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Bivariate Copula-TGARCH Model for Real Option Valuation

Tesis para recibir el título de Doctorado en Ciencias Financieras
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México D.F., 13 de Febrero de 2013

Abstract

The valuation methodologies commonly used by the project managers, based on the discount cash flow perspective, follow unrealistic assumptions that barely relate to business conditions. Their inability to consider elements such as flexibility, uncertainty and irreversibility leads to an undervaluation of the investment projects. The real option analysis is gaining popularity in the academic and practitioner arenas because it outperforms the traditional valuation techniques; specifically, by its treatment of uncertainty and strategic thinking incorporation.

A fundamental matter for the project manager is the volatility treatment of the variables involved in the investment opportunity; for they determine its viability. Therefore, valuation techniques must focus on this matter. The proposed methodology is an alternative procedure for investment project valuation that seeks to enhance the benefits of the real option analysis by incorporating volatility treatment through a powerful tool that captures and describes the nature of the financial series in a more efficient way: a Copula-TGARCH model..

Considering the effect of two underlying assets, this methodology can be summarized in three steps. First, the volatility and terminal value for the two underlyings assets is obtained, followed by the estimation of a measure of association between them. Finally, that information is being used as inputs in the real option analysis context. The structure of this work is as follows: Sections 1 and 2 present the properties of the real option analysis and its main advantages over traditional valuation methods based on the discount cash flow perspective. Section 3 presents a discussion of the importance of the volatility in the financial context and describes its common treatments. Sections 4 and 5 define and describe copula modeling and its financial applications. Section 6 describes the proposed methodology and applies it in a Mexican natural gas investment project. The results from this valuation are shown in Section 7, while Section 8 presents final conclusions and remarks on this matter.

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Real Option Analysis: Definition, Properties and Requirements

Introduction to Real Option Analysis (ROA)

When addressing the valuation of investment projects, there is a generalized trend to apply traditional approaches based on the discount cash flow (DCF hereafter) such as net present value (NPV hereafter) and decision tree analysis (DTA hereafter). Even though these techniques have been criticized because their lack of adaptation to the existent business environment, they are still the most used tools of investment project valuation; particularly the NPV. There is a problem that arises directly from the use of these techniques. The assumptions they follow barely relate to real business conditions, as they do not consider elements such as flexibility, uncertainty and irreversibility in the investment decision process.

As DCF techniques were originally designed to value passive investments such as bonds or stocks, their active management was not required. Consequently, they do not take into account the particular treatment of either the market or the manager's responses into the investment decision. Traditionally, business schools and managers of investment projects operate under the perception that investment decisions cannot he reversed or postponed. In other words, each project is a now-or-never situation–including irrevocable decisions on the future–and does not diverge from their intended result. In presence of an uncertain environment, making irreversible investment decisions is risky. This risk becomes even higher if, as in these approaches, the project's life is fixed, and the possibilities of changing the nature of it (like expansions, abandonments or shifts), in order to respond to unanticipated conditions, are not even considered. As a result, the DCF techniques systematically undervalue the investment projects. By rejecting some strategic investment decisions that could bring better results in the medium or long run for the company, the manager induces a miss-allocation of the company's factors of production.

The real options analysis (ROA hereafter) provides a framework of how the valuation of investment projects can be managed so to address the business conditions, and improves the results presented in the traditional valuation techniques; particularly the treatment of uncertainty and strategic thinking. Being able to adapt decisions as new information presents itself helps the manager to reduce the risk in which the company is incurring by continuing the investment project. Even though the ROA is perceived as a highly complex method in order to completely substitute the traditional ones, it is not until one understands their limitations, that the advantages of this approach become apparent.

The incorporation of option pricing theory into the valuation of investment projects has gained importance in both academics and practitioners over the last thirty years. The construction and development of the real option theory has rapidly expanded, and presents an actively growing field of research. It has revolutionized the way to acknowledge investment projects by explicitly integrating flexibility into the manager's decisions, and by providing strategic insights into the risks the company is incurring as new information is presented. Is not just a methodology but a new perspective of the dynamics of investment decisions. For this, the ROA represents a better alternative of valuation as it incorporates the effect of the abovementioned variables and can be applied in multiple real-world situations. Moreover, advances in technology have enabled option-pricing theory to flow out of the academic environment and permeate into the companies' valuation perspective.

Review of Traditional Valuation Methods

DCF: Characteristics and Assumptions

An investment valuation process tries to generate an extrinsic monetary or intrinsic strategic figure in order to determine the future profitability of the project. The calculation provides the manager a decision tool to select whether to undertake an investment or not. Traditionally, three approaches are used: the market, cost, and income perspective. The first one focuses on the comparability of a financial asset in the marketplace so to establish a corresponding price for the project that derives from market forces. The second approach focuses on the total amount of costs in which the company will incur if they decide to replace or duplicate the asset. The third approach, by far the most common, uses a potential profit perspective in order to forecast, by discounting net,[1] free cash flows to present value, *i.e.* the future income of the company as a result of the investment decision. This perspective is known as the DCF valuation approach and its objective is to determine the present value of a project, and decide whether to undertake the investment or not.

The traditional DCF approach to valuing investment projects involves the determination of a discount rate that reflects the risk of the cash flows and defines their expected result in present terms. Typically, this is made through a firm-specified hurdle rate, a weighted average cost of capital (WACC) calculation or a risk-adjusted discount rate. It represents a widely accepted and almost unmodified tool,[2] for the past two decades, to calculate an expected present value and decide whether if a project can be undertaken or not. Its generalized use comes directly from its practicality and easiness to apply. Under this approach, investment projects are passively managed. As a result, no subjective considerations, such as accounting conventions or risk profiles from investors or managers, are made. Therefore, the criterion considered to invest is clear, consistent and homogeneous for all types of projects. This presents economically and quantitative rational results that are relatively simple to explain and to apply as it considers a risk-adjusted, cash flow based, and a multi-period viewpoint.

---

[1] Cost related to the implementation, acquisition, and development of the asset are deducted.

[2] Apart from capital asset pricing model (CAPM).

The main assumptions the DCF approach follows are:

- Now-or-never investment decisions with an individual project perspective.
- Cash flow streams are fixed, hence predictable and determined for the future.
- The discount rate is proportional to non-diversifiable risk and all risks are accounted in it.
- Known factors that could affect the outcome of the project are already considered in the valuation.
- Unknown, intangible or immeasurable factors are valued at zero.

In general terms, the traditional DCF approach yields good valuation result if a precise estimate of the future cash flows can be made and the time horizon is short. However, these characteristics are not commonly founded on real life situations. Investment decisions can be postponed in order to wait for better external and internal conditions. A deterministic perspective for the cash flows cannot be established due to the stochastic and risky nature of the financial tools used in the investments. Some benefits for companies are intangible and the factors involved in each project cannot be entirely considered, since sources of risks are constantly evolving. Finally, active management is, in fact, a common activity in companies along with project interconnection. To sum up, in a stochastic environment, presented in the use of a deterministic model, may potentially lead to miss valuations of investment projects. For that, a brief description of the valuation process involved in the NPV and DTA will be made in order to point out the specific disadvantages they present in terms of the ROA.

Net Present Value

Resulting from the DCF approach, the NPV follows a basic idea: to accept projects for which the sum of the expected profits or cash flows are positive, rejecting those that do not meet the criteria. A representation on how the valuation is made is described in the following figure.[3]

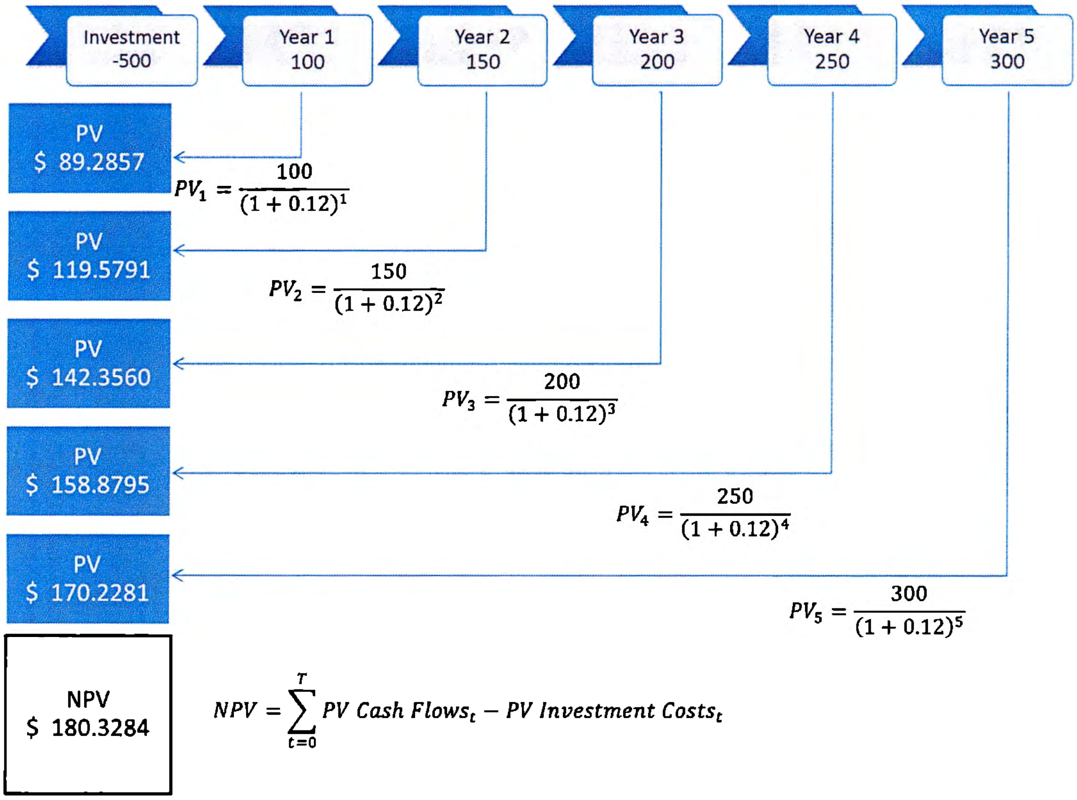
Investment
-500
Year 1
100
Year 2
150
Year 3
200
Year 4
250
Year 5
300
PV
$ 89.2857
PV
$ 119.5791
PV
$ 142.3560
PV
$ 158.8795
PV
$ 170.2281
NPV
$ 180.3284
$PV_1 = \frac{100}{(1+0.12)^1}$
$PV_2 = \frac{150}{(1+0.12)^2}$
$PV_3 = \frac{200}{(1+0.12)^3}$
$PV_4 = \frac{250}{(1+0.12)^4}$
$PV_5 = \frac{300}{(1+0.12)^5}$
$NPV = \sum_{t=0}^{T} PV\ Cash\ Flows_t - PV\ Investment\ Costs_t$

Figure 1: DCF approaches, NPV.

As shown, the expected cash flows are discounted by a specific rate on each period (12%). The present values of the expected cash flows are added, and then a subtraction of the investment costs is made in order to determine the NPV of a project. A common consideration is that, if chosen correctly, the discount rate used to establish the cash flows, not only represents the opportunity cost of capital for a particular project, but it also reflects its systematic risk. Therefore–for this valuation to work effectively–two issues need to be addressed: how to determine the expected profits and costs of the project; and how to choose the discount rate that will be used. The typical treatment for these issues is to calculate a statistical forecast of the flows, and to estimate the WACC and the Internal Rate of Return (IRR=23.29% for this example). A particular concern that arises in this kind of treatment is that both present values, cash flows and investment costs, are discounted using the same rate, usually, the same WACC rate. The second one should consider a risk free rate instead, because the market's rate will not compensate the company for incurring on private risks.

---

[3] The presented values have merely an illustrative purpose; particular models are developed to forecast the cash flows and the discount rate.

It becomes clear that, in order to follow the NPV rule, two main assumptions must be presented. First, a fixed operating strategy in which a company undertakes and concludes a project is anticipated; no contingencies in the project's life are considered. This represents a disadvantage, as they will be exposed to symmetrical uncertainty; that is, they cannot amplify the profits if conditions are favorable, or reduce the losses if adverse. Another consequence of this perspective is that it ignores the capability of the project to create future cash flows. Secondly, investment decisions are either reversible or, if irreversible, it is a now-or-never proposition. This is rarely presented in real life, as in most projects investments are irreversible and also possibly delayed.

This approach cannot include the asymmetry introduced by managerial flexibility, leading to an undervaluation of the project and possible erroneous investment decisions. Modifications to the initial considerations have been made based on the idea that the NPV rule needs to be adapted to include uncertainty. One type of effort is the NPV with scenarios;[4] the structure is practically the same as the static NPV, but it now considers the different paths that the project can follow and a probability for each to happen. Even though it incorporates uncertainty, every possibility path remains fixed on a particular expected outcome. Since it is not possible to choose between scenarios or to mix results between them, it is still considered a minor correction to the main concerns of this approach.

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[4] The representation is very similar to Figure 1 but it incorporates probabilities for the expected cash flow calculations; therefore, different NPV results are presented. Commonly, this version of NPV is known as the DTA approach because they both use the same information as inputs. However, the first one depicts, usually, situations in which moving between scenarios is not possible.

A second effort is presented in a dynamic perspective for the NPV.[5] It is based on a more careful treatment of uncertainty in the cash flows determination, and the discount rate election at each time and in each state. A higher involvement of the manager is demanded, as it is required to lay out all important future modifications, along with the probability of the original considered paths completion. Just as the static NPV, the estimations reflect the information available at the present time, but, in order to include a dynamic standpoint, the behavior of similar projects is measured as input for future cash flow and probability estimations. Nonetheless, this analysis still uses a single fixed discount rate within periods and makes subjective probability assessments. For that, no mayor correction is presented. A proper application of dynamic NPV may require a use of different discount rates at each period of analysis.

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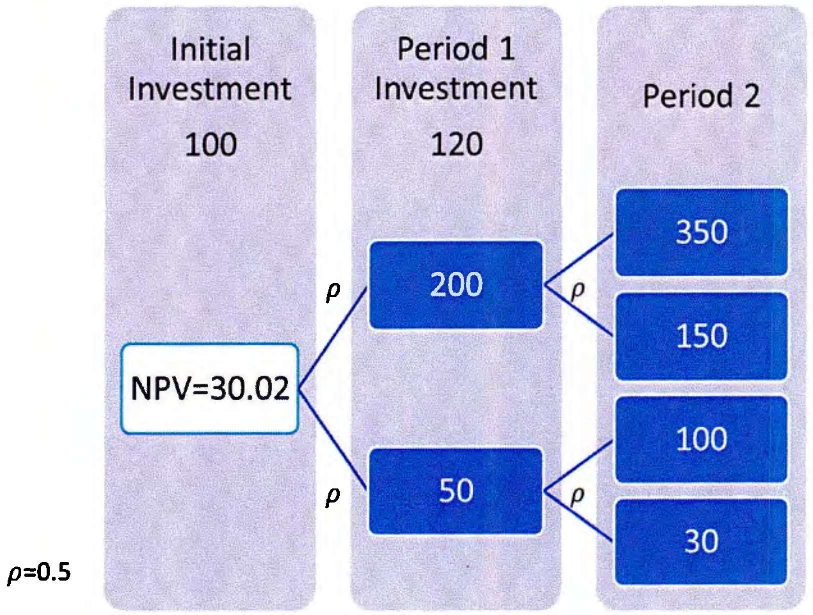
[5] In this case, the present values are recalculated as new information arrives.

Decision Trees

The DTA can be seen as another effort to modify the NPV approach, particularly in the flexibility issue. It decomposes the investment project into sequential states, resulting in a tree-like structure of decisions. The DTA differs from a dynamic NPV regarding the use of the discount factor as a decision tool rather than only an estimation instrument at each state of the project's life. In order to reflect the manager's period preference, a risk-free rate is employed to calculate the expected utility of the state, so that decisions can be used to determine the risk-adjusted value of an investment project. In an uncertain environment, this perspective gives the opportunity to trace an optimal decision strategy, for the company adapts as a response to changes both in the external and internal conditions. A representation on how the valuation is made is described in Figures 2 and 3.

