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IOP Conf. Series: Journal of Physics: Conf. Series **936** (2017) 012070 doi:10.1088/1742-6596/936/1/012070

Modeling the dynamics of multipartite quantum systems created departing from two-level systems using general local and non-local interactions

Francisco Delgado

Departamento de Física y Matemáticas, Escuela de Ingeniería y Ciencias, Tecnológico de Monterrey, Campus Estado de México, Atizapán, Estado de México, CP. 52926, México.

E-mail: fdelgado@itesm.mx

Abstract. Quantum information is an emergent area merging physics, mathematics, computer science and engineering. To reach its technological goals, it is requiring adequate approaches to understand how to combine physical restrictions, computational approaches and technological requirements to get functional universal quantum information processing. This work presents the modeling and the analysis of certain general type of Hamiltonian representing several physical systems used in quantum information and establishing a dynamics reduction in a natural grammar for bipartite processing based on entangled states.

1. Introduction

Quantum computation [1, 2, 3] and quantum cryptography [4, 5] are based on disruptive quantum phenomena: superposition and entanglement. Binary nature of computer science is connected with quantum computation using two-level quantum systems through the qubit, a unipartite minimum element of information adopting two simultaneous states of existence. Multipartite systems obtained from the combination and interaction of qubits exhibit a notable dynamics complexity. There, Heisenberg-Ising model written in the Bell states basis reproduces quantum dynamics for magnetic systems exhibiting $SU(2)$ block decomposition for two-qubits [6, 7]. Hilbert space $\mathcal{H}^2$ becomes a direct sum of two subspaces, each one spanned by a pair of Bell states and $U \in U(1) \times SU(2)^2$, reducing the dynamics to well-known procedures in control theory [8, 9]. This paper demonstrates that such decomposition is achievable for larger systems. The second section states the Hamiltonian and the basis to reach the decomposition. The third section discusses the architecture and the structure of U. Last section includes the conclusions.

2. Hamiltonian for $2d-$partite two level systems and basis to reach $SU(2)$ reduction

Heisenberg-Ising model is exploited in developments as D-Wave and IBM-Q quantum computers because it reproduces the exchange information on spins. With $SU(2)$ reduction, a set of universal gates has been achieved with a notable degree of fidelity [10, 11, 12]. As first approach:

$$H = \sum_{\{i_k\}} h_{\{i_k\}} \bigotimes_{k=1}^{n} \sigma_{i_k} = \sum_{\mathcal{I}=0}^{4^n-1} h_{\mathcal{I}_4^n} \bigotimes_{k=1}^{n} \sigma_{\mathcal{I}_{4,k}^n} \tag{1}$$

for n qubits, with $\{i_k\} = \{i_1, i_2, ..., i_n\}$ and $i_k \in \{0, 1, 2, 3\}$. There, $h_{\{i_k\}}$ is a set of time-dependent real functions. $\{i_k\}$ could be represented as the number $\mathcal{I} \in \{0, 1, ..., 4^n - 1\}$ in base-4 with n digits, $\mathcal{I}_4^n$. Then, $\mathcal{I}_{4,k}^n = i_k$ for $k = 1, 2, ..., n$. σ_{i_k} are the Pauli matrices in the computational basis $|0\rangle, |1\rangle \in \mathcal{H}$ for each qubit k. Hamiltonian obeys the Schrödinger equation for the evolution operator U. This Hamiltonian contains some unphysical terms, but we will restrict H to pair spin coupling like interactions H_{nl_i} together with local operations H_{l_i} generated by fields on each qubit, for an even number of them $n = 2d$. The Bell basis fits the evolution in the $U(1) \times SU(2)^2$ decomposition of $SU(4)$ for bipartite systems [6]. Here, the most obvious candidates are the generalized Bell states (GBS) for $n = 2d$ [13], a tensor product of Bell states settling an orthogonal basis of non-maximal entangled states for $2d$ qubits. Such elements are:

$$\left|\Psi_{\mathcal{I}_4^d}\right\rangle = \frac{1}{\sqrt{2^d}} \sum_{\mathcal{E},\mathcal{D}=0}^{2^d-1} (\tilde{\sigma}_{i_1} \otimes ... \otimes \tilde{\sigma}_{i_d})_{\mathcal{E}_2^d,\mathcal{D}_2^d} \left|\mathcal{E}_2^d\right\rangle \otimes \left|\mathcal{D}_2^d\right\rangle \tag{2}$$

