{g}$ in equation 10.118. And we took the pressure as the magnitude of the average force per unit area

$$P=|<\vec{F}>|/A \tag{10.121}$$

Nice! The pressure is just the magnitude of the averaged poynting vector divided by the speed of light. However, this is the pressure for a very specific case, when all the energy and momentum of the electromagnetic waves was transferred to the electric charges of a perfect absorber. Such material does not exist! This idealized absorber is called as *black body*. Now, suppose a surface that reflects all light! In such case the pressure doubles!

$$\boxed{P=2\frac{|<\vec{S}>|}{c}} \tag{10.122}$$

because the momentum change is due now to the incident light and the reflected light with exactly the same speed as the incident light. So, any surface material between a perfect reflector and a perfect absorber, the pressure exerted on the surface lies between those two values

$$\boxed{\frac{|<\vec{S}>|}{c}<P<2\frac{|<\vec{S}>|}{c}} \tag{10.123}$$

Example 7: Cosmic dust in our Solar system

Figure 10.19: Picture taken from [29]

In our Solar system, there are cosmic dust particles. These particles could range from any size, from extremely tiny ones of size of molecules up to little dust grains like tiny rocks. However, a tiny amount of cosmic dust in our Solar system is of a size lower than $0.2\mu m$. Show why it is that this fact takes place.

Solution:

We have learned that light exerts a force given that it carries momentum. So, the dust particles in the Solar system are under the influence of two forces. The gravitational force and the radiation force. The radiation pressure pushes the dust particles away from the Solar system, while the gravitational force attracts the dust particles towards the Sun. So, only those particles where the gravitational force prevails against the radiation pressure will stay in the Solar system. Those which the radiation force is bigger than the gravitational force will be accelerated away from our Solar system. So, for any dust particle to stay at the solar system then

$$F_G > F_R \tag{10.124}$$

where F_G is the magnitude of the gravitational force and F_R is the radiation force due to the momentum carried by the electromagnetic waves from the sun. Now, giving some numbers. We have that the luminosity of the Sun is approximately $L = 3.8 \times 10^{26}\text{W}$. That is the amount of energy per unit time generated by the nuclear reactions of the sun. So, this energy flies away from the sun in electromagnetic radiation. Now, the average of cosmic dust density is approximately $\rho = 2 \times 10^3 \frac{\text{kg}}{\text{m}^3}$. So, the mass of the cosmic dust particles in the Solar system is

$$m = \rho \frac{4\pi R^3}{3} \tag{10.125}$$

where R is the radius of the dust particles and we are modelling them as spherical dust particles for simplicity. Now, if the dust particles are at a distance r from the sun, we can say then that the magnitude of the average pointing vector is

$$|<\vec{S}>| = \frac{L}{4\pi r^2} \tag{10.126}$$

We obtained last equation analyzing the following, the poynting vector is the *energy flow per unit area per unit of time*. Given that the luminosity is the energy per unit of time transported by the electromagnetic waves of the sun, then by diving by the area that these electromagnetic waves are crossing, we obtain the units of the poynting vector. If we take that the electromagnetic waves emitted by the sun propagate in all directions, we can take as the area that crosses the electromagnetic waves as the area of a sphere. Recalling that the area of the sphere is $4\pi r^2$ we obtain the last equation. Now, from equation 10.120 and equation 10.121, we have that the magnitude of the radiation force exerted on the cosmic dust particles is

$$F_R = \frac{|<\vec{S}>|}{c}A = \left(\frac{L}{4\pi r^2}\right)4\pi R^2 \tag{10.127}$$

where A is the cross sectional area of the dust particles. Therefore, we have that equation 10.124 becomes

$$G\frac{M_\odot\left(\rho\frac{4}{3}\pi R^3\right)}{r^2} > \left(\frac{L}{4\pi c r^2}\right)4\pi R^2 \tag{10.128}$$

where $M_\odot$ stands for the mass of the Sun, R is the radius of the dust particles, the term $4\pi R^2$ is the cross sectional area of the dust particles, and we took into consideration as the dust particles as idealized perfect *blackbodies*. So, if we isolate the radius of the cosmic dust particles we have

$$R > \frac{3L}{16\pi G M_\odot \rho c} \tag{10.129}$$

so any cosmic dust particle with a radius lower than whatever value we obtain from last equation, will be pushed away from our solar system due to electromagnetic radiation! Beautiful! Plugging in the values we have

$$R > \frac{3\cdot 3.8\times 10^{26}\mathrm{W}}{16\pi\left(6.67\times 10^{-11}\frac{\mathrm{Nm^2}}{\mathrm{kg^2}}\right)(1.98\times 10^{30}\mathrm{kg})\left(2\times 10^{3}\frac{\mathrm{kg}}{\mathrm{m^3}}\right)\left(3\times 10^{8}\frac{\mathrm{m}}{\mathrm{s}}\right)} \approx 0.28\mu m \tag{10.130}$$

So, all cosmic dust particles whose radius approximately is

$$R > 0.28\mu m \tag{10.131}$$

will remain in the Solar system, those with radius lower than that will be pushed away from our solar system due to radiation pressure! Beautiful! Radiation pressure for those tiny cosmic particles is big enough to throw them away! Light is like a cleaner of tiny cosmic particles for our Solar system.