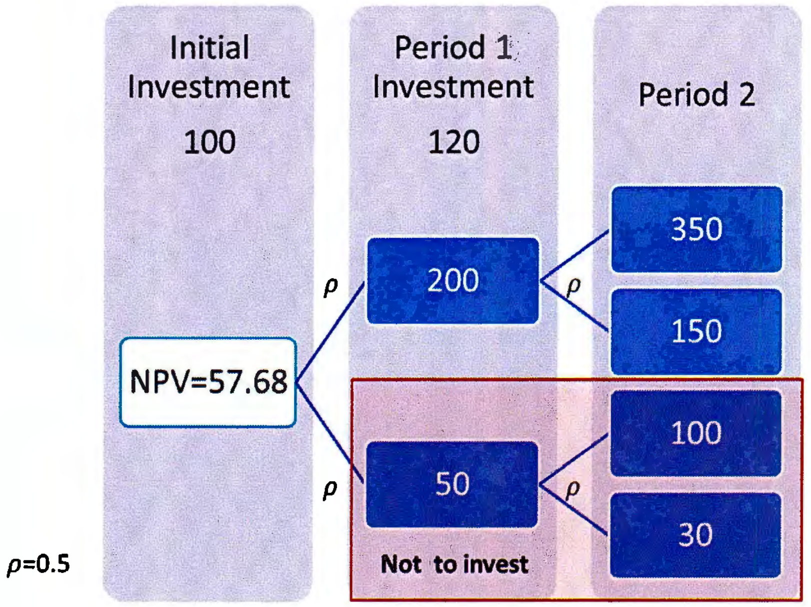
Figure 2 shows a DTA when no flexibility is considered into the investment decision. That is, the manager pre commits to undertake either both investments or none of them at the beginning. Under this perspective, with a discount rate of 12% and a probability of occurrence of 0.5 in each period, the NPV is 30.02. As shown in Figure 3, by considering flexibility, the payoffs will rationally imply that no investment will take place if the environment is not convenient. Therefore an expected payoff of zero is allocated in the "not to invest" area. With that, it is obvious that a higher NPV can be reached; it is 57.68 under the same discount rate and probability structure.


Initial Investment
100
Period 1 Investment
120
Period 2
NPV=30.02
ρ
200
ρ
50
ρ
350
150
ρ
100
30
ρ=0.5

$$PV_1 = \frac{(0.5 * 200) + (0.5 * 50) - 120}{(1+0.12)^1} \qquad PV_2 = \frac{(0.5^2 * 350) + (0.5^2 * 150) + (0.5^2 * 100) + (0.5^2 * 30)}{(1+0.12)^2}$$

$$PV_1 = 4.4643 \qquad PV_2 = 125.5580$$

Figure 2: DCF approaches, DTA with no flexibility.


Initial Investment
100
Period 1 Investment
120
Period 2
NPV=57.68
ρ
200
ρ
50
Not to invest
ρ
350
150
ρ
100
30
ρ=0.5

$$PV_1 = \frac{(0.5 * (200 - 120)) + (0.5 * 50)}{(1+0.12)^1} \qquad PV_2 = \frac{(0.5^2 * 350) + (0.5^2 * 150) + 0}{(1+0.12)^2}$$

$$PV_1 = 58.0357 \qquad PV_2 = 99.64.92$$

Figure 3: DCF approaches, DTA with flexibility.

It can be seen that the DTA is an advantageous tool to depict strategic pathways that a firm can take. It obligates the manager to acknowledge, by graphically stating a decision road map, the interdependence between the initial and subsequent investment choices. The application of this approach can range from a simple expected value calculation, to sophisticated Bayesian probability updating methods. This is because the valuation of each node is determined by the calculation of the expected outcomes for the cash flows based on their probability of occurrence, and a discount rate for the particular period. Even though the DTA addresses the flexibility issue better than the NPV, it introduces subjectivity matters into the investment decision; how to assess probabilities and discount rates along with the manager's preferences at each node. Cash flow values and probabilities are based again on the available information at the time, but a critical problem in how to determine the appropriate discount rate occurs under this perspective.

If a constant risk adjusted discount rate is used throughout the project's life, it is implied that the risk effects in each period are constant and uncertainty can be resolved continuously. Due to the evolving nature of the risk factors, this clearly cannot be considered. Moreover, when either a change of conditions is presented or a managerial decision is made, a modification of the future cash flows along with the risk structure of the project occurs. Therefore, in order to correctly apply the DTA valuation, different discount rates should be calculated in order to consider the effects of each possible managerial strategy and states of nature at each time periods. As a response to it, simulation tools are used to account possible paths of behavior. Nevertheless, it is very difficult to allocate an optimal value if the terminal date is not proximate.

The estimation difficulties of the probabilities and the appropriate discount rate for each node represent the main source of error under the DTA analysis. Still, its implementation gives the opportunity to set the foundations for the ROA. As it will be described, DTA and ROA are closely related. In a general sense, ROA can be understood as the inclusion of modifying discount rates into the DTA approach. In order to fully understand how the properties of the ROA approach enhance the results from traditional DCF methods, a clear description of their main disadvantages must be included.

Disadvantages

The set of disadvantages of the traditional DCF approaches are derived from the conditions for which they were initially design to valuate. Originally, the objective of these approaches was to value financial passive investments such as stocks and bonds. As they do not present modifications through their life, the assumptions were completely followed, leading to a precise, easy, and concrete valuation tool. The problems arose when the methodologies were implemented into investment project valuation as their nature and environmental characteristics do not correspond to the established assumptions. The DCF approaches are not effective as intended due to their inability to work into new business conditions; they are characterized by strategic investments and uncertainty under a dynamic environment. In the better case scenario, traditional DCF techniques roughly yield a first rough estimation of flexibility into the valuation process. However, is almost impossible for them to account for effects such as competition and interaction of multiple investment options.

Authors such as Myers, Trigeorgis, Mason, Kulatilaka, and Ross had noted various defects of the DCF approaches. These focus particularly in two directions: the overlook of strategic thinking, and the manner in which the assumptions prevent the incorporation of flexibility into the valuation process. The general perspective is that, under traditional valuation methodology, the discount cash flows do not incorporate intrinsic attributes of the underlying assets or investment opportunities they can generate over time. The final result is, as the discount rate is not able to reflect changes in the risk structure, the DCF approaches present an undervaluation of investment opportunities that might lead to myopic decisions, underinvestment and loss of competitive advantage.

Ross (1995) pointed out that the traditional NPV rule is limited to the cases in which is a now-or-never investment opportunity with no other alternative. His critical consideration is that, in most of investment projects, they can, at least, compete with themselves in a delayed period. Therefore they need to be treated as a series of investment options rather than a single-staged decision. Myers (1984) describes that the finance theory used in the traditional DCF methods, presents a significant gap that limits the valuation of investment decision that present strategic considerations. On the same direction, Kulatilaka (1995a) describes that traditional DCF approaches are incapable to reduce risk incurrence as companies are unable to trace strategies that accurately manage uncertainty and flexibility effects on the investment decisions. Trigeorgis and Mason (1987) pointed out that, in order to incorporate strategic thinking and flexibility, and the methodology should be improved. With that, the ROA can be understood as a methodology correct version of the DTA approach that will be better suited to value operating and strategic options.

Since the DCF approaches do not permit the project to adapt in response to unexpected market developments, they make implicit assumptions concerning a particular expected scenario with permanent characteristics. Moreover, they are unable to incorporate variance across the different scenarios. As most of investment opportunities are characterized by uncertainty and irreversibility, standard valuation tools such as NPV and DTA do not work effectively in this situation. The main potential problems presented in the utilization of DCF approaches as valuation tools can be summarized as follows:

- More effective in short run and, somewhat, deterministic investment decisions.
- Imposes a constant and skewed nature for the discount rate throughout the project's life.
- Does not consider, as part of the valuation, assets that currently do not produce cash flows but in might as well do in the future.
- Methodological errors inherent to the forecast process use to estimate the life of the project, the future cash flows and probabilities of the different states.
- Does not explicitly address the difference of discount rates for variables with market risks (market risk-adjusted) from the ones that present private risks (risk-free).

Definition of ROA

Mun (2006) presents the most comprehensive and yet accurate definition of the ROA, as he establishes that it is a technique that systematically incorporates financial theory, economic analysis, management science, decision science, statistics, and econometric modeling into the application of the option pricing theory as a valuation tool of real assets. He does this under a dynamic business environment characterized by uncertainty, flexibility, and strategic investment decisions. He develops an eight step framework for its implementation that contains elements such as time-series and regression forecasting, real options modeling as well as portfolio and resource optimization.[6] The intention is to internalize the business conditions into the investment project valuation process. It drops the rigid assumptions made on traditional DCF approaches, and provides a sound statistical tool that identifies multiple decision pathways and optimally selects one. This, as a consequence of the effects of uncertainty, combined with irreversibility and flexibility in the prioritizing and election of a strategy path that can be modified, period after period, as new information appears and feeds the model back.

Trigeorgis (1993a), in order to take advantage of the simplicity of the NPV as a valuation tool and enhance its properties by the ROA benefits, introduces the concept of Expanded Net Present Value (ENPV hereafter) defined as the addition of the traditional NPV along with the value added by the active (dynamic) management of the product.

$$ENPV = NPV + Real\ Option\ Premium$$

The ENPV quantifies the value of options resulting from active (dynamic) management and can be understood as a collection of real options (call or put) that take the gross project value of the DCF technique as an underlying asset. For this approach to work, two conditions are needed. First, the expansion has to be viewed as an option, that is, the NPV has to be compared to an optional ENPV accounting for the additional market value that comes from the flexibility. Secondly, in order to determine the option premium, an analogy between financial and real options has to be established; along with a proper clarification of assumptions that will be used. Special attention will be paid to these points in the following sections.

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[6] For a complete description of this proposed framework please refer to chapters 4-8 in Mun (2006).

Types of Real Options

Real options represent investment opportunities to be made or modified in the future depending on each period's economic, market and company's conditions. At the established date, the company holds the right–not the obligation–to execute certain investment decision based on the comparison of the cash flow value of an underlying asset against the cash flow value of an exercise asset. That is, by including the new available information of each period, the decision will be reconsidered more accurately than in previous ones. Different types of real option exist as a result of the diverse natural characteristics of the investment projects and the dynamics they can follow. In other words, almost any kind of managerial flexibility can be designed and understood in the ROA. The most representative types are:

- **Option to defer**, also known as learning options, are applied when the investment opportunity can wait a specific amount of time in order to see if future conditions are favorable enough to undertake the project later on.[7] The intrinsic value of this option is:

  $$c_d(S_T, T; K) = \max(S_T - K, 0)$$

  Where S_T is the value of the underlying asset, T the decision time and K the investment cost.

- **Option to switch inputs or outputs** are applied when, due a change on the market conditions, the company has the opportunity to either use the same factors of production in order to produce another good or service, or produce the same good or service but with different factors of production.[8] The intrinsic value of this option is:

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[7] Suggested classical readings on this type are Ingersoll and Ross (1992), Paddock, Siegel and Smith (1988) and McDonald and Siegel (1986).

$$c_s(S_T, T) = \max(S_{2T} - S_{1T} - K, 0)$$

Where S_{1T} is the value of the underlying asset with the current structure at time T, S_{2T} is the value of the underlying asset with the alternative structure at time T and K the investment cost associated with the switch.

- **Option to abandon,** applied when market or economic conditions are not favorable anymore and, in order to stop the loss, the company has the opportunity to sale their assets in a secondhand market.[9] The intrinsic value of this option is:

  $$c_a(S_T, T) = \max(S_T, V_T)$$

  Where S_T is the value of the underlying asset at time T and V_T is the salvage value of the project.

- **Option to alter operating scale** are applied when, based on changes on the market or economic conditions, the company has the opportunity to enlarge or reduce the size of the project in order to fit the new environment. Some examples of this type of options are to expand, shutdown, contract, or restart operations.[10] The intrinsic value of some of this options are:

  Expansion $c_e(S_T, T; \alpha, K) = \max(\alpha S_T - K, S_T)$

  Contraction $c_c(S_T, T; \beta, N) = \max(\beta S_T + N, S_T)$

  Temporary shutdown $c_x(S_T, T; X_T, C, a) = \max(S_T - X_T - a, S_T - C - a)$

---

[8] Suggested classical readings on this type are Kulatilaka and Trigeorgis (1994) and Margrabe (1978).

[9] Suggested classical reading on this type is Myers and Majd (1990).

[10] Suggested classical readings on this type are McDonald and Siegel (1985), Trigeorgis and Mason (1987), Brennan and Schwartz (1985) and Pindyck (1988).

Where α and β represent an expansion and contraction proportion of the project with a cost of K or saving of N. X_T represents the variable costs in which the project will incur with its normal activity, a the fixed costs and C is the cost derived from the temporary shutdown.

- **Time-to-built option**, also known as compound option or staged investment, is applied when a series of investments are planned and the company has the opportunity to manage and evaluate them as individual projects, while its subsequent value depends on previous stages.[11] The intrinsic value of this option is:

$$c_{com}(S_{T_1}) = \max\left(K_1, c(S_{T_1}, T_2 - T_1; K_2)\right)$$

  Where T_1 and T_2 are the decision times for project one and two respectively, K_1 and K_2 the investment costs associated with each project. Note that $c(S_{T_1}, T_2 - T_1; K_2)$ represents the intrinsic value of another option.

- **Growth option**, also known as interproject compound option, may be applied as part of a strategic decision opportunity for the company in which the original project structure is modified in order to generate new products or processes from it. This type of option is usually a result of a research and development scheme.[12]

- **Multiple interaction options** can be applied when the company has the opportunity to combine several of the abovementioned real options. In this type of option, a strategy to hedge may be accompanied by one to enhance the production capability. It is noteworthy that the combined option value may defer from the addition of them separately.[13]

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[11] Suggested classical readings on this type are Carr (1988), Majd and Pindyck (1987) and Trigeorgis (1993).

[12] Suggested classical readings on this type are Pindyck (1988) and Kester (1984, 1993).

[13] Suggested classical readings on this type are Brennan and Schwartz (1985), Kulatilaka (1995) and Trigeorgis (1993).

Properties Considered in ROA

The main elements needed to incorporate managerial flexibility in the investment project valuation are uncertainty, irreversibility, and discipline. As it will be pointed out, the addition of flexibility is the central advantage of the ROA over the traditional DCF approaches. A brief description of the properties is presented to illustrate its benefits.

Flexibility

A static valuation typically results in a symmetric risk profile that undervalues the project as the probability structure of positive and negatives outcomes remains unmodified through the project's life. Flexibility, in context of investment valuation, is understood as the situation in which the project offers the company the option to adjust the course of the investment decision depending on new internal or external conditions. This perspective gives the sense of sequential modes as it provides the opportunity to switch the investment paths, at given costs, when necessary. With that, the risk profile of the project will be skewed to the right; tending to positive outcomes of the project, hence, provides a higher valuation. Inflexible valuation approaches most of the time will present a value below zero as a result of their incapability to adapt to new information.

Much of the research work, in terms of valuation approaches, has described the beneficial effects of the inclusion of flexibility in the process of investment project valuation. Kulatilaka and Trigeorgis (1994) showed that the value of a flexible project could be seen as the value of a rigid project plus the sum of values of the options to switch in future periods; they also highlighted that there is an increase in value of the project derived from the possibility of having potential investment choices in the future. Kulatilaka (1995a) presented a general model of flexibility under a dynamic programming framework that illustrates how, by incorporating ROA, we can eliminate the need to deduce risk-adjusted discount rates. The benefit from this approach comes in the enhanced ability of the company to cope with uncertainty; characteristically presented in flexible investment environments such as growth, waiting-to-invest, time-to-build, abandonment, and shutdown options.

Dixit and Pindyck (2000) developed a continuous-time model of capacity choice, in a stochastic environment, where the company has narrow possibilities to expand or contract their activity. They noted that, when flexibility is considered into the valuation process, the project could be seen as call or put option on incremental inputs of capital that affect the future investment decisions of the company. In the same direction, Brekke and Schieldrop (2000) examined how operating flexibility in the decision of fuel technologic development, affects the optimal timing of investment. They established that as uncertainty usually leads to the delay of the investment decisions, when flexibility conditions are considered, earlier investment might be optimal as uncertainty of future outcomes can be solved and added to the investment decision's course.

There are only two situations in which flexibility does not prove its advantages: when there is perfect information of the company and market future conditions, that is, when the optimal path of the project is predetermined; and when the investment decisions are completely reversible. Under these conditions, the valuation process is simply reduced to a discount of future cash flows with the riskless interest rate, being the DCF approaches perfectly suited for that. Since most of the investment environments clearly do not present these characteristics, any technique that is used to quantify the investment project value needs to incorporate flexibility in an explicit manner. Therefore, the ROA approach must be used when the sources of uncertainty are clearly identified, and the irreversible conditions considered. Following this perspective, the effect of flexibility in the development of a managerial strategic thinking can be understood easily. Furthermore, it can present a defensive–when looking to downsize the losses–or offensive–when exploiting the upside potential–character in the way it enhances the company's performance and provides strong arguments in favor of its long-term survival under an environment characterized by uncertainty and irreversibility. For that, an explanation of their properties is offered.

Uncertainty and Irreversibility

In the economic environment, as in most situations in life, there is an ignorance of the future behavior of the present variables. Even though sophisticated mathematical tools are used, no forecast can accurately determine a specific result. In the investment project context, when a decision is taken, a non-diversifiable risk exists and, if we want to incorporate uncertainty in the valuation process, the best we can do is to assess the probabilities of the alternative outcomes that can be achieved. Along with the establishment of a specific discount rate, the traditional DCF approaches use scenarios to trace the possible results, and value the project with their constant parameters. This approximation clearly undermines the value of a project as it does not incorporate the dynamic of uncertainty into the valuation process.