At this point, $\tilde{\sigma}_i$ is a unitary factor of the traditional Pauli matrices (in fact, $\tilde{\sigma}_{i\neq 2} = \sigma_i, \tilde{\sigma}_2 = i\sigma_2$) [13]. $\mathcal{E}_2^d, \mathcal{D}_2^d$ are numbers in base-2 with d digits ($\mathcal{E}, \mathcal{D} \in \{0, 1, ..., 2^d - 1\}$) representing $\{\epsilon_1, ..., \epsilon_d\}, \{\delta_1, ..., \delta_d\}$ respectively, where $\epsilon_s, \delta_s \in \{0, 1\}$ (digits appear reversed explicitly in $\mathcal{E}_2^d$ or $\mathcal{I}_4^d$ expressions). Noting $\left\langle \Psi_{\mathcal{I}_4^d} | \sigma_{j_1} \otimes ... \otimes \sigma_{j_{2d}} | \Psi_{\mathcal{K}_4^d} \right\rangle = \frac{1}{2^d} \prod_{s=1}^{d} \mathrm{Tr}(\tilde{\sigma}_{i_s}^* \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T)$, then:

$$\left\langle \Psi_{\mathcal{I}_4^d} | H | \Psi_{\mathcal{K}_4^d} \right\rangle = \frac{1}{2^d} \sum_{\mathcal{J}=0}^{4^{2d}-1} h_{\mathcal{J}_4^{2d}} \prod_{s=1}^{d} \mathrm{Tr}(\tilde{\sigma}_{i_s}^* \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T) \tag{3}$$

with $\mathcal{J} \in \{0, 1, ..., 4^{2d} - 1\}$. $\tilde{\sigma}_{i_s}^* \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T$ has some properties inherited from Pauli matrices because $\sigma_1, \sigma_2, \sigma_3$ are traceless and $\sigma_i^T = \pm\sigma_i$ (negative sign for $i = 2$). $\mathrm{Tr}(\tilde{\sigma}_{i_s}^* \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T)$ is non-zero only if i_s, j_{d+s}, k_s, j_s are: a) completely different among them, or b) equal by pairs. For non-local interactions, if there are non-local interactions between pairs s and $s + d$, for $s = 1, 2, ..., d$, then $\sigma_{j_s} = \sigma_{j_{d+s}}$, implying $\tilde{\sigma}_{i_s} = \tilde{\sigma}_{k_s}$ to get $\left\langle \Psi_{\mathcal{I}_4^d} | H | \Psi_{\mathcal{K}_4^d} \right\rangle \neq 0$. Then these terms are diagonal. These pairs will be called *correspondent* pairs in the following. For local interactions, only one matrix between $\sigma_{j_s}, \sigma_{j_{d+s}}$ will be different from σ_0, namely σ_i, then the remaining factors will be σ_j, σ_k, where i, j, k are a permutation from $1, 2, 3$. It implies they will give $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle \neq 0$ only if $\mathcal{I}_4^d$ and $\mathcal{K}_4^d$ in $\left|\Psi_{\mathcal{I}_4^d}\right\rangle, \left|\Psi_{\mathcal{K}_4^d}\right\rangle$ differ in only one digit, following the exchange rules: $0 \leftrightarrow i$ or $j \leftrightarrow k$. This aspect, together with the diagonal entries of non-local interactions generate the 2×2 blocks in H (and U as a consequence).This is only possible if direction i is unique for the local interactions, otherwise the block structure becomes destroyed. Some remarks about the notation are convenient: a) $\tilde{\sigma}_i = \alpha_i \sigma_i$, $\alpha_i \in \{1, i\}$; b) $\sigma_i^T = \beta_i \sigma_i$, $\beta_i \in \{-1, 1\}$; c) $\sigma_i \sigma_j = \gamma_{i,j} \sigma_j \sigma_i$, $\gamma_{i,j} \in \{-1, 1\}$. Then, $2c_{j_s,j_{d+s}}^{i_s,k_s} \equiv \mathrm{Tr}(\tilde{\sigma}_{i_s} \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T) = \alpha_{i_s} \alpha_{k_s} \beta_{j_s} \beta_{k_s} \gamma_{k_s j_s} \gamma_{k_s i_s} \mathrm{Tr}(\sigma_{i_s} \sigma_{k_s} \sigma_{j_{d+s}} \sigma_{j_s}) \in \{0, \pm 2, \pm 2i\}$. Formulas for $\alpha_i, \beta_i, \gamma_{i,j}, c_{j_s,j_{d+s}}^{i_s,k_s}$ can be obtained from the Pauli matrices properties.