Example 8: IKAROS (Interplanetary Kite-craft Accelerated by Radiation Of the Sun)

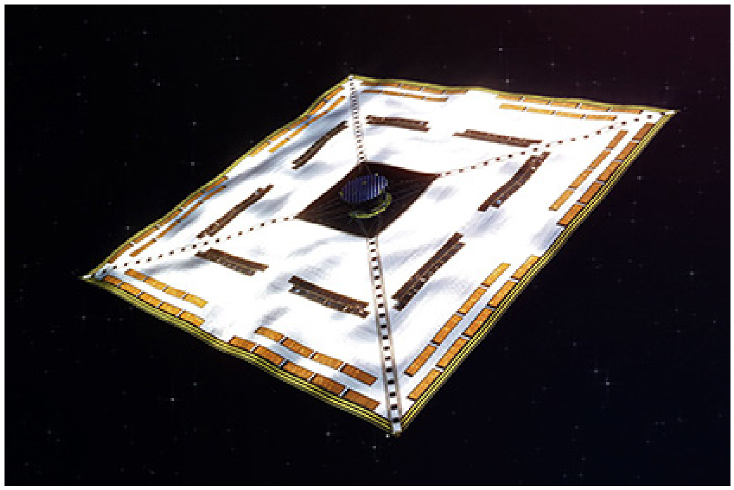


Figure 10.20: IKAROS picture. The original picture is from reference [30]

IKAROS is a spin-stabilized spacecraft. Its unique feature is a huge solar sail of an ultra-flexible structure, $14m \times 14m$ *in size and* $7.5\mu\mathrm{m}$ *in thickness (...)On June 9, 2010, IKAROS successfully deployed its sail fully. (...) The data clearly indicated that IKAROS started accelerating just after sail deployment with an acceleration amount of* $3.6\times10^{-6}m/s^2$. *This value is just that we predicted as solar-light pressure acceleration. It was this very moment that we confirmed start of cruising by solar-sail in deep space for the first time in the world. (...) IKAROS acquired acceleration of 100m/s from the solar-light pressure over six months until it passed Venus. This value is equal to propellant amount consumed for orbital adjustment of deep-space exploration missions of usual ballistic flight. Moreover, the advantage is simply proportional to flight duration. Our next technical target is a sail area 10 times that of IKAROS and mission duration over five years. This translates to an acceleration capability of several km/s, which is virtually the same as obtaining rocket acceleration capability without fuel. Furthermore, by using the large sail as a generator, we plan to drive an electric propulsion system with high-specific impulse and high power, eventually add more freedom to mission planning. I hope that you see how solar-power sail technology can drastically change deep-space exploration in the future.*

— Professor Yuichi Tsuda [31]

The extract just shown is from the official website of the JAXA (Japan Aerospace Exploration Agency). It is incredible how radiation pressure was used to make IKAROS move without the use of fuel! Just think about the possibilities! I want to finish this chapter with this exercise. Making a simple calculation to get a grasp how useful can be exploiting radiation pressure. Let's use the dimensions of the IKAROS. So, the area where the electromagnetic radiation is incident is roughly

$$A = 14\text{m} \times 14\text{m} = 196\text{m}^2 \tag{10.132}$$

Now, calculating the poynting vector at a distance away from the sun of 1UA (average distance from the Sun to the Earth) and with the luminosity of the Sun

$$|<\vec{S}>| = \frac{L}{4\pi r^2} = \frac{3.8\times 10^{26}\text{W}}{4\pi\left(149597870700\text{m}\right)^2} = 1351.21\frac{\text{W}}{\text{m}^2} \tag{10.133}$$

where we consider that electromagnetic waves from the sun go in all directions, so we took as the area crossed by the electromagnetic waves as an sphere. So, the radiation pressure is

$$P = 2\cdot\frac{|<\vec{S}>|}{c} = 9\times 10^{-6}\frac{\text{N}}{\text{m}^2} \tag{10.134}$$

where we now consider as a perfect reflector the surface of IKAROS (the IKAROS surface was designed with a extremely good reflector to double the radiation pressure). Therefore, the magnitude of the exerted force on the IKAROS is

$$|\vec{F}| = PA = \left(9\times 10^{-6}\frac{\text{N}}{\text{m}^2}\right)\left(196\text{m}^2\right) = 1.765\times 10^{-3}\text{N} \tag{10.135}$$

Now, the mass of the IKAROS is about 310kg. Therefore, the magnitude of the acceleration of IKAROS due to radiation pressure is

$$|\vec{a}| = \frac{1.765\times 10^{-3}\text{N}}{310\text{kg}} = 5.69\times 10^{-6}\frac{m}{s^2} \tag{10.136}$$

which is a little above the acceleration registered from the IKAROS experiment. However, we are making various assumptions, firstly that IKAROS is a perfect reflector, the pointing vector is calculated by a surface by a perfect sphere, and we have not taken into account also the gravitational forces exerted on IKAROS. So, the approximation of our calculation to the experimental data is beautifully close enough to know that the acceleration registered by IKAROS was due to radiation pressure. Now, probably you would say "*What is the big deal? The acceleration is ridiculously small*". Well, indeed, however the IKAROS being accelerated constantly by one year, then its speed would be

$$|\vec{v}| = \left(5.69\times 10^{-6}\frac{m}{s^2}\right)\cdot(31536000\text{s}) \approx 180\frac{\text{m}}{\text{s}} \approx 648\frac{\text{km}}{\text{hr}} \tag{10.137}$$

A little bit more than half of the speed of sound at sea level (approximately 1atm), and 20°C! And even better, all that speed gained from free, no fuel used to obtain such speed. Beautiful indeed!

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