Risk can be controlled but never eliminated, that is, although uncertainty may be solved throughout time, the exposure to unknown situations holds; hence the incurred risk. As uncertainty requires managers to assess and account for risk with a more sophisticated approach, the ROA becomes very attractive. It creates managerial flexibility as it incorporates the dynamic of uncertainty when new conditions and information are presented period after period. This perspective has clear implications on an active management, along with strategic thinking. With them, a greater value will be allocated on investments that create options in the future rather than the ones that focus on undertaking them in the present. As long as contingencies keep incurring in the probability that the investment presents losses, the option to wait will be valuable.

An investment is considered irreversible when its initial/installation cost cannot be recovered if the company tries to take back that decision. This idea is commonly exemplified in the definition of the sunken costs undertaken in each project. The recognition that most investment decisions are irreversible, gives the manager the perspective that an optimal choice of timing is fundamental. A sense of postponement possibility is created and, with it, an option approach can be understood. Irreversibility and the possibility of delay are, in fact, very relevant characteristics in the investment environment.

Irreversibility is mainly caused by the specificity of investments and the inefficiency of secondary markets for investments goods. This happens because, in most investment projects, sunk costs are rarely recovered entirely as they are product, industry or company specific: they are originally designed to fit the particular requirements, to produce a good or service, from a specific company. Moreover, even if they are not custom-made investments, still there is a sunk cost notion involved because is very difficult to recover the total cost if the assets are sold on a used-product market. Another source can be found in institutional arrangements, government regulations or differences in corporate culture.

The postponement possibility created by irreversible investments can be perceived as financial call option, because a company can decide whether to undertake the investment today or in the future: its cost is here understood as the exercise price and the project, with a specific value, as the purchased asset. Recognizing this analogy can help managers to acknowledge the crucial role uncertainty plays in the timing of their decisions and the value of the projects. When a firm decides to take a venture, it eliminates the option possibility in the future. A trade-off between the investment and the opportunity of waiting for new information to arrive in order to be included in the valuation process is made. This factor has to be acknowledged as an opportunity cost and must be included in in the value of a project. Dixit and Pindyck (1994) showed, with an emphasis on the option-like characteristics of investment projects resulting from irreversibility under uncertainty, how optimal investment rules can be derived from methods that have been developed for pricing options in financial markets. Particularly, they described that this opportunity cost is highly sensitive to uncertainty, as the changing market and economic conditions drastically affect the riskiness perception of the future cash flows; hence to the value of the project.

McDonald and Siegel (1986) demonstrated that the simple NPV rule, accept when the $V > I$, with V beign the value of the project and I the initial cost of investment, is incorrect most of the times because the valuation process does not include the effect of irreversibility and timing decision. As future values of V are unknown, an opportunity cost in today's investment decision appears. This effect of uncertainty needs to be incorporated as an additional element in the value of the project, so as to conceive a critical value, say V^*. The rule then needs to be modified for us to undertake the project if V is at least equal to V^* and exceeds I. If not, an undervaluation of the projects is presented most of the time.

Discipline and Strategic Perspective

The ROA approach is not only considered a tool to improve the accuracy of valuation, but the restructuration of a way of thinking and deciding in the investment context. The contingent decision perspective involved in it gives the manager an active and flexible role in the valuation of the investment project. Constant involvement needs an environment characterized by structure and rigorous analysis of the current situation and the future expected outcomes. Managerial flexibility can be then defined as the ability to react to unpredicted changes in order to reach or modify a goal, by adopting new tactics in response to new information. This provides a sense of strategic thinking in the course of the project and also of the company's future success. The manager and the valuation process must be disciplined if the work is to be effective. According to Leslie and Michaels (1997) there are four ways in which the discipline, involved in the ROA, improves the strategy of a company. It emphasizes the logic of strategic opportunism as it needs a constant comparison of every incremental opportunity derived from existing investments. It promotes strategic leverage by encouraging those that keep the company present as relevant actor in the market. It maximizes the acquirements of rights so as to have opportunities in the future. It gives more protection as no obligations are stated when adverse conditions are presented.

Discipline, in the ROA context, can be understood as the state in which three elements are present: the investment decision is structured based on the options it creates, it uses financial market inputs and concepts to either acquire options or lessen risk, and it takes into account all relevant information available. The DCF approach is clearly undisciplined as it is grounded on the value the project can create today instead of the one in the future; there is no extra information requirement rather than the initial, and focuses only on a particular discount rate fixed in the different periods. Most of the projected cash flows become positive toward the end on the forecast period providing an opportunity to manipulate the terminal value easily. The use of the ROA approach gives the manager a disciplined involvement in the project that creates two links between the strategic investment of the latter and the company's strategic vision. When analyzed from the top of company to the particular project, it helps to discover the uniqueness of the firm in terms of the aggregate value opportunity they enhance. On the opposite direction, it provides a structure on how to accumulate, the value and risk of each project, into the company as a whole.

The strategic thinking in the ROA does not reside only on a specific project or a particular company. Most valuation processes do not take into account either the interaction between projects or the effect of other companies' decisions. In the real investment environment, options are not usually isolated, and there is affection on the conditions derived from the different firms present on the industries. The ROA approach into strategic investing needs to be extended to a market structure perspective. Therefore, a comprehensive business strategy needs to be understood as a series of parallel options coexisting in a competitive market.

In the last years, special attention has been paid in this direction by applying an integrated ROA and game-theoretic industrial organization framework into the strategic investment context. With this perspective, the valuation process acknowledges the crucial impact the dynamic competitive environment has over the result of the investment project. Noteworthy work on this matter can be found in Smit and Ankum (1993), Grenadier (1996), Kulatilaka and Perotti (1998) and Smit and Trigeorgis (2001). Further work of these authors is considered fundamental in today's growing literature of game and industrial organization theory in the ROA context. A comprehensive synthesis of this research direction can be located in Chevalier-Roignant, et al. (2010).

The beginnings of the focus over the interaction between projects can be traced to the work of Brennan and Schwartz (1985) in which they determined the combined value of the options to shut down or abandon a mine. Even though they did not directly address the effect on the value that came from that interaction, this work pointed the way into its consideration in the ROA context. A notable expansion of the literature came after the work of Trigeorgis (1993, 1993a) over the nature and conditions in which individual options interact. He pointed out the importance of properly accounting it in order for the strategic and flexibility senses to be enhanced. Individual options can be acknowledged as part of an investment decision sequence; therefore, their value affects the ones of the following projects and conversely. One of his main findings is that the interaction leads to a non-additive valuation in real options such as deferral, abandonment, contraction, expansion and switching. Another key work on this matter is presented in Kulatilaka (1995) where he develops a numerical example to illustrate the interdependencies between two options, in a dynamic programming framework, and how their interaction affects their strategy and selection of the optimal exercise date. He pointed out that, even though the value of the project increases when considering the introduction of additional ones, the increment can be diminished or enhanced depending if the other option is a substitute or complement project. Childs, Ott and Triantis (1998) developed a real option model that examined the effect of project interrelationships, comparing sequential and parallel developments, over investment decisions and project valuation. Herath and Pak (2002) extended the binomial lattice framework to model a multi-stage investment as a compound real option when several uncorrelated variables exist. They developed a theoretical framework for estimating the volatility parameter of underlying assets using Monte Carlo simulation technique. Even though they use different sources of uncertainty, no attention is paid to its interaction only to their presence at different phases.

Advantages of ROA

Brennan and Trigeorgis (2000) described three types of models that can be used in the valuation process of investment opportunities: static, controllable cash flow, and dynamic. With this work, a very clear distinction is made on how appropriate is to use DCF techniques under specific characteristics. For the first situation, under a fully defined environment, the traditional techniques work as intended. In the controllable cash flow, when active management is considered, the usefulness of DCF approaches depends on their capacity to estimate the future cash flows and the probability distribution. However, they provide a very limited notion of how contingencies affect the project's risk structure. Finally, due the characteristics of the DCF techniques, is almost impossible to consider them an accurate dynamic tool as they do not account for active management and market structure effects. By incorporating uncertainty, irreversibility, and discipline into the valuation, the ROA must be understood as an enhanced version of the traditional DCF techniques. The same fundamental principles underlie both of them but, as ROA explicitly considers managerial flexibility, it provides a value-added insight to the investment decision making process that correctly adjusts for risk. Therefore, it provides an optimal functionality as a controllable cash flow or dynamic model.

In general terms, the main advantage of the ROA follows two directions: how it overcomes the obstacles presented under the traditional DCF techniques, and its capacity to quantify strategic implications into the valuation process. The following advantages represent the arguments to think that the ROA has the potential to narrow the breach between strategic management and capital market theory:

- Obeys the law of one price; *eliminates arbitrage possibilities.*
- Uses market information[14] as inputs in order to provide quantitative analysis for sensibility, uncertainty, and volatility; *eliminates estimation problems.*
- Combines the value of the financial and real options along with the manager's skills and the company's strategy; *promotes a comprehensive management.*

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[14] Such as future prices, the standard deviation of the return rate of an underlying asset' risk-free interest rates and equivalent probabilities.

- Values and makes a strategic distinction between the initial investment opportunities and the additional embedded in it; *eliminates the pre commitment notion of investments.*
- Flexibility, presented in multistage investments, is explicitly taken into account as it maps out the relevant courses the project can follow and identifies the optimal path to follow in each period; *enhances the scenario perspective.*
- Accounts for different levels of risk incurrence in the cash flow evolution; *avoids discretionary risk selection of the real assets.*

When to use ROA? Requirements and Conditions

Even though the advantages of the ROA–as a superior investment decision valuation approach in the presence of uncertainty and irreversibility–are clear in works such as Leslie and Michaels (1997), Luehrman (1998, 1998a), Amram and Kulatilaka (1999, 1999a), Dixit and Pindyck (1994, 1995), Copeland and Keenan (1998, 1998a), there is still a general reluctance of its use in the practitioner world. This can be explained either by its complexity, as not all managers master the mathematical tools needed in the approach, or because there is an accentuated use in the commodity markets such as gold, gas, and oil.

What most of the critics of this approach argue is that the examples and assumptions used in research and applications lack real life characteristics, as they are seen more as an academic exercise rather than a business decision tool. Particular attention has been paid in the complete capital market hypothesis which requires that the underlying asset of the option, or a perfectly correlated portfolio of assets, is traded in the financial markets such that the stochastic component of it can be duplicated. Noteworthy is the fact that the applicability of this approach cannot be left aside because of that limitation. If markets are incomplete, the ROA perspective obligates the investment manager to express its risk preferences towards the source of uncertainty.

The benefit comes as it promotes the constant involvement of the manager, with a continuous review of the financial market conditions as source of new information inputs in order to modify the project course. Even though there is no available solution to this problem, the ROA gives, at least, a better treatment of it when compared to traditional DCF approaches.

Another critique is that it is considered just as another methodology focused on the discretionary increase of the value as a justification to undertake higher risk investment projects rather than necessary ones. This is because, as sometimes argued, by letting the manager to get involved, classical capital budgeting fundamentals are contradicted, particularly the results obtained by DCF methods. It is important to remember that in order to have a scientific growth, new methodologies need to overcome previous weaknesses. For that, a new methodology cannot be critiqued by the lack of correspondence with new ones, but for its incapability to adapt to new conditions.

For that matter, a final common critique is found on the valuation process itself. As it is very difficult to identify all possible future options, the uncertainty handling may be limited by a numerical approximation. Also, most of the options do not come isolated; therefore a treatment of option interaction must be done. With that, most probably, the valuation process will not yield a closed-form solution. As a result of it, current research has emphasized the treatment of uncertainty and volatility effects over the valuation process. For that reason, it has to be understood that a series of basic requirements and conditions are needed in order exploit the advantages of the ROA.

First, a financial model must be developed, that is, an analogy with financial markets has to be presented and their information must be included in the valuation. Second, contingent investment decisions opportunities have to be offered and the concepts of uncertainty and irreversibility considered. That is, uncertainty has to exist and it has to affect the decisions of the manager as well as the results of the financial model. As not all industries are exposed to uncertainty to the same extent, this approach could lose power when valuing projects in low-uncertain industries. Third, the project permits updates and mid-course strategy modifications when actively managed. Managerial flexibility is particularly attractive when the DCF approach yields values close to zero as with the ROA perspective the result of the decision may change.

This approach, as any other, has its limits. Therefore, the results obtained by the ROA can differ from the theoretical optimal answers. Some examples of these limitations are: not having enough objective information of conditions and prices of the financial markets; a low trading volume on the underlying assets is normally presented; problems involved on how the manager chooses the underlying variables and proxies; and an emphasis on private risk rather than strategic thinking. Because of them, the ROA valuation has the challenge of fitting more commonly presented characteristics in the business environment such as large amounts of time periods, multiple interacting real options, and multiple uncertainties. Also a more extensive work has to be done so as to make practitioners, corporate managers, and other decision-makers understand and implement it in a better way. The crucial challenge is presented in how to transfer models developed for financial markets into actual investment decisions. For that, a brief discussion on how to address the analogy between them is developed.

Financial Options: Characteristics and Valuation

Analogy between the Financial and Real Options

The relationship between the financial and real options initiated with the development of the option pricing theory by Black and Scholes (1973), Merton (1973), and Cox and Ross (1976). They introduced the concept of pricing securities by arbitrage methods, based on the technique of risk-neutral or equivalent martingale pricing, and showed how to value a claim whose payoff is directly connected to the structure of the underlying stochastic process. Their model priced elements of the firm's capital structure, under Modigliani-Miller theorem's conditions, and led to the consideration of individual securities as options or contingent claims for the company. Therefore, under a risk neutral environment, they can be priced as such since the option is valued in relation to the underlying asset and, in principle, can be replicated synthetically. Even though equivalent martingale pricing techniques are appropriate for both financial and real options, it is more difficult to determine the equivalent martingale measure in the last case as the cash flows typically cannot be reduced to claims on traded assets. The conceptual analogy between financial and real options rests on their payoff and risk structure similarities. For that, if conditions exist for having a valuation process for one, it is feasible to do so for the other. Mun (2006) summarized the similarities and differences between them in Figure 4.

Assumptions for Valuing Real Options as Financial Options

Black and Scholes described the general conditions that must be assumed in order to value financial options under their model. If met, the seller of the option can mimic its payoff structure with a replicating portfolio consisting of a combination of other investment tools along with the underlying asset. The assumptions are:

- Markets are complete and frictionless.
- The risk-free interest rate is constant over the life of the option and is the risk-free rate of interest under a risk-neutral environment.
- Dividends are known in size and date.
- The underlying asset follows a known stochastic process.
- Investors are rational.

	Financial Options	**Real Options**
Maturities	Short	Long
Underlying variable	Equity price or price of a financial asset	Elements that affect the free cash flows
Manipulation	Cannot manipulate stock prices	Can manipulate option value through strategic decisions
Values	Usually small	Usually in millions/billions
Competitive or market effects	Irrelevant to its value and pricing	Drive the value of a strategic option
Markets	Well established, more than four decades	Recently established, within the last two decades
Solved by	Closed-form partial differential equations and simulation/variance reduction techniques for exotic options	Closed-form equations and binomial lattices with simulation of the underlying variables
Marketable, comparable and complete pricing information	Yes	No
Management role on the valuation	Inexistent	Fundamental

Figure 4: Comparison between Financial and Real Options.

It is clear that the most critical assumption concerns market completeness.[15] Therefore, in order to trace a valuation analogy between financial and real options, the keystone is to guarantee that the underlying asset for the real option, or another perfectly correlated portfolio, is being traded at capital markets. With it, the risk structure of the real option's underlying asset can be duplicated and, consequently, its value can be determined. This implies that–for the analogy to stand–the main concern is to identify a financial contract with the same risk structure as the real option's underlying.

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[15] Condition of presenting as many uncorrelated primary securities as sources of risk, along with their capability of being completely hedged by a replicating portfolio existing in the market.

The ROA can then be understood as an extension of the option pricing theory but with real assets. This because the characteristics embedded in the investment strategy, just as the ones presented on the financial contracts, must be also identified. Moreover, the multiple sources of uncertainty, distinctive of this approach, make the valuation process more difficult for the real assets than for the financial ones. Therefore, it becomes clear that for the valuation to work properly under the risk-neutral framework, the manager has to consider three elements: take into account flexibility, include all the relevant information from the financial market, and enhance strategic thinking. Only this can lead to a disciplined strategy that aligns the objectives from the management perspective with the ones from the shareholders.

Valuation Process

As noted, most of the traditional valuation methods have no simple correction to their disadvantages. Even though the ROA has limitations, still represents a better tool to value opportunities embedded in strategic investment. Following the analogy, the real options can be perceived as financial call options due to the fact that companies decide whether to undertake the investment today or in the future. As most investment projects and industry conditions are unique, there is no fixed methodology to find a financial analogy. The best solution for this is to construct one from the available information at the financial markets. Therefore, the correct estimation of the following parameters, required in the option pricing theory, is fundamental for the ROA to work properly:

- Current value of the underlying asset, S

  Value at which the underlying asset is purchased; it results of an estimation of the present value of the expected cash flows related with the investment project. Typically, as described in Dixit and Pindyck (1994), they are modeled as continuous diffusions processes, where the risk follows a Brownian motion–also known as Itô diffusion process.