3. Hamiltonian conditions to get $SU(2)$ reduction and related dynamics

For larger systems, analytical formulas for $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{nl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$ and $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$ are useful for computer simulation optimality. We get them for the $SU(2)$ reduction case, with:

$$H = \bigoplus_{i=1}^{2^{2d-1}} \mathbf{S}_{Hi} \quad \Rightarrow \quad U = \bigoplus_{i=1}^{2^{2d-1}} \mathbf{S}_{Ui} \tag{4}$$

with each $\mathbf{S}_{Hi}$ a 2×2 matrix. The structure is preserved under matrix products, then inherited to U due to the time ordered integral: $\mathbf{S}_{Ui}=\tau\{e^{-\frac{i}{\hbar}\int_0^t \mathbf{S}_{Hi}dt'}\}\equiv e^{-i\Delta_i^+ t}\tau\{e^{-\frac{i}{\hbar}\int_0^t \mathbf{S}_{H_i^0}dt'}\}$, where $\mathbf{S}_{H_i^0}=\mathbf{S}_{Hi}-\Delta_i^+\mathbf{I}_i$ is the part of $\mathbf{S}_{Hi}$ free of $\mathbf{I}_i$. $\mathbf{S}_{Ui}\in U(1)\times SU(2)$ as it will be seen. To get the $SU(2)$ decomposition and a closer approach to the Heisenberg-Ising model in [6], we state:

$$H^{(j,k')}=H_D+H_{ND}^{(j,k')}\quad\text{with}:\quad H_{ND}^{(j,k')}=\sum_{t'=0}^{1}h_{(j4^{k'+dt'-1})_4^{2d}}\bigotimes_{s=1}^{2d}\sigma_{(j4^{k'+dt'-1})_{4,s}^{2d}} \tag{5}$$
$$H_D\equiv\sum_{i'=1}^{3}\sum_{k=1}^{d}h_{(i'(4^{k-1}+4^{k+d-1}))_4^{2d}}\bigotimes_{s=1}^{2d}\sigma_{(i'(4^{k-1}+4^{k+d-1}))_{4,s}^{2d}}$$

generating the $SU(2)$ blocks with the diagonal entries coming from non-local interactions between correspondent parts and the non-diagonal terms from local interactions (only applied on the pair $[k',k'+d]$). Figure 1a shows the architecture being considered. Introducing $\delta^{\mathrm{S}}_{\mathcal{IK}}\equiv\prod_{s\notin S}\delta_{i_s k_s}$, with the set S of scripts excluded in the product and defining the exchange factor in the diagonal-off entries (Einstein summation convention understood):