- Exercise price/cost, X

  Predetermined value at which the option can be exercised; it is a market equivalency to the present value of the expected fixed costs involved in the investment project. Even though, typically it is consider as a constant value during the lifetime, some models include factors such as contractual conditions or depreciation into its calculation.

- Cash payouts or dividends, D

  Amounts paid regularly to stockholders during their holding of the option. In terms of the ROA, they can be considered as the opportunity cost incurred in the preservation of them as the option is not executed and prevents a possible cash inflow. On this matter, McDonald and Siegel (1994) demonstrated how to establish the value of D through a dividend yield or rate of return and its final implication into the option valuation process.

- Time for the decision/expiry, t

  Period in which the option can be exercised or the investment project is valid. It depends on the characteristics of the particular company or the industry. Also, in terms of the valuation process, δt is defined as the decision steps involved in the calculation within t. That is, how many stages or reviews are going to be considered during the project's life. The work of McDonald and Siegel (1986) explores the implication over the value of the option, derived from the selection of an investment decision time, when the value of the project and its investing costs are stochastic.

- Risk-free rate of interest, r

  Yield of a riskless asset or security with the same t as the financial or real option. Its importance in the model lies on the capability to replace, if the complete market assumption stands, the underlying's expected rate of return. The latter is typically assumed to be known and constant in the financial options; however, in the case of real options, this assumption only stands for short periods. When longer periods are presented it has to be modeled as a stochastic variable. Ingersoll and Ross (1992) developed a stochastic model to show the effect of interest rate uncertainty over the cash flows, investment timing and their consequences over macroeconomic decisions.

- Volatility of the underlying asset, σ

  Standard deviation of future cash inflows' growth rate associated with the stock. In the financial context, it represents a measure of the unpredictability of future stock price movements. In terms of real options, this element measures and explains the risk incurrence inherent to the stochastic process of the underlying source of uncertainty. Most volatility models use historical market information to roughly perform its estimation. Davis (1998) proposed a fixed-output model of production in order to provide closed-form valuation equations that lead to a volatility and dividend yield calculation. From the mentioned variables, this is the most difficult to estimate. Therefore, a review of the most relevant volatility models is presented in a following section.

In practice, most real option problems must be solved using numerical methods. In terms of the valuation procedure, generally there are two types of numerical techniques that are used: the ones that directly approximate the stochastic process of the underlying asset, and those approximating the resultant partial differential equations (PDE's). The most representative solutions under the ROA perspective are: path-dependent simulation (typically Monte Carlo), closed-form models, PDE's, and binomial/multinomial approaches. The advantage of the solutions is that they not only provide a value for the project, but also illustrate the optimal strategy to follow in the investment opportunity. Their selection relies on the project's characteristics; the most straightforward solution can be found in the binomial/multinomial approach while the Monte Carlo simulation and PDE's are more complicated methods.

Valuation Examples under ROA Perspective

The binomial option valuation model will be used to develop the following examples. This discrete time lattice-based model was developed by Cox, Ross and Rubinstein (1979) with the intention to provide a simple representation of the evolution of the underlying asset value and how it generates a change in the option's value. Is noteworthy that this multiplicative binomial model of uncertainty, for European options without dividends, converges with the Black–Scholes formula values as the number of time steps increases. It is neither the scope of this section, nor this work, to compare the abovementioned tools, but to illustrate how the valuation process is taken under this perspective; for that, five examples are described. First a standard European call option is presented, followed by four real option scenarios: option to abandon, to expand, to contract, and to choose from them simultaneously.

Figure 5 shows a five year evolution of an underlying asset, in this case a European Call Option, and the valuation of the option derived from it. Step 1 is developed by calculating an upper bifurcation $S_0u = S_0 * u$ and, correspondingly, a lower bifurcation S_0d. The calculations consider the following values for $S_0 = 100$, $\sigma = 20\%$, $\delta t = 1$ for 5 years, $r_f = 12\%$ and $X = 100$; yielding a value for $u = 1.22140$, $d = 0.81873$, and $p = 0.76679$.

Step 2 is a backward induction process which starts with a comparison of the final value of the underlying asset with the opportunity to execute the option. It follows the maximization rule between executing the option, in the final nodes, $S_0F - X$ or letting it expire. The intermediate values from the final to the initial are derived from the past nodes and the risk-neutral probability. For example, the node with the value 133.86 results from $133.86 = [(p * 171.82) + (1 - p) * 82.21] * e^{-r_f \delta t}$. This process is repeated until the first node, representing the option's value at 46.17, is reached.

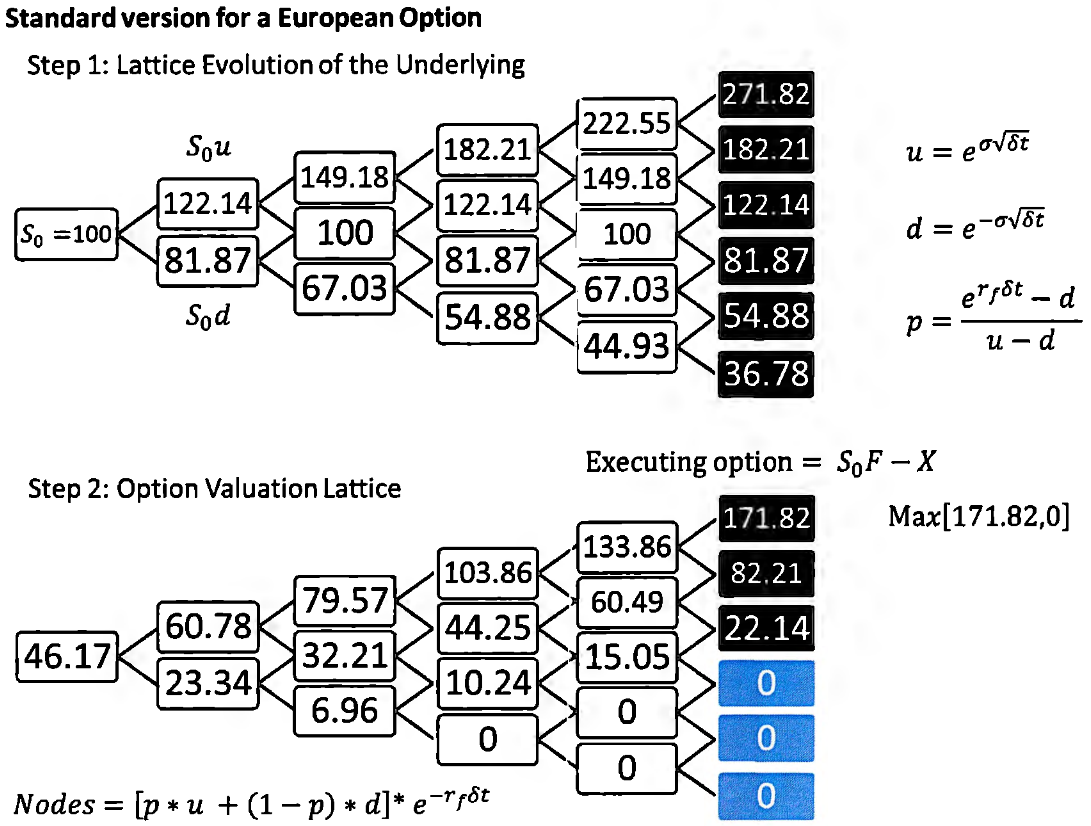

Standard version for a European Option
Step 1: Lattice Evolution of the Underlying
S_0u
S_0d
S_0 =100
122.14
81.87
149.18
100
67.03
182.21
122.14
81.87
54.88
222.55
149.18
100
67.03
44.93
271.82
182.21
122.14
81.87
54.88
36.78
$u = e^{\sigma\sqrt{\delta t}}$
$d = e^{-\sigma\sqrt{\delta t}}$
$p = \frac{e^{r_f \delta t} - d}{u - d}$
Step 2: Option Valuation Lattice
Executing option = $S_0F - X$
Max[171.82,0]
46.17
60.78
23.34
79.57
32.21
6.96
103.86
44.25
10.24
0
133.86
60.49
15.05
0
0
171.82
82.21
22.14
0
0
0
$Nodes = [p * u + (1 - p) * d]* e^{-r_f \delta t}$

Figure 5: Standard version of the ROA for a European Call Option.

Applying the same methodology and, for illustrative purposes, maintaining the same structure for the underlying asset, Figure 6 shows the valuation for a real option based on the opportunity to abandon a project or continuing it. In this case, the maximization rule lies on the comparison between the value of the project in the final nodes and a salvage value of 70. Calculations for intermediate nodes follow the same structure than Figure 5 but now each node needs to verify the maximization rule in order to allow previous decisions rather than to wait until the last period to do so. The abandonment option for this example is taken in year 4 if node A is reached, as no improvement can be expected in the consequent paths. All the others nodes provide arguments to continue with the project. The result of the valuation is $0.19 million ($100.19M-$100M).

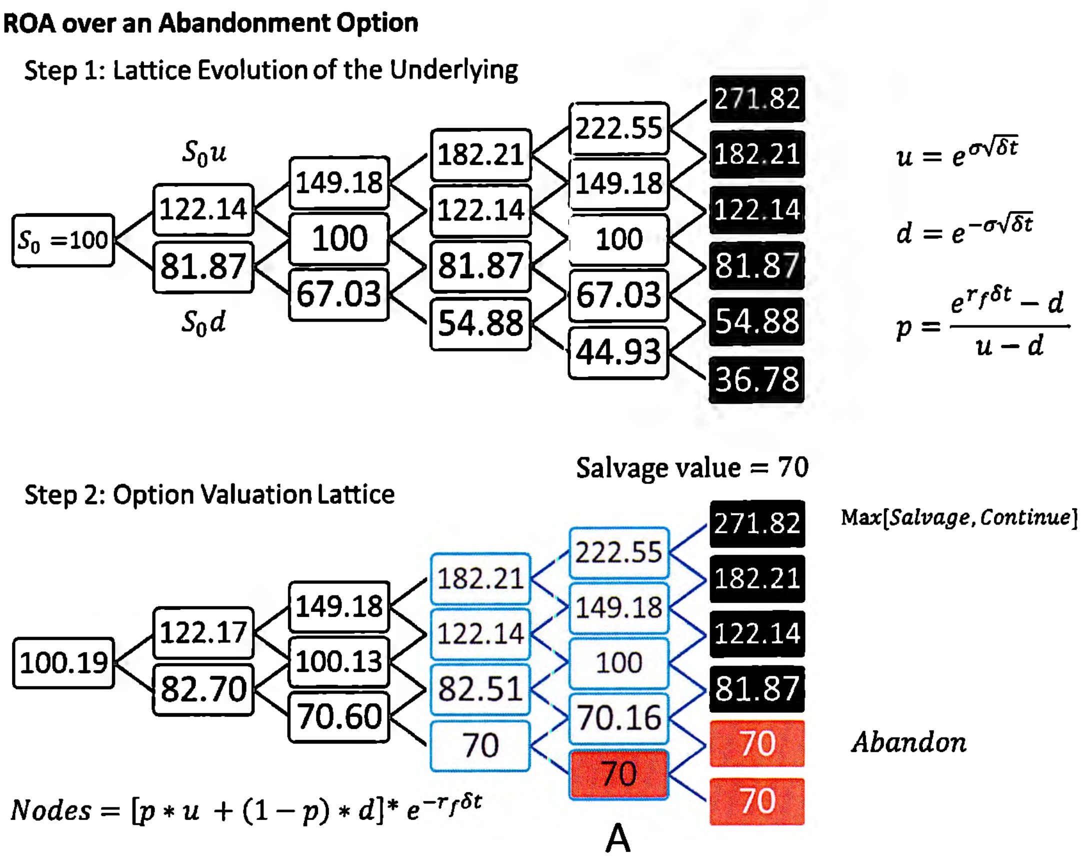
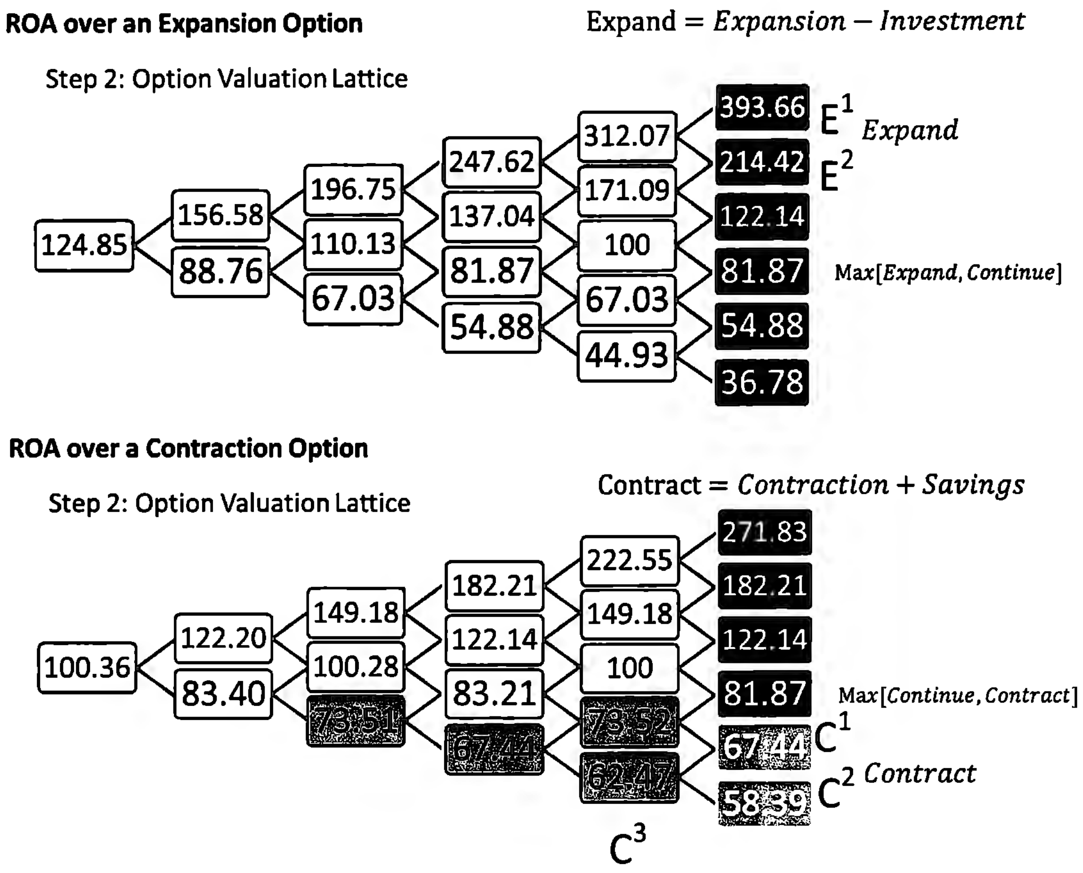

ROA over an Abandonment Option
Step 1: Lattice Evolution of the Underlying
S_0u
S_0d
S_0 =100
122.14
81.87
149.18
100
67.03
182.21
122.14
81.87
54.88
222.55
149.18
100
67.03
44.93
271.82
182.21
122.14
81.87
54.88
36.78
$u = e^{\sigma\sqrt{\delta t}}$
$d = e^{-\sigma\sqrt{\delta t}}$
$p = \frac{e^{r_f \delta t} - d}{u - d}$
Step 2: Option Valuation Lattice
Salvage value = 70
100.19
122.17
82.70
149.18
100.13
70.60
182.21
122.14
82.51
70
222.55
149.18
100
70.16
70
271.82
182.21
122.14
81.87
70
70
Max[Salvage, Continue]
Abandon
$Nodes = [p * u + (1 - p) * d] * e^{-r_f \delta t}$
A

Figure 6: Example of ROA valuation-Abandonment option.

On the other hand, Figure 7 shows two valuations: the opportunity to expand and the opportunity to contract. In both options, a specific factor should be established in order to determine how much capacity is estimated to increase (decrease) in terms of the current productive situation and the cost (savings) derived from taking the option. For the first case, the factor of expansion is $F_e = 2$ at a cost of \$150M, as for the contraction $F_c = \frac{1}{2}$ with a saving of 40M.



ROA over an Expansion Option
Step 2: Option Valuation Lattice
Expand = $Expansion - Investment$
124.85
156.58
88.76
196.75
110.13
67.03
247.62
137.04
81.87
54.88
312.07
171.09
100
67.03
44.93
393.66
214.42
122.14
81.87
54.88
36.78
E^1 $Expand$
E^2
$\mathrm{Max}[Expand, Continue]$
ROA over a Contraction Option
Step 2: Option Valuation Lattice
Contract = $Contraction + Savings$
100.36
122.20
83.40
149.18
100.28
73.51
182.21
122.14
83.21
67.44
222.55
149.18
100
73.52
62.47
271.83
182.21
122.14
81.87
67.44
58.39
$\mathrm{Max}[Continue, Contract]$
C^1
C^2 $Contract$
C^3

Figure 7: Examples of ROA valuation-Expansion and Contraction options.

Step 1 is omitted in Figure 7 as the underlying structure is unchanged. Step 2 now follows the maximization rule between continuing with the current conditions (calculation for intermediate nodes) and enhancing (reducing) the project, along with a deduction (addition) of the investment (savings). With that, the values for expanding follow the form $Expand = (F_e * S_0) - Investment$, and for contracting $Contract = (F_c * S_0) + Savings$. The resultant valuation establishes that, for the expansion option it is \$24.85M, while for the contraction option it is \$0.36M. The decision of expanding is made in nodes E^1 and E^2. In the case of the contracting option, C^1, C^2, and C^3 are the decision triggering nodes but the other ones marked in orange are preliminary stages that may give to the manager a sense that the trend would unlikely change to a better path; thus leading to a contract decision.

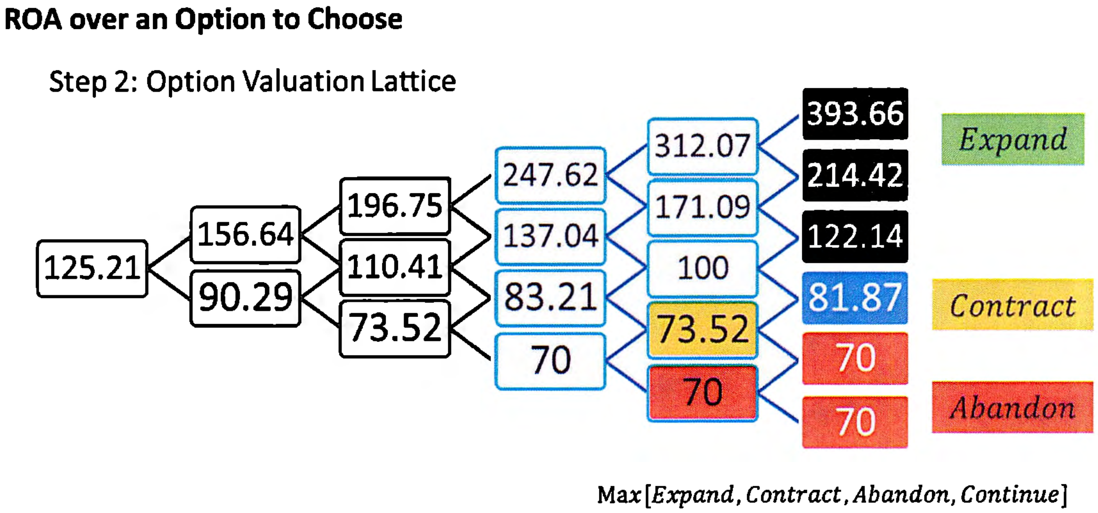
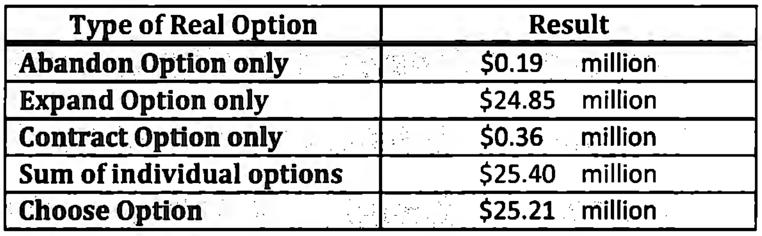

ROA over an Option to Choose
Step 2: Option Valuation Lattice
125.21
156.64
90.29
196.75
110.41
73.52
247.62
137.04
83.21
70
312.07
171.09
100
73.52
70
393.66
214.42
122.14
81.87
70
70
Expand
Contract
Abandon
Max[Expand, Contract, Abandon, Continue]

Figure 8: Example of ROA valuation-Choose option.

Finally, Figure 8 shows the Step 2 valuation if the four decisions are permitted: expand, contract, abandon, or continue. In this case, the maximization rule considers the value of the four possibilities at the same time. With this modification the manager can decide to expand (nodes in green), contract (orange), abandon (red), or continue (blue/white). The value under this panorama is $25.21M which is superior to the previous ones as Figure 9 illustrates.



Type of Real Option	Result
Abandon Option only	$0.19 million
Expand Option only	$24.85 million
Contract Option only	$0.36 million
Sum of individual options	$25.40 million
Choose Option	$25.21 million

Figure 9: Comparison between ROA valuation examples.

It is important to note that the sum of individual options is clearly bigger than the Option to Choose. However, is an incorrect value, as it does not consider their interaction in the same panorama. Under the "addition" case, the firm can have arguments to decide whether to contract or abandon (or any combination) in the same node at the same time. This clearly lacks sense since the valuation does not capture the mutually exclusive and independent nature of the specific options. This can only be reached if the Choose Option panorama is considered. Even though the biggest valuation, under the described conditions in the examples, is reached with this option, it still presents limitations as more precise considerations are needed, particularly in the interaction area. This and other limitations are described in the next section.

Limitations of the Option Pricing Analogy and Valuation Process

In order to fully apply the financial analogy along with an acceptance of the valuation into the practitioners' world, special attention must be paid regarding the following subjects:

- Parameter treatment

  Probably one of the main arguments against the use of the valuation under the ROA perspective is that, in most textbook or papers' examples, the inputs are hypothetical parameters that vaguely represent real information of the market. In most of them, the methodology is explained with detail but no special mention is presented on how to obtain and work with the needed parameters.
- Time delay effect

  A typical assumption in valuation examples, for both financial and real options, is that the decision to execute the option takes place immediately. In the financial environment this can be done most of the time, but in real options, the assumption is rarely seen. The idea is clear in, for example, options to expand. There is a lag between the moment when the decision takes place and when the building is ready; and, for valuation purposes, when it is capable to start generating cash flows.
- Urgency of decision

  The valuation must also reflect when investment projects are mutually exclusive and present different priority levels for the company. With that, the manager can have a tool that distinguishes between those that need an immediate (expiring) decision and ones that can be deferred.

- Competitive structure

  There is a need for the valuation tool to incorporate the market structure and its competitive characteristics. Most of the real options base their value on the competitive advantage they inherently present. Moreover, the pressure applied by competitors might force the manager to accelerate or defer a decision even though the ROA tool might not support it.
- Interaction between projects and underlying assets

  Externalities derived from investment decisions are presented in other projects. The execution of an option might generate the value of another to change, originating an alteration of the traced strategy for it. On the other hand, most of the time projects' values do not rely on a single underlying asset only. The interconnection of the variables involved is critical to understand their behavior and, of course, their impact on the value of the project. For that, an analysis of the intra/inter project compoundness and the possibility to include multiple underlying assets must be included in the valuation process.
- Uncertainty resolution

  In most valuations, the identification of all the sources of risk is rarely implemented, leading to a limited tracking on how they perform over time. Moreover, the treatment for them is standardized despite their differences in nature. However, is in the premium assignment where the biggest limitation appears. As some are not market priced, there is no risk premium that could be assigned from the capital markets. Therefore, it is crucial for this process to provide the best possible approach, to take special care in this matter.

Following these observations, Amram and Kulatilaka (1999) suggest treatments that must be executed, in three levels, when designing a ROA: valuation, analysis, and implementation. In the first level they suggest to focus on the impact that uncertainty and cost measurement have over the value of the option. For the second level, attention must be paid on how to design a simple, dynamic, and transparent framework that is aware of the evolution of all the relevant information from the financial markets. Finally, in the third level, the suggestion is to perform a specific treatment of private and market risks at all time.

It can be seen that, in order to apply the analogy between financial option pricing theory and ROA perspective, the most relevant aspect to consider is the process of identification, treatment, and quantification of the sources of uncertainty. The key step is to choose, based on the particular investment project characteristics, the appropriate mathematical framework to perform this process. For that, the present work focuses on how to treat two particular limitations: the interaction between underlying assets and how to address uncertainty. The next sections will describe the previous work done on the matter, followed by my proposal of treatment as well as the results for an application on a real investment project valuation.

Volatility

Definition

In the financial context, volatility is defined as the standard deviation of future cash inflows' growth rate associated with the stock. It is considered as a measurement of the unpredictability of the price movements, typically expressed using continuous composition. Following the financial analogy for the real options environment, it can be understood as an explanatory measurement of the risk incurrence inherent to the stochastic nature of the underlying source of uncertainty; in this case a daily composition is customarily used. The treatment of volatility is, probably, the most relevant aspect within the real option analysis, for it considers the second moment of the distribution of returns on the underlying asset. Mathematically speaking, this is the element that differentiates the ROA valuation process from the traditional DCF methods. Davis (1998) formalizes some concepts for estimating a project's volatility and dividend yield when valuing options to invest or abandon a project. He establishes that the instantaneous rate of volatility of the project, σ_V^j, is directly linked to the one of the price of the project's output good, σ_S , via a positive elasticity term ε^j following the form: $\sigma_V^j = \varepsilon^j \sigma_S$. Noteworthy is the fact that, if there is not a correct treatment of it, managers can be tempted to manipulate the parameter σ_S in order to alter the value of the project.

Conceptually, volatility illustrates the uncertainty factors that do not dissolve during the projects' lifetime. Following Amram and Kulatilaka (1999), it is treated as a constant in the ROA approach, as they noted that, in most of the cases, real options are virtually unaffected by unexpected changes in the volatility during the life of the project. They even state that including the stochastic nature of the volatility often leads to more errors in the final valuation result rather than major improvements in it. Another common practical error is to use the terms risk and uncertainty interchangeably. As described by Hung and So (2011), most of the valuation inaccuracies come directly from it. In order to perform this distinction, they propose a method to filter the risk of the project without the influence of uncertainty, using an adjusted Black-Scholes pricing formula.

The characteristics that make volatility require a special focus are:

- Is not directly observable.
- Tends to present a performance known as *clustering*, *i.e.* large returns (positive or negative) are followed by large returns and conversely.
- Does not diverge to infinity, *i.e.* the variation is presented in a fixed range.
- Jumps are rare as it tends to present a continuous evolution over time.
- Presents a negative correlation with the returns typically known as *leverage effect*, *i.e.* high variability is followed from large negative returns.

The development of different volatility models comes as a response to the incorporation of the effect each characteristic has into the valuation process. In some cases, they were designed to improve specific weaknesses of previous models or even to address a particular characteristic. Due to the relevance of this variable within the ROA context, there is a current research trend in relation to its treatment. For that, a review of the most common methods is presented in the following section. However, I highly encourage future works to continue in this direction.

Volatility Models

There are several ways to estimate the volatility used in the option models.[16] The most common are:

- ***Logarithmic stock price returns***, this approach uses the historic data so as to calculate their logarithmic returns and finally their volatility. This model assumes that the returns follow a lognormal distribution and are defined by:

  $$r_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$$

  With volatility defined by:

  $$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2}$$

  The main advantage of this approach is that it is a mathematically valid procedure with computational ease and transparency, as no simulation or extra assumption is required. Even though it is widely used in the financial context, there is a key drawback that makes this method not suited to the ROA environment: the presence of negative returns. The natural logarithm of a negative value does not exist; therefore, the volatility estimation will not fully capture the price dynamic accurately. Another disadvantage is noted when using historic volatility data to describe its future behavior; it does not necessarily presents the same performance.

- ***Management assumptions***, this approach relies on an educated (or industry experience) guess made by the manager. In order to do so, a correct identification of the range of outcomes must be performed. In most of the cases, the value can be derived from the next form:

  $$volatility = \frac{Percentile\ Value - Mean}{Inverse\ of\ the\ Percentile \times Mean}$$

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[16] For a detailed explanation of each, please refer to Mun (2006) in the Appendix 7A.

The key advantage of this model is, clearly, the simplicity of calculation and explanation within the valuation process. On the other hand, the main drawback is that it is subjective and, therefore, unreliable estimation as it comes from a non-formal, non-statistical method incapable of transforming intuition into a robust parameter.

- ***Market proxy using comparables***. Under this approach the manager focuses on the behavior of a particular index, venture, or twin security and associates its volatility to the one of the project. Its main assumption is that inherent risks in the comparables are identical to the ones presented in the project. Therefore, the selection of the proxy has to be supported by commonalities in their characteristics, as well of a proven correlation between them. This method has been recommended by Trigeorgis (1996) to be used when an appropriate twin security can be traced in the market, typically when comparing liquid and non-liquid assets. Two advantages can be identified: it enhances a constant and well-informed involvement of the manager, and incorporates the financial market activity into the model. The main drawback is that it not only relies on a subjective identification of the proxy but also fails to consider market interactions that can be presented in the comparable although the project might not be exposed to them.

- ***Implied volatility***. This approach, instead of presenting an internal calculation made by the manager or the company, looks for the expected future volatility value of the underlying asset as it is presented in the market under the statement that both present the same implied risk. This procedure uses available information regarding the price of an option, exercise and initial stock price, expiry, risk free rate, and, by inverting the Black-Scholes formula, infers the volatility value. *I.e.* if the market price for the option is, say X, the value of the implied volatility is the one, when substituted in the BS formula, gives the precise same X. This method assumes that the price of the underlying asset follows a geometric Brownian motion; therefore, its option market value represents a well-informed prediction about the future of the volatility of the underlying asset. The key advantages of this method are that it does not require a sophisticated mathematical basis and also it does not depend on past data to forecast future volatility values. Drawbacks are similar to the market proxy approach as available information might not be entirely suited to represent or characterize the specific project that is valued.

- ***Logarithmic present value returns,*** this approach tries to fix some disadvantages of the previous ones by focusing on building a formal statistical method, based on simulations of the inputs in order to improving the volatility estimation accuracy. A notable methodology is found in Copeland and Antikarov (2001) where they apply Monte Carlo simulation, under the binomial tree perspective. The goal is to incorporate volatility into the ROA approach by considering the following definition:

$$X = \ln\left(\frac{\sum_{i=1}^{n} PVCF_i}{\sum_{i=0}^{n} PVCF_i}\right) = \ln\left(\frac{\frac{CF_1}{(1+D)^0}+\frac{CF_2}{(1+D)^1}+\frac{CF_3}{(1+D)^2}+\cdots+\frac{CF_N}{(1+D)^{N-1}}}{\frac{CF_0}{(1+D)^0}+\frac{CF_1}{(1+D)^1}+\frac{CF_2}{(1+D)^2}+\cdots+\frac{CF_N}{(1+D)^N}}\right)$$

In this approach, instead of establishing constant values for the parameters involved in the calculation of the present value of the cash flows (*PVCF*), a simulation of them is presented. With it, the inputs of the model represent potential sources of uncertainty and, therefore, the volatility of the project is not necessarily the same as the volatility of its inputs. The first advantage of this method is that, unlike the logarithmic stock price returns approach, it includes negative cash flows. The second one is that its rigorous and conservative perspective leads to a more accurate estimation of the volatility. This holds if, and only if, there is a proper treatment of the inputs' variability. Two main drawbacks are noted with this approach. First, as Monte Carlo simulation is used, refined computational system is required; most of the time, it cannot be applicable for highly traded assets. The second and probably the strongest objection is that the value, as seen in Samuelson's proof, presents a strong dependence on the variability of the used discount rate.

- ***Generalized autoregressive conditional heteroskedasticity models (GARCH)***. This approach considers that the volatility is not constant during the series. It is an extension of the autoregressive conditional heteroskedasticity models (ARCH)[17], and assumes that the lagged information of a variable and its conditional volatility are determinants for present and future behavior of the series. Engle (1982) first introduced the ARCH model, and Bollerslev (1986) properly presented its extension. He did so by following a similar extension from an autoregressive model (AR) to an autoregressive moving average model (ARMA) in order to describe the behavior of the error variance. The specification for this model is:

ARMA (p,q) process for the returns

$$r_t = \varphi_0 + \varphi_1 r_{t-1} + \cdots + \varphi_q r_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q} = \varphi_0 + \epsilon_t + \sum_{i=1}^{p} \varphi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j}$$

And a GARCH (p,q) process for the volatility

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_q \epsilon_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

Typically, these models are used for working with liquid assets, such as stock and oil prices, as they capture volatility clustering and, by avoiding over adjustment, they give a more parsimonious description of the behavior of the series. The advantage of this approach resides on a rigorous statistical analysis so as to calculate the best-fitting volatility curve with a lesser probability of breaking the non-negativity restriction. Nevertheless, it requires a large amount of data and advanced econometric expertise in order to be operated properly. The weaknesses of these models are:

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[17] Please refer to Hamilton (1994), Greene (2008) and Tsay (2005) for a comprehensive review of these models.

- Treats positive and negative returns identically.
- Imposes high restrictions.
- Does not explain the sources of variation for the behavior of the conditional variance, it only describes them.
- Tends to present an over-prediction of volatility, as it responds slowly to large isolated shocks of the returns.
- Does not incorporate the leverage effect or the possibility that the equilibrium level of variance change over time.
- Does not provide a good statistical fit.