$$\mathcal{F}_{i,k'}^{j,k}=\delta_{i_{k'}0}\delta_{k_{k'}i}c_{j,k}^{0,i}+\delta_{i_{k'}i}\delta_{k_{k'}0}c_{j,k}^{i,0}+\epsilon_{ii'i''}(\delta_{i_{k'}i'}\delta_{k_{k'}i''}c_{j,k}^{i',i''}+\delta_{i_{k'}i''}\delta_{k_{k'}i'}c_{j,k}^{i'',i'}) \tag{6}$$

by using the properties $c_{0,0}^{i_s,i_s}=1$ and $c_{i,i}^{i_{k''},i_{k''}}=(-1)^{\delta_{i,2}+(1-\delta_{i,i_{k''}})(1-\delta_{0,i_{k''}})}$, then:

$$\left\langle\Psi_{\mathcal{I}_4^d}|H^{(j,k')}|\Psi_{\mathcal{K}_4^d}\right\rangle=\delta_{\mathcal{IK}}\sum_{i'=1}^{3}\sum_{k''=1}^{d}c_{i',i'}^{i_{k''},i_{k''}}h_{(i'(4^{k''-1}+4^{k''+d-1}))_4^{2d}}+$$
$$\sum_{t'=0}^{1}\delta_{\mathcal{IK}}^{\{k'\}}\mathcal{F}_{j,k'}^{j\delta_{0,t'},j\delta_{1,t'}}h_{(j4^{k'+dt'-1})_4^{2d}}\equiv H_{D\mathcal{IK}}+H_{ND}^{(j,k')}{}_{\mathcal{IK}} \tag{7}$$

$H_{D\mathcal{IK}}, H_{ND}^{(j,k')}{}_{\mathcal{IK}}$ denotes the diagonal and non-diagonal terms of the whole interaction. This formula shows that the pair of diagonal entries for each $SU(2)=\{h_{ij}\}, i,j=1,2$ block are different: it is formed switching only one index $i_{k''}$ in the rows labels, then for $i'\neq j$ the terms in $H_{D\mathcal{IK}}$ have a sign change, implying $h_{11}\neq h_{22}$. We should note that the exchange of entanglement will be allowed only between correspondent pairs through the driven local interactions.

Hilbert space $\mathcal{H}^{2d}$ becomes the direct sum of 2^{2d-1} subspaces generated by pairs of GBS basis $\{\left|\Psi_{\mathcal{I}_4^d}\right\rangle,\left|\Psi_{\mathcal{K}_4^d}\right\rangle\}$ (those having only one index k' exchanged with the rules $0\leftrightarrow i$, $j\leftrightarrow k$). As a consequence, only two different blocks are independent through the complete dynamics. There, the dynamics mixes the probabilities leaving unchanged those among subspaces. If $|\psi_0\rangle=\sum_{s=1}^{2^{2d-1}}|\psi_{0s}\rangle$ is the initial state with amplitudes $|\psi_{0s}\rangle=\psi_{0s,\mathcal{I}_4^d}\left|\alpha_{\mathcal{I}_4^d}\right\rangle+\psi_{0s,\mathcal{K}_4^d}\left|\alpha_{\mathcal{K}_4^d}\right\rangle$, then the evolved component for each $s=1,2,...,2^{2d-1}$ fulfills $\|\,|\psi_{ts}\rangle\,\|=\|\mathbf{S}_{Us}\,|\psi_{0s}\rangle\,\|=\|\,|\psi_{0s}\rangle\,\|$. Figure 1b depicts the dynamics evolving separately in Bloch spheres for each pair. To extend the exchange of entanglement between different pairs, a rearrangement among parts conforming the correspondent pairs should be generated, remixing the original pair probabilities. Exchange dynamics operates on the information states, not directly on physical parts of the system. Blocks in the rows $\mathcal{I}$ and $\mathcal{I}'$ have the form (for time independent Hamiltonian):