Despite the weaknesses, these models still present a generalized acceptance in the practitioner and academic environment. The development of corrections for them led to an expansion of econometric research applied to financial time series models, such as TARCH, EGARCH, NGARCH and IGARCH. Also, the most important research direction in this matter is in the development of multivariate GARCH models presented in Engle & Sheppard (2001) and Engle (2002). Noteworthy is the work of Duan and Pliska (2004) where they used a multivariate GARCH under an equilibrium-based option pricing approach to show the effect co-integrated assets have over the option value when changing volatility is considered. They conclude that, when volatilities are deterministic, the option prices do not depend on the co-integration parameters. Conversely, when stochastic, the value explicitly depends upon them.

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Multiple Underlying Assets

Considerations over the Volatility Treatment

Most of the ROA research and application focuses on the presence of one underlying asset. However, this is unrealistic as options rarely arise in isolation, *i.e.* investment projects depend on more than one underlying asset or even other projects. The inclusion and treatment of this effect is the center of current literature in the ROA analysis. The biggest challenge in real-life project valuation is to correctly identify the collection of multiple real options and underlying assets involved, as well as measuring the interaction between them. A familiar problem arises when letting more underlying assets to be part of the analysis: how the inclusion of two or more volatilities into the model can be addressed. Hence, the valuation process needs to focus on the volatility of the underlyings and their impact to the value and volatility of the project. Volatility treatment with multiple underlying assets is typically made through a spread perspective (σ_S) which is assessed by using a bivariate lognormal distribution with a constant correlation factor ρ, typically expressed as:

$$\sigma_S = \sqrt{\sigma_{f_1}^2 + \sigma_{f_2}^2 + 2\rho\sigma_{f_1}\sigma_{f_2}}$$

Where σ_{f_1} and σ_{f_2} represent the volatility measure of the individual underlying assets. Copeland and Antikarov (2003) noted that the standard deviation for each can be estimated from the residuals of the individual time series regression. However, they must be adjusted as confidence bands widen for out-of-sample forecasts. On the other hand, Mbanefo (1997) pointed out that, typically in spread option valuation models, the volatility and the co-movement structures are not treated adequately when analyzing two underlying assets. Most of the assumptions made in the spread models[18] are not adequate when applying them in practice, particularly in the energetic industry.

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[18] Such as considering that the difference of two correlated lognormal variables is also lognormal.

Therefore, he suggests that a special treatment over these elements is required in order to have a better implementation of these models.

Rainbow Options

In the financial context, an option that incorporates two or more underlying variables is known as a rainbow option. A similar approach has been undertaken in the ROA context by defining it as the real option whose payoff depends on several underlying assets. In this perspective, the behavior of the involved underlying assets is compared, and the value of the project depends on a particular decision rule. The typical examples for European rainbow options, including their payoff at the expiry, are described in Figure 10.

Type of European Rainbow Option	Description	Payoff
Maximum of assets	Selects the best within several assets	$\max(S_1, S_2, \ldots, S_n, K)$
Call on the maximum	Gives the holder the right to buy the maximum asset at the strike price	$\max(\max(S_1, S_2, \ldots, S_n) - K, 0)$
Call on the minimum	Gives the holder the right to buy the minimum asset at the strike price	$\max(\min(S_1, S_2, \ldots, S_n) - K, 0)$
Put on the maximum	Gives the holder the right to sell the maximum asset at the strike price.	$\max(K - \max(S_1, S_2, \ldots, S_n), 0)$
Put on the minimum	Gives the holder the right to sell the minimum asset at the strike price.	$\max(K - \min(S_1, S_2, \ldots, S_n), 0)$
Put B and Call A	Gives the holder the right to exchange one asset with another one.	$\max(S_1 - S_2, 0)$

Figure 10: Examples of European rainbow options.

The beginning of the rainbow option analysis can be traced back to the work of Margrabe (1978) in which he evaluated a European option to exchange one asset for another. His idea was developed by the work of Stulz (1982,) where analytical formulas are presented for pricing a put and call European option when considering the maximum or the minimum of two risky assets. He transformed the double integral of the bivariate density function into a cumulative bivariate normal distribution. These results showed that a call option on the minimum of two risky assets, considering zero as its exercise price, can be evaluated with the same formula used to price an option to exchange one asset for another. Those results were extended by Johnson (1987) in order to define a solution for the general case of an option on several assets through an intuitive approach founded on the Black-Scholes formula. The inclusion of these ideas in the ROA context has led to an expansive research trend. Sødal, Koekebakker and Aadland (2008) modeled, under the valuation of a switching option context, the price spread as a mean-reverting process between the co-integrated dry and wet bulk markets for a combination carrier. Pimentel, Azevedo-Pereira and Couto (2008) used a high speed rail project to develop a partial differential equation model to address the impact multiple sources of uncertainty have over the optimal investment decision; hence, over the valuation process.

Initial approaches on the presence of multiple underlying assets only considered the European option case. Still, in the last twenty years, interesting advances have been offered for the American case.[19] Most of the research considers pricing methods of options with a finite number of exercise opportunities, known as Bermudan options, as an approximation to the Americans. Tan and Vetzal (1995) analyzed how elements such as the nature, time to mature, correlation, and volatility of assets, modify the exercise region for American options on the maximum and minimum of multiple underlyings. Barraquand and Martineau (1995) developed a numerical method that combined Monte Carlo simulation with a partitioning method for the underlying assets' space called Stratified State Aggregation. By doing so, an approximation of the prices of American securities with multiple underlying assets can be calculated; determining the exercise strategy. Longstaff and Schwartz (2001) developed a simulation model in order to have an approximation to the value of American options with multiple factors. They used the least squares Monte

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[19] Fu, Laprise, Madan, Su, and Wu (2001) provide a comprehensive review of this research direction.

Carlo approach to estimate the conditional expected payoff of the option holder from cross-sectional information found on the market. Broadie and Glasserman (2004) introduced a stochastic mesh method for pricing American options whose value depends on multiple assets, providing bounds and confidence intervals for their results. García (2003) presents an extensive and detailed description of the different numerical methods in the American option pricing theory. Ibáñez and Zapatero (2004) introduced a Monte Carlo simulation method for pricing multidimensional American options, on the maximum of up to five underlying assets, based on the computation of their optimal exercise frontier.

Despite the work of Dockendorf and Paxson few efforts have been made in order to develop a "rainbow real option" approach. In his doctoral work, Dockendorf (2010) developed two sequential rainbow option models, one for the best of two stochastic assets, and the other on the mean-reverting spread between two co-integrated assets. Dockendorf and Paxson (2010) incorporated two sources of uncertainty into the ROA valuation by working on the spread of two co-integrated variables into a continuous rainbow option model.

Recognizing and accounting for multiple underlying assets is only the first step to introducing a realistic perspective into the ROA analysis. As noted, one of the most serious obstacles for the widespread adoption of this perspective is the lack of financial option pricing techniques to deal with the complex environment around real life investment projects. The development of new techniques within this context is needed in order for them to address the complexity without dropping the link with capital markets; only then the gap between theory and application will be narrowed. The following section will describe a sophisticated, yet accurate, tool that will help to fulfill this objective: copula modeling.

Copula Modeling

Introduction to Copula Modeling

The current financial situation has drawn the attention towards new alternatives for hedging the risk of the companies, particularly for those derived from the interaction of multiple assets. An important effort has been performed in the development of rainbow options as tools for this objective. However, its application in the ROA context is somehow limited. One of the main arguments in favor of the ROA is that it allows the manager to include more realistic conditions into the valuation of an investment project. Nonetheless, most of the empirical work only considers the univariate underlying asset case. An increasing research trend has emphasized the relevance of multivariate contingent claim pricing as it is clear that real options often deal with two or more random variables interacting simultaneously. For that to be introduced in the valuation model, the key determinant must be the treatment of the relationship that exists in the multiple underlying assets, either as a dependence structure or as a measure of association. Therefore, a sophistication of the current mathematical techniques is required in order to include the effect that the interaction has over the valuation of investment projects. A critical element to address this requirement is to understand, and measure, the dynamic of the interaction of the underlying assets through the analysis of the co-movements of their processes.

Typically the multivariate normal distribution is used to describe this interaction. Yet, it restricts the association measure between margins, the covariance and correlation, to be linear. That is far from being a realistic characterization. When working with derivatives, three main problems appear: the analysis has to move away from the assumptions of normality and market completeness, along with the presence of credit risk. In other words, the traditional tool does not fit in the typical characteristics of the financial world: uncorrelated but dependent returns, heavy tailed and asymmetric distributions, volatility effects, along with the presence of clusters. It becomes clear that, in trying to maintain the intention of the ROA, the inclusion of other association measures is needed. Due to its characteristics, copulas constitute as an alternate measure of stochastic dependence, which addresses the limitations of the correlation as a dependence measure.

The copula concept was introduced by Sklar (1959) as a tool to model the dependence between random variables. It is derived from *copulare*, a Latin word that refers to the connection or joint of objects. Its probabilistic metric space context is taken as a point of departure in financial applications. During the last fifteen years, copula modeling has gained popularity in the academic and practitioner world. This is because it is a mathematical tool that presents advantages addressing, particularly, the multiple asset interaction within the financial context, as it enables to deal with specification of marginal univariate distributions separately from the one of co-movement and dependence structure. Literature on the copula founding concepts, statistical properties, and financial applications has developed rapidly. Joe (1997) and Nelsen (1999) are excellent and highly technical introductory texts, while Frees and Valdez (1998) provide an introduction to the statistical properties of copulas and their applications to the actuarial world. Bouyé, Durrleman, Nikeghbali, Riboulet and Roncalli (2000), along with Cherubini, Luciano and Vecchiato (2004), cover relevant material on copula application in financial econometrics.

Copula modeling is appealing to statisticians and analysts because its characteristics fit better into the financial context than other approaches. First, they allow the treatment of nonlinear dependence. An important property of copulas is that their measures of dependence are invariant to increasing transformations of the individual series. As a result, they represent a way to study scale-free measures that are used to construct multivariate distributions on a next step. Noteworthy is the fact that the main focus of these measures is in tail dependence or extreme events, which makes them appropriate within the financial context. Secondly, they constitute a flexible tool that permits to use and treat any type of marginal distributions and a consequent specification of a link representing the dependence structure between them. The freedom of working with any marginal distribution gives the opportunity to select the copula family that better fits to describe the joint structure they present. The separation suggests that the estimation can be organized in two steps, first to work with the margins and then in the fitting of a copula family. This can be done in a relatively easy process as copulas are a very computational versatile tool that had been enhanced by the advances of the current econometric software.

For that, copulas are a superior approach, in the financial perspective, as they provide a flexible tool to analyze nonlinear and asymmetric dependence structure between markets and risk factors, preserving the specification of the individual marginal distributions and eliminating their influence in the joint structure. Mikosch (2006) pointed out some drawbacks in copula modeling that had to do, directly, with the application process: difficulties to estimate copulas from real data, static dependence results obtained, and the "arbitrariness" involved in the copula, and margin distribution selection. Genest and Rémillard (2006) stated that, despite his scathing review, copulas had greatly improved the modeling of dependencies in practice as they are a mathematical consistent tool. The application of copulas has been a great success in risk management, particularly in calibration and stress testing processes. This because, complicated interaction observed in markets, can be acknowledged and treated, into the interpolation of extreme cases of dependency made by this approach. In order to work with dependence structures, a review of their concepts, properties, and treatments is needed. The following section will focus on that discussion, along with the description on how the copula modeling has been applied to provide a more accurate measure in the financial world.

Dependence

The association between random variables has been one of the most studied concepts in statistics, probability, and therefore, in the financial context. In order to characterize the nature of the dependence structure between financial time series, measures of association are needed. This is usually done through the Pearson coefficient widely known as the linear correlation measure. As it will be described, its properties and assumptions fail to be applied in the financial environment, especially the ones regarding nonlinearity and non-normality properties of the series. In order to fit this environment, two other dependence measures must be considered: the rank correlation and the tail dependence. For that, alternative nonparametric measures are needed, i.e. the Kendall's tau and Spearman's rho, characteristic measures in the copula context. Even though copulas are a less known approach to describe dependence, two observations make its use appealing in financial literature. Commonly we have more information about the margins distributions rather than the joint and, in a bivariate context, they are useful for defining a nonparametric measure of dependence. A very relevant work on this matter is the one presented by Embrechts, Lindskog and Mc Neil (2003) in which they provide an eloquent and detailed coverage of the dependence concept and its treatment trough copulas.

Two random variables (X, Y) are said to be associated or dependent if $F(X,Y) \neq F_1(X)F_2(Y)$, that is, they are not independent. For that, as defined by Embrechts, McNeil and Straumann (2002) a measure of dependence, say δ in the bivariate case $\delta(X,Y)$, can be understood as a scalar that summarizes the structure of two random variables meeting the following four properties:

- Symmetry: $\delta(X,Y) = \delta(Y,X)$
- Normalization: $-1 \leq \delta(X,Y) \leq 1$
- Comonotonic: $\delta(X,Y) = 1 \Leftrightarrow X,Y$
    - Countermonotic: $\delta(X,Y) = -1 \Leftrightarrow X,Y$
- For $T : \mathbb{R} \rightarrow \mathbb{R}$ strictly monotonic transformation on the range of X:

$$\delta(T(X),Y) = \begin{cases} \delta(X,Y) T \text{ increasing} \\ -\delta(X,Y) T \text{ decreasing} \end{cases}$$

In order to measure association between variables, apart from the linear correlation, it is necessary to define and examine complementary concepts: quadrant dependence, rank correlation, concordance, and tail dependence. Next section will do so.

Linear Correlation

The most popular association concept is the Pearson correlation coefficient between a pair of random variables (X, Y), and it is defined as

$$\rho_{XY} = \frac{\text{cov}[X, Y]}{\sigma_X \sigma_Y}$$

Where $\text{cov}[X, Y]$ represents the covariance defined as $E[XY] - E[X]E[Y]$. σ_X and σ_Y denote the finite $(\sigma_X, \sigma_Y > 0)$ standard deviations of X and Y. It is a symmetric measure of linear dependence with a range $-1 < \rho_{XY} < 1$ and $|\rho_{XY}| = 1$ representing perfect linear dependence. Another interesting property is that it is invariant with respect to linear transformations of the variables, that is, $\rho(\alpha X + \beta, \gamma Y + \delta) = \text{sign}(\alpha\gamma)\rho_{XY}$ for $\alpha, \gamma \in \mathbb{R} \setminus \{0\}$ and $\beta, \delta \in \mathbb{R}$. Finally, if the pair (X, Y) follows a bivariate normal distribution, then the correlation coefficient fully informs about their joint dependence and $\rho_{XY} = 0$ implies independence. This does not necessarily stand with other distributions.

As noted, the Pearson's correlation is a very straightforward calculation of a natural scalar measure of dependence in elliptical distributions; for most of the bivariate distributions, it only requires obtaining their second moments in order to derive it. However, a series of disadvantages can be found. The main one is that its use can only be appropriate when working with elliptical distributions, such as the normal or any derivation from it. It also requires the variances to be finite for it to be defined and is not invariant under nonlinear strictly increasing transformations. Even though financial applications, particularly in the risk management area, extensively use the correlation concept to describe dependence between variables, the characteristics of the data disqualifies the use of it in this area. As, most of the financial random variables are not jointly elliptically distributed, using linear correlation as a measure of dependence seems inappropriate. This is because it is not sufficiently informative when working with heavy-tailed distributions or in presence of asymmetric dependence, characteristic of the financial data. Moreover, as shown in Fréchet (1957), it may not be bounded by 1 in absolute value and they even change between different distributions, making it an unsuitable measure of dependence when nonlinear relationships are presented.

Embrechts, McNeil and Straumann (2002) highlighted fallacies concerning the treatment of correlation in models other than the typical multivariate normal. They pointed out that the main problem constructing multivariate distributions is to make them consistent with given marginal distribution and correlations, for that, they emphasized the use of copula representation to clarify the dependence concept. As for its application in the financial context, Chen, Fan, and Patton (2004) provided two simple goodness-of-fit tests so as to apply copula models into multivariate financial time series. Copula modeling represents an appropriate alternative as they offer a non-parametric, scale-invariant measure which is independent from the margins' distributions. In order to define and construct them, a review of two important concepts, Fréchet bounds and concordance, is required.

Fréchet-Hoeffding Bounds

A fundamental element in dependence structures within the copula construction is the bound concept proposed by Hoeffding (1940) and developed by Fréchet (1957). Consider a copula $C(u) = C(u_1, \dots, u_d)$, the Fréchet-Hoeffding bounds are defined by

$$\max\left\{\sum_{i=1}^{d} u_i + 1 - d, 0\right\} \leq C(u) \leq \min\{u_1, \dots, u_d\}$$

Therefore, according to this definition, every bivariate copula has to lie inside the surface given by the lower bound (counter monotonicity copula) $C(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$ and the upper bound $C(u_1, u_2) = \min(u_1, u_2)$. The reason for this is the presence of extreme cases of dependency. Dependence properties and measures of association are interrelated. The most known scale-invariant measures of association are the Kendall's tau and the Spearman's rho rank correlation, both measures of concordance.

Concordance

The concordance concept describes that the probability of having simultaneous large (small) values for X and Y, is high, while having an opposite value is low. Two observations (x_i, y_i) and (x_j, y_j) from a vector (X, Y) of continuous random variables are said to be concordant if $(x_i - x_j)(y_i - y_j) > 0$, and discordant if $(x_i - x_j)(y_{i},- y_j) < 0$. Similarly, two-random vectors (X_i, Y_i) and (X_j, Y_j) are said to be concordant if,

$$P[(X_i - X_j)(Y_i - Y_j) > 0] > P[(X_i - X_j)(Y_i - Y_j) < 0]$$

From this definition, two important measures of dependence can be established: the Kendall's tau and the Spearman's rho rank correlations.

Kendall's tau and Spearman's rho

Nonparametric statistics concentrate on the ranks of given data rather than on the data itself. Therefore, working with rank correlation leads to scale-invariant estimates that allow fitting copula modeling into the obtained data. The two most known rank correlation measures are the Kendall's tau and Spearman's rho. They represent the best alternative to linear correlation coefficient as they measure the degree of monotonic dependence within the non-elliptical context. Since both are not moment-based correlations, manipulation over the variance-covariance structure is not permitted.

The Kendall's tau is defined as the probability of concordance minus the probability of discordance:

$$\tau = \tau_{X,Y} = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]$$

And it can be expressed[20] as

$$\tau_C = \tau_{X,Y} = 4 \iint C(u,v)dC(u,v) - 1$$

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[20] Proof can be found in Nelsen (1999) p. 127.

As noted, the abovementioned integral is the expected value of the random variable $C(U,V)$ where $U,V \sim U(0,1)$ with joint distribution function C, that is:

$$\tau_{X,Y} = 4E(C(U,V)) - 1$$

For the definition of the Spearman's rho, three independent random vectors with a common joint distribution function H, with margins F and G, are considered; say $(X_1,Y_1),(X_2,Y_2),(X_3,Y_3)$. It is defined to be proportional to the probability of concordance minus the probability of discordance for a pair of vectors with the same margins while the components of another are independent; (X_1,Y_1) and (X_2,Y_3) for example. The representation[21] in this case is:

$$\rho_C = \rho_{X,Y} = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0])$$

It can be expressed[22] as

$$\rho_C = \rho_{X,Y} = 12 \iint C(u,v)dudv - 3$$

As noted by Nelsen (1999) Spearman's rho is often called the grade correlation coefficient (population analogy for rank) and for a pair of continuous random variables X and Y is identical to Pearson's product-moment correlation coefficient for the grades of X and Y; that is, the variables $U = F(X)$ and $V = G(Y)$.[23]

$$\rho_C = \rho_{X,Y} = \frac{E(UV) - 1/4}{1/12} = \frac{E(UV) - E(U)E(V)}{\sqrt{Var(U)}\sqrt{Var(V)}} = E(UV) - 1$$

Even when the Kendall's tau and Spearman's rho are measures of the probability of concordance between random variables with a given copula, their values are different. As expressed by Nelsen (1999), the Spearman's rho can be interpreted as a measure of "average" quadrant dependence[24] while the Kendall's tau can be as an "average" of likelihood ratio dependence.

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[21] The coefficient 3, is a normalization constant.

[22] Refer to Nelsen (1999) p. 135 for proof.

[23] The grades u and v are observations from the uniform (0, 1) random variables $U = F(X)$ and $V = G(Y)$ with a joint distribution function of C.

[24] X And Y are said to be positive quadrant dependent if the probabilities that they are simultaneously small (large) is at least as great as it would be if they were independent. That is, their joint probability at each point must be not smaller than the independence one (product) $F(x,y) \geq F_1(x)F_2(y)$.

Tail Dependence

A very particular concern arises when working in the financial context: how to measure the concordance between extreme values. Tail dependence can be understood as an asymptotic measure that relates the degree of dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a bivariate distribution. It is essentially the relation of the conditional probability of having one variable exceeding a particular value given that the other exceeded another. It is important to note that, if the marginal distributions of these random variables are continuous, then the conditional probability measure is a copula function; hence, it also will be invariant under strictly increasing transformation.

The tail dependence measure, for standard uniform random variables u_1 and u_2, can be expressed in terms of a joint survival function $S(u_1, u_2)$ where $\lambda_L$and λ_U are measures of lower an upper dependence defined as,

$$\lambda_L = \lim_{v \to 0^+} \frac{C(v,v)}{v}$$

$$\lambda_U = \lim_{v \to 1^-} \frac{S(v,v)}{1-v}$$

$S(v,v) = \Pr[U_1 > v, U_2 > v]$ represents the joint function with $U_1 = {F_1}^{-1}(X)$ and $U_2 = {F_2}^{-1}(Y)$. Therefore, the upper tail dependence can be understood as a measure that limits the conditional probability $\Pr[U_1 > v | U_2 > v]$. Conversely, the lower tail dependence limits $\Pr[U_1 < v | U_2 < v]$. Following the definitions of the four tail monotonicity conditions[25] made by Esary and Proschan (1972), the positive quadrant dependence can be strengthened by adding a non-increasing (decreasing) property to the function v. As noted by Capéraà and Genest (1993), the most relevant consequence of this is that the bounds for the Kendall's tau and Spearman's rho can be narrowed when one random variable presents a left tail decreasing behavior while the other shows a right tail increasing one.

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[25] Left tail decreasing, left tail increasing, right tail increasing and right tail increasing. For a further explanation of this implication refer to Nelsen (1999).

Definition and Estimation

As defined by Nelsen (1999), a two dimensional subcopula is a certain class of grounded 2-increasing function C' with margins following the properties:

- $Dom\ C' = S_1 \times S_2$ where S_1 and S_2 are subsets of I containing 0 and 1;
- C' is grounded and 2-increasing;
- For every u in S_1 and every v in S_2, $C'(u,1) = u$ and $C'(1,v) = v$

The copula can be understood as a two dimensional subcopula with domain I^2. It is defined as a function C from I^2 to I that presents the properties:

- For every u, v in I, $C(u,0) = 0 = C(0,v)$, $C(u,1) = u$ and $C(1,v) = v$
- For every u_1, u_2, v_1, v_2 in I such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$$

The first property, describing the grounded characteristic, explains that if one of the two events has a zero probability to occur, the joint probability that both events occur must be it as well. Conversely, if one of the arguments is certain to occur, the function must yield the other argument as it is the one that will determine the joint probability. The second property explains that if the probabilities of both events increase, the joint probability should also do so and, for sure, it cannot be expected to decrease. This final element implies that the function must be 2-increasing. With that, is easy to define an n-variate copula $C(u_1, \dots, u_n)$ as a cumulative distribution function (CDF) with uniform margins on the unit interval.

A fundamental theorem in the copula construction was proposed by Sklar (1959) and it clarifies the role copulas play in the relationship between multivariate distribution functions and their univariate margins. It states that if $F_j(x_j)$ is the CDF of a univariate continuous random variable X_j then $H(x_j) = C(F_1(x_1), \dots, F_n(x_n))$ is an n-variate distribution for $X = (X_1, \dots, X_n)$ with marginal distributions F_j for $j = 1, \dots, n$. Conversely, if F is a continuous n-variate CDF with univariate marginal $F_1, \dots, F_n$ CDFs then there exists a unique n-variate copula C such that $F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n))$.

This theorem states that any joint probability distribution can be stated in terms of a copula function taking the marginal distributions as arguments and, conversely, any copula function taking univariate probability distributions as arguments, yields another distribution. That means that any choice for the marginal distributions will be consistent with the copula approach, but also, that the resulting function will provide a separated description of the margins and their dependence structure. This conclusion represents an attractive feature that directs practitioners to its application in finance. Another important consequence of Sklar's theorem it that the inequality $C^- \leq C \leq C^+$ with C^-, C^+ being the minimum and maximum copulas, commonly known as the Fréchet lower and upper bounds, can be rewritten as:

$$\max(F_1(x_1) + F_2(x_2) - 1,0) \leq F(x_1, x_2) \leq \min(F_1(x_1), F_2(x_2))$$

This is known as the Fréchet-Hoeffding inequality for distribution functions and its consideration is crucial in tail dependence analysis, also a key aspect in the financial context.

One of the most challenging tasks in copula modeling is the correct method selection in order to fit observed market data. Due to the characteristics of the copula function, much of the classical statistical theory cannot be used as part of its estimation process. This is commonly developed in the bivariate iid context through asymptotic maximum likelihood estimation (MLE). The following methods are the most used in the literature:[26]

- **Exact maximum likelihood (EMLE).** In this method, the set of all parameters, both from the margins and the copula, are calculated jointly, therefore it is a computational intensive process.
- **Inference for the margins (IFM).** As a way to computationally ease the process, this method estimates the parameters in two steps. First, the parameters of the univariate marginal distributions and then the copula parameter, both through MLE typically using a bootstrap technique.

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[26] Refer to Cherubini, Luciano and Vecchiato (2004) for a comprehensive description of the estimation methods.

- **Canonical Maximum Likelihood (CML)** This method is a special case of the IFM; it avoids assumptions about the marginal distributions by not making an *a priori* specification of them. In the first stage, empirical distributions are used in order to estimate the margins and then proceeds as the IFM.
- **Non-parametric**, these methods avoid assuming a particular parametric copula based on the idea that its form will converge to the underlying dependence structure in a probabilistic sense. Empirical distributions are used trough kernel structures. The functional form is chosen depending on the smooth properties that are needed and is used as a building element to reach the desired estimator.

A clear benefit from the first three options is that flexibility is presented in the way that the selection of the marginal distribution is based on best fitting the sample for the data series and, consequently, the selection of the copula will be made pursuing desirable properties. Nonetheless, this poses a potential problem on building effective criterions on how to select from a set of margins and copula combinations. On the other hand, the non-parametric methods arise as an attractive alternative to reduce this problem as it lets the dataset express the copula without any subjective choice. Still, in order for them to work properly, large amounts of data are needed. As it will be seen later on this section, most of the research advances in copula modeling use non or semi parametric methods in the estimation process. In order to acknowledge the variety of possible copula fitting decisions, a review of the representative copula families is needed; next section will do so.

Copula Families: Representation and Characteristics

As it has been discussed, the joint distribution between underlying assets can take many forms. When selecting a copula, it is very important to acknowledge that the nature of the dependence structure has to be the determinant argument in order to allocate a specific functional form to the relationship. As noted by Frees and Valdez (1998), identifying the appropriate copula family is not a trivial task. In most financial applications, the real challenge consists in finding a convenient distribution to fit some stylized facts expected for the underlying asset behavior. For that, this section presents a review of the commonly used copula families in financial applications. As most of them consider the effect from two underlying assets, the following representations are established in the bivariate case. However, the multivariate case can be easily traced from them.

Product

This representation is the simplest copula that can be found and is typically used as a benchmark for the development of other families as it depicts independence between the underlying assets u_1, u_2. It has the form,

$$C(u_1, u_2) = u_1 u_2$$

Where u_1 and u_2 take values in the unit interval f of the real line.

Farlie-Gumbel-Morgenstern

First proposed by Morgenstern (1956), the Farlie-Gumbel-Morgenstern (FGM) copula is a generalization of the product copula. A dependence parameter θ is introduced to describe the relation between the variables. It takes the form,

$$C(u_1, u_2; \theta) = u_1 u_2 \big(1 + \theta(1 - u_1)(1 - u_2)\big)$$

Is easy to notice that if independence is considered, that is θ equals zero, the FGM copula takes the form of the product copula. Despite its simplicity, it is a restrictive family because the θ can only take modest values in order for it to work adequately.[27]

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[27] Prieger (2002) recommends its use in modeling health insurance plans.

Marshall-Olkin[28]

Proposed by Marshall and Olkin (1967), this class of copulas is useful to describe the dependence structure of components that are subject to shocks that lead to failure of either one or both of them. It is usually applied using Poisson processes when modeling insurance losses in natural disasters[29]. These shocks are assumed to be independent exponential random variables denoted by parameters $\lambda_1,\ \lambda_2,\ \lambda_{12} \geq 0$ and with occurrence time $Z_1,\ Z_2, Z_{12}$.The probability of the component lasting longer than a specific x_i from the component's lifetime X is then,

$$\mathbb{P}\{X_1 > x_1, X_2 > x_2\} = \mathbb{P}\{Z_1 > x_1\}\mathbb{P}\{Z_2 > x_2\}\mathbb{P}\{Z_{12} > \max(x_1, x_2)\}$$

This leads to the construction of the copula family known as the bivariate Marshall-Olkin copula[30] which presents both an absolutely continuous and a singular component and, with $\alpha_i \epsilon [0,1]$, the form,

$$C_{\alpha_1,\alpha_2}(u_1, u_2) = min\left(u_1^{1-\alpha_1} \cdot u_2, u_1 \cdot u_2^{1-\alpha_2}\right)$$

Elliptical

This class of copulas considers that u_1 and u_2 present elliptical distributions. The main argument is that, as they share most of the tractable properties of the multivariate normal distribution, it makes possible to model other forms of non-normal dependences. Using the Sklar's Theorem, simulation from elliptical distributions can be done easily. However, the main drawbacks when applying this class of copulas in finance are that they do not have closed form expressions, only considers one type of distribution for the margins, and, as they are restricted to have radial symmetry[31], the tail dependence cannot be modeled with them. The most characteristic elliptical copulas, as they can be easily parameterized by the typical linear correlation matrix, are the Gaussian and the Student´s t-copula.

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[28] Refer to Embrechts, Lindskog and Mc Neil (2003) for a comprehensive review on their properties and algorithms to generate them.

[29] Refer to Lindskog and McNeil (2003) for a description of this type of models.

[30] Some texts also refer to it as the *generalized Cuadras-Augé* copula.

[31] The coefficient or degree of upper and lower tail dependence are equal.

Gaussian (Normal)

The normal copula takes the form,

$$C(u_1, u_2; \theta) = \Phi_G(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \theta)$$

It can also be expressed as,

$$\int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\theta^2)^{1/2}} \exp\left\{\frac{-(s^2-2\theta st+t^2)}{2(1-\theta^2)}\right\} dsdt$$

Where Φ is the cumulative distribution function (CDF) of the standard normal distribution, and $\Phi_G(u_1, u_2)$ is the standard bivariate normal distribution with correlation parameter θ restricted to the interval $[-1, 1]$. In this copula, θ represents the usual linear correlation coefficient of the corresponding bivariate normal distribution and, when it approaches -1 and 1, it reaches the Fréchet lower and upper bound respectively. Since the normal copula allows equal degrees of positive and negative dependence they are not able to work with not have lower or higher tail dependence.

Student′s t

The t-copula has the form,

$$C^t_{v,R}(u_1, u_2) = t_{v,R}\left(t_v^{-1}(u_1), t_v^{-1}(u_2)\right)$$

Where R denotes the correlation of the margins and $t_{v,R}$ the CDF. For the bivariate case with two dependence parameters (θ_1, θ_2), v degrees of freedom and correlation ρ, this expression can be written as,

$$C^t_{v,R}(u_1, u_2;\ \theta_1, \theta_2) = \int_{-\infty}^{t_{\theta_1}^{-1}(u_1)} \int_{-\infty}^{t_{\theta_2}^{-1}(u_2)} \frac{1}{2\pi(1-\theta_2^2)^{1/2}} \left\{1+\frac{s^2-2\theta_2 st+t^2}{v(1-\theta_2^2)}\right\}^{-(\theta_1+2)/2} dsdt$$

Where $t_{\theta_1}^{-1}(u_1)$ denotes the inverse of the CDF of the standard univariate t-distribution with θ_1 degrees of freedom. This parameter controls the heaviness of the tails; noting that if $\theta_1 < 3$ variance does not exist and, with $\theta_1 < 5$, the fourth moment does not exist. The coefficient of upper tail dependence is increasing in θ_2 and decreasing in θ_1. Noteworthy is that, as $\theta_1 \to \infty$, the t-copula $C^t_{v,R}(u_1, u_2;\ \theta_1, \theta_2)$ approximates to the Gaussian copula.

Archimedean

This type of copulas is one of the most used in financial applications mainly because of their easy construction and because they allow working with a variety of different dependence structures. Moreover, in contrast to elliptical copulas, they commonly present closed-form expressions. Its origins can be traced into the study of probabilistic metric spaces, but the reason for their explosive academic evolution in finance lies on the acknowledgement that, as pointed out by Schmidt (2007), interesting parametric families of copulas can be generated from interpolating between certain other; particularly the family of Archimedean. Genest and Rivest (1993), assuming uniform margins, suggested a nonparametric method for estimating the dependence function which determines an Archimedean copula. Later on, Genest, Ghoudi and Rivest (1995) extended the idea to a semi parametric method for estimating the dependence parameters in a family of multivariate distributions. Noteworthy is the work of Armstrong, Galli, Bailey and Couët (2004) that developed an innovative Bayesian updating form on Archimedean copulas by analyzing how the incorporation of new information affects the distribution the parameters and allowing them to take continuous distributions.