$$\mathbf{S}_{H\mathcal{I},\mathcal{I}'}=\frac{h_{11}+h_{22}}{2}\mathbf{I}_{\mathcal{I},\mathcal{I}'}+\mathbf{n}\cdot\mathbf{s}_{\mathcal{I},\mathcal{I}'} \tag{8}$$
$$\mathbf{S}_{U\mathcal{I},\mathcal{I}'}=e^{i\mathbf{S}_{H\mathcal{I},\mathcal{I}'}\frac{t}{\hbar}}=e^{i\frac{h_{11}+h_{22}}{2\hbar}t}(\cos\omega t+i\sin\omega t\mathbf{n}\cdot\mathbf{s}_{\mathcal{I},\mathcal{I}'})\in U(1)\times SU(2) \tag{9}$$

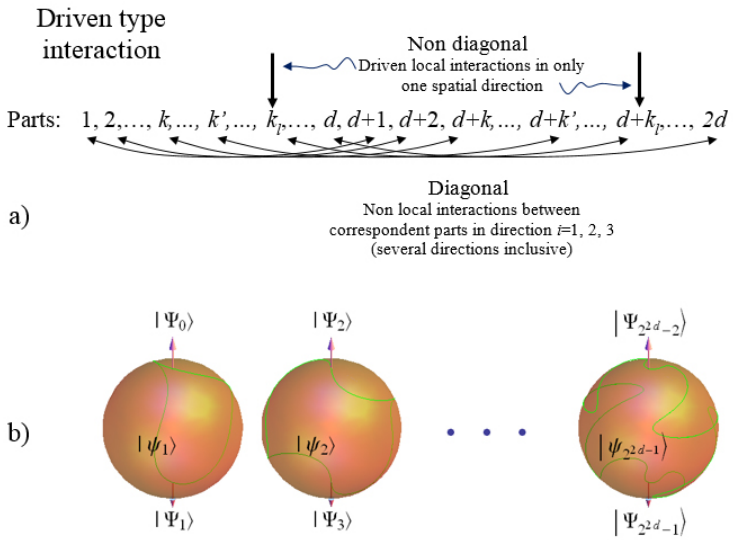

Figure 1. a) Interactions to achieve the $SU(2)$ decomposition, and b) probability exchange in temporary pairs evolving in separate Bloch spheres.

Here, $\{\mathbf{I}_{\mathcal{I},\mathcal{I}'}, \mathbf{X}_{\mathcal{I},\mathcal{I}'}, \mathbf{Y}_{\mathcal{I},\mathcal{I}'}, \mathbf{Z}_{\mathcal{I},\mathcal{I}'}\}$ is a Pauli basis for the $SU(2)$ block, $\mathbf{s}_{\mathcal{I},\mathcal{I}'} = (\mathbf{X}_{\mathcal{I},\mathcal{I}'}, \mathbf{Y}_{\mathcal{I},\mathcal{I}'}, \mathbf{Z}_{\mathcal{I},\mathcal{I}'})$, $\hbar\omega\mathbf{n} = (\mathrm{Re}(h_{12}), -\mathrm{Im}(h_{12}), \frac{h_{11}-h_{22}}{2})$ and $\mathbf{n}$ is a unitary vector defining ω. $\mathcal{F}_{j,k'}^{j_d,j_{d+s}}$ is imaginary only if j_d or j_{d+s} are 2, then only one n_1 or n_2 is non-zero. It completes the dynamics description.

4. Conclusions

The generic Hamiltonian modeled in this work comprises characteristic elements of the systems used for quantum information processing. Here, only a special kind of systems exhibiting the $SU(2)$ reduction have been modeled through of Hamiltonian (5), but still other possible interactions including this property should be considered as extension [14]. $SU(2)$ reduction lets understand the quantum processing as a simple series of operations manipulating directly the quantum information underlying the system. GBS basis works as a universal grammar for the group of analyzed systems having adaptable control elements as the driven local interactions presented here. Additional research to develop this grammar should be still developed.

Acknowledgments

The support from Tecnológico de Monterrey to develop this research is acknowledged.

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