The Archimedean copulas behave like a binary operation on the interval I, in which the copula C assigns to each pair u, v, in I a number $C(u, v)$ also in I. This class of copula is constructed through a generating function φ, being a continuous, strictly decreasing function from [0, 1] to $[0,\infty]$ such that $\varphi(1) = 0$. Typically it is called an additive generator of the copula[32]. If and only if φ is convex, Archimedean copulas have the form

$$C(u,v) = \varphi^{[-1]}\big(\varphi(u) + \varphi(v)\big)$$

With $\varphi^{[-1]}$ being the pseudo-inverse of the function φ.[33]

One important aspect of Archimedean copulas is that, as they are consistent with bivariate extreme value theory, they are inherently fitted to work with tail dependence; a key aspect in financial applications. Because of this, I will focus my analysis in the three main one-parameter Archimedean copulas, constructed using the generator $\varphi_\theta(t)$, the Frank, Clayton, and Gumbel copula families; with a particular emphasis on the last two as they exhibit asymmetric dependence. However, an extensive description of the entire set of this class can be found in Joe (1997) and Nelsen (1999). A summary of the main characteristics of these copulas is presented in Figure 11.

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[32] If $\varphi(0) = \infty$ it is known as a strict generator.

[33] For a proof and the description of the properties of φ and $\varphi^{[-1]}$ see Nelsen (1999) pp.90-91.

Copula	**Function $C(u_1, u_2)$**	$\theta - domain$	$\varphi_\theta(t)$	**Kendall's τ**	**Spearman's ρ**
Normal	$\Phi_G(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \theta)$	$-1 < \theta < 1$	-	$\frac{2}{\pi}\arcsin(\theta)$	$\frac{6}{\pi}\arcsin\left(\frac{\theta}{2}\right)$
Clayton	$\max\left\{(u_1{}^{-\theta} + u_2{}^{-\theta} - 1)^{-1/\theta}, 0\right\}$	$[-1, \infty)\backslash\{0\}$	$\frac{1}{\theta}(t^{-\theta} - 1)$	$\frac{\theta}{\theta + 2}$	*
Gumbel	$\exp\left\{-[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{1/\theta}\right\}$	$[1, \infty)$	$(-\ln t)^\theta$	$1 - \frac{1}{\theta}$	*
Frank	$-\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right)$	$(-\infty, \infty)\backslash\{0\}$	$-\ln\left(\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}\right)$	$1 - \frac{4(1 - D_1(\theta))}{\theta}$	$1 - \frac{12(D_1(\theta) - D_2(\theta))}{\theta}$

Figure11: Summary of characteristic Gaussian and Archimedean copulas. The asterisk indicates that the expression is complicated and $D_k(x)$ is the Debye function given by $D_k(x) = \frac{k}{x^k}\int_0^x \frac{t^k}{e^t - 1} dt$.

Frank

The Frank (1979) copula takes the form,

$$C(u_1, u_2; \theta) = -\theta^{-1} \log\left\{1 + \frac{(e^{-\theta\, u_1} - 1)(e^{-\theta\, u_2} - 1)}{e^{-\theta} - 1}\right\}.$$

The dependence parameter θ may take any real value in $(-\infty, \infty)$. It reduces to the product copula if $\theta = 0$, and reaches the lower and upper Fréchet bounds for $\theta \to -\infty$ and $\theta \to +\infty$ respectively. This copula is popular because it allows negative dependence between the margins, θ is symmetric in both tails, akin to the Gaussian and Student-*t* copulas, and it includes both Fréchet bounds as permissible dependence values. Still, under this copula, the strongest dependence located in the middle of the distribution and, as pointed out by Embrechts, Lindskog and Mc Neil (2003), dependence in the tails tend to be weak in relation to the Gaussian copula. This suggests that the Frank copula is better suited for margins that exhibit weak tail dependence.

Clayton

The Clayton (1978) copula takes the form,

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

In which the dependence parameter θ is restricted to $(0, \infty)$. As θ approaches zero, the margins become independent, that is, it gives the product copula. On the other hand, as it approaches infinity, the copula attains the Fréchet upper bound. Under this form, the lower Fréchet bound $max(u_1 + u_2 - 1{,}0)$ can be described when $\theta = -1$; therefore, this copula family is not able to describe negative dependence. Because it exhibits strong left tail dependence (lower tail) and relatively weak right tail dependence, the Clayton copula is highly used in correlated risk studies.

Gumbel

Also known as the logistic copula, the Gumbel (1960) copula takes the form,

$$C(u_1, u_2; \theta) = \exp\left(-\left(\tilde{u}_1^{\theta} + \tilde{u}_2^{\theta}\right)^{1/\theta}\right)$$

Where $\tilde{u}_j = -\log u_j$ and θ being restricted to $[1, \infty)$. If $\theta = 1$ it follows the product copula form, while for $\theta \to +\infty$ it reaches the Fréchet upper bound $min(u_1, u_2)$. Similar as in the Clayton copula, it describes positive association only. Yet, it presents strong right tail dependence (upper tail) and relatively weak left tail dependence. Consequently, it is an appropriate copula to be applied when u_1 and u_2 are expected to be strongly correlated at high values but less at low ones.

It is important to note that there is no defined rule for copula selection. The choice is strictly influenced by the nature of the considered data; as each copula implies a different dependence structure between the variables. Therefore, the final selection is commonly derived from the analysis of several distribution functions and the comparison of which one yields the best fit according to the provided information.

Copula Modeling in Finance

Review of the Financial Applications

Copula modeling has gained popularity in the financial sector due to their advantages capturing dependence structures between two or more variables, specifically tail dependence, and providing a robust methodology to risk management studies. Currently, we can find them in derivative pricing, insurance and value-at-risk models. However, there is an important effort to expand its application into the real option valuation.

In the last ten years, there has been a notable expansion of academic literature regarding the application of copula modeling in the bivariate and multivariate financial context. A noteworthy effort to enhance this trend is found in Cherubini and Luciano (2002, 2002a); who make a comprehensive description on how to price bivariate and multivariate digital options trough copula modeling. This work led to an interesting development, focused on rainbow options, presented in Cherubini, Luciano and Vecchiato (2004). An application of copula modeling in rainbow option pricing was made by Knox and Ouwehand (2006) where they estimated the marginal risk-neutral asset returns distributions of two South Africa's market indexes.

As it can be noted, most of the financial series cover reasonably long time periods. This represents a potential problem in the dependence structure estimation as a prolonged exposure to economic factors increases the possibilities of its modification. Therefore a dynamic perspective is needed in the copula approach for pricing financial instruments. Goorbergh, Genest and Werker (2005) developed a dynamic copula GARCH method in order to examine the behavior of bivariate option prices when an association between the underlying assets is presented. The method describes the dynamics of the copula by letting the Kendall's tau to evolve over time. In a similar way Zhang and Guegan (2008) applied a dynamic copula model, with time-varying parameters, in order to price the bivariate contingent claims under a GARCH process. The central idea is that, as the association between the underlying assets may be altered over time, the dynamic copula, allowing for time-varying parameter, offers a better approach to the description of it.

Over the last six years, notable copula applications in finance can be found, particularly in the risk management area. Patton (2006, 2006a) extended Sklar's theorem to multivariate conditional distributions and, by so, introduced the concept of conditional copulas; including a time-varying conditional density for each individual variable plus the conditional dependence between them. He developed a bivariate GARCH model that used a Multi-stage Maximum Likelihood Estimator (MSMLE), based on the asymptotic theory and sample simulations, and used a Symmetrized Joe-Clayton copula to test for asymmetry in tail dependence for the Deutsche Mark and the Japanese Yen exchange rates in relation to the US Dollar. Jondeau and Rockinger (2002, 2006) used a similar conditional Copula-GARCH model to measure changes in the dependency of stock markets; particular attention is paid in extreme events such as crisis periods. A variation is presented in Johansson (2011) as he used an EGARCH model, in order to incorporate the asymmetry in log-return volatility of stock indices, into the conditional copula framework. His analysis was centered on how regional financial markets, in Europe and Eastern Asia, were affected in the 2008 crisis.

Chen and Fan (2006, 2006a) introduced a class of semi parametric copula-based multivariate dynamic models (SCOMDY). They specify the conditional mean and variance of a multivariate time series parametrically, such as in the GARCH model with errors generated by a Gaussian or a Student's t-copula, but specifies the multivariate distribution of the standardized innovation semi parametrically. In order to do so, copula modeling is used to capture the simultaneous dependence between multivariate innovations and their marginal distributions. One interesting result is that the asymptotic distribution of the dependence parameter is not affected by the estimation of parameters $\hat{\theta}$. Chiou and Tsay (2008) applied a univariate GARCH model to describe the marginal distributions of the return of two assets. They used copulas to model the dependence between them, and finally applied it in the pricing of derivatives within the VaR context.

Copula Modeling in Real Option Analysis

The early stage of real option literature is based on the modeling single uncertainty sources through closed-form solutions. The current challenge for the academic research is to incorporate more complexities in the models in order to close the gap between theoretical valuations and real applications. Complex real options often consider the presence of multiple underlying assets whose return distributions may exhibit tail dependencies. Therefore, copula models are useful in this context as they capture nonlinear dependencies that arise in this situation. In order to apply copula models in real options theory, two preconditions are needed: irreversibility and the postponement possibility. In that sense, Herath and Kumar (2007) discussed its importance in order to develop a measure dependence that can be applied on engineering economics; particularly in risk simulation and forecasting areas.

The application of copula models into the real options theory and decision-making under uncertainty, in the context of new investments in power generation technologies, is gaining popularity in current research directions.[34] The main reason is that energy derivatives tend to present non-linear dependencies derived from an increasingly intertwined commodity markets. Armstrong, Galli, Bailey and Couët (2004) used an Archimedean copula base model to include technical uncertainty in the valuation of expansion projects in the oil industry. Grégoire, Genest and Gendron (2008) studied the dependence structure between prices for futures on crude oil and natural gas using a copula approach and discussed an appropriate copula family selection for these markets.

Denault, Dupuis and Couture-Cardinal (2009) used a copula model to analyze the diversification effect of energetic generation plants when considering a combination of inflows. They determined that the risk value of a project which considers a mixed hydro-and-wind generator is lower that when considering an all-hydro project. Valizadeh Haghi, Tavakoli Bina, Golkar and Moghaddas-Tafreshi (2010) developed a copula approach to study the planning and operation characteristics of renewable energy generation in Iran. Even though Fleten and Näsäkkälä (2010) did not worked within the copula modeling environment, they develop an interesting model to determine thresholds for energetic prices in which it will be optimal to make an investment decision in gas-fired power plants under the ROA context.

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[34] Westner and Madlener (2010) include a comprehensive literature review on the matter.

In terms of considering multiple underlying assets as determinants for the valuation of an investment project, Herath, Kumar and Amershi (2011) applied the copula methodology to price refinery crack[35] spread options. This is used as a base for risk management in the volatile commodity markets as they allow refiners to hedge their operating margins while letting them to participate in any future widening of their refining margins. They concluded that a Clayton copula model is more appropriate to describe this particular spread option. A similar approach is presented by Benth and Kettler (2006) as they developed a non-symmetric copula to model the spark[36] spread options following a bivariate non-Gaussian autoregressive process. Similarly, Westner and Madlener (2010) applied a specific spread[37] copula-based real options approach in order to determine if an investment project of a power generation plant should work without heat utilization technology or should it be a plant with combined heat-and-power (CHP) generation. They showed that power plants with CHP generation present a lower real option value than those without heat utilization.

As noted, the current approaches in copula modeling instead working with the underlying assets, they consider the combination of financial options and/or real options. On the other hand, the described option valuation approaches commonly use the spread to describe the dependence structure of the underlying assets. To the best of my knowledge, there has been yet no application of Copula-GARCH methodology into the ROA context considering the effect that two (or more) underlying assets have over the value of a project. Therefore, the intention of the proposed methodology is to contribute to the abovementioned literature by illustrating its application with an expansion real option for a natural gas pipeline project in Mexico.

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[35] Also known as a refinery spread, refers to the purchase (sale) of crude oil against the purchase (sale) of refined petroleum products.

[36] It refers to the comparison between electricity and natural gas prices.

[37] Is the difference between the price of the output (electrical power) and the costs of the input factors (e.g. fuels), that is, the contribution margin that a plant operator earns for converting fuels into power.

Empirical Application

Project Description: Los Ramones Natural Gas Pipeline

Pemex Gas y Petroquímica Básica (PGPB hereafter), a branch of *Petróleos Mexicanos* (PEMEX hereafter), through its subsidiary, Mex Gas International Enterprises, Ltd (MGI), is studying the possibility to enter in a 20-year take-or-pay contract seeking to invest and operate a 1,021 kilometer (635 miles) and 2.2 billion cfd (cubic feet per day) natural gas pipeline in Mexico. The main objective of this investment project is to expand the National Gas Pipeline System (SNG for its acronym in Spanish) by increasing the natural gas distribution in the country; mainly in the central-west area.

The project is known as "Los Ramones Natural Gas Pipeline" and is designed to transport natural gas from the U.S.A.-Mexico border (between Texas and Tamaulipas) to Aguascalientes, Querétaro and Guanajuato as shown in Figure 12. PEMEX estimates that the project will provide approximately 23% of the natural gas consumption of the Midwest region, encompassing the states of Aguascalientes, Colima, Guanajuato, Jalisco, Michoacán, Nayarit, Querétaro, San Luis Potosí and Zacatecas.[38]

The estimated total cost of construction, including rights-of-way and line fill, is $3.1 billion USD and is expected to be constructed in four phases through a combination of 48, 42, and 24-inch pipelines and five compression stations that will expand the current capacity of 21,250 horsepower compression up to 247,200. As described by Figure 13, it includes a stage that will distribute natural gas from Los Ramones to Aguascalientes, and could be operational by 2016, with an estimate capacity of 400 MMcfd (million cubic feet per day). The project will grow to another stage to a capacity of 850 MMcfd by 2022, covering the area from San Luis Potosi to San Jose Iturbide. The length of the pipeline from Los Ramones to Aguascalientes is estimated at 660 km with a diameter that could go from 36 to 30 inches and from San Luis Potosi to San Jose Iturbide of 160 km with a diameter of 16 to 24 inches.[39]

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[38] PEMEX (2012a)

[39] PEMEX (2012)

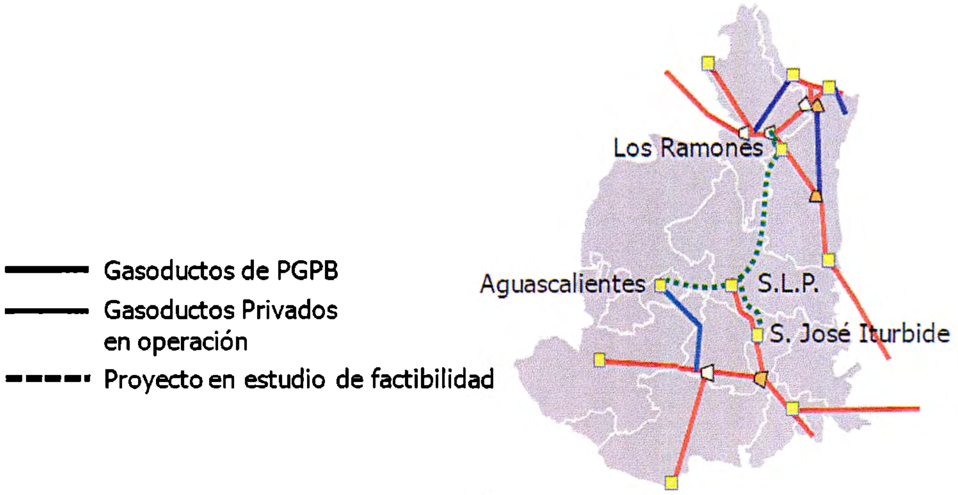
Gasoductos de PGPB
Gasoductos Privados en operación
Proyecto en estudio de factibilidad
Los Ramones
Aguascalientes
S.L.P.
S. José Iturbide

Figure 12: Los Ramones Natural Gas Pipeline Project.[40]

This project is still in the feasibility study stage but MGI has finalized the front-end engineering design, secured the most critical right-of-ways lands and expects to receive required permits and licenses in the fourth quarter of 2012. Also, it is currently negotiating an Engineering, Procurement and Construction agreement, acquiring additional right-of-ways and finalizing debt funding arrangements.

Phase	Description	Estimated Cost	In-service by
1	**USA-Mexico Border – Los Ramones** 112 kilometers of 48-inch pipeline	$295 million USD	July 2014
2	**Los Ramones – Apaseo del Alto & Compression** 728 km of 42-inch pipeline, compression of 216,300 HP	$2.63 billion USD	May 2015
3	**San Luis Potosí – Aguascalientes** 181 km of 42 and 24-inch pipeline	$120 million USD	May 2017
4	**Additional Compression** Compression: 30,900 horsepower	$54 million USD	2020

Figure 13: Los Ramones Natural Gas Pipeline phases description.[41]

Figure 13 and Figure 14 describe the four phases considered in the project, from which it can be seen that the expansion factor considered in its design is 11.63 times with a cost of the expansion of $3.1 billion USD approximately. By May 2015, 88% of the total expansion and 94% of the total cost of the project is scheduled.

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[40] PEMEX (2012a)
[41] PEMEX (2012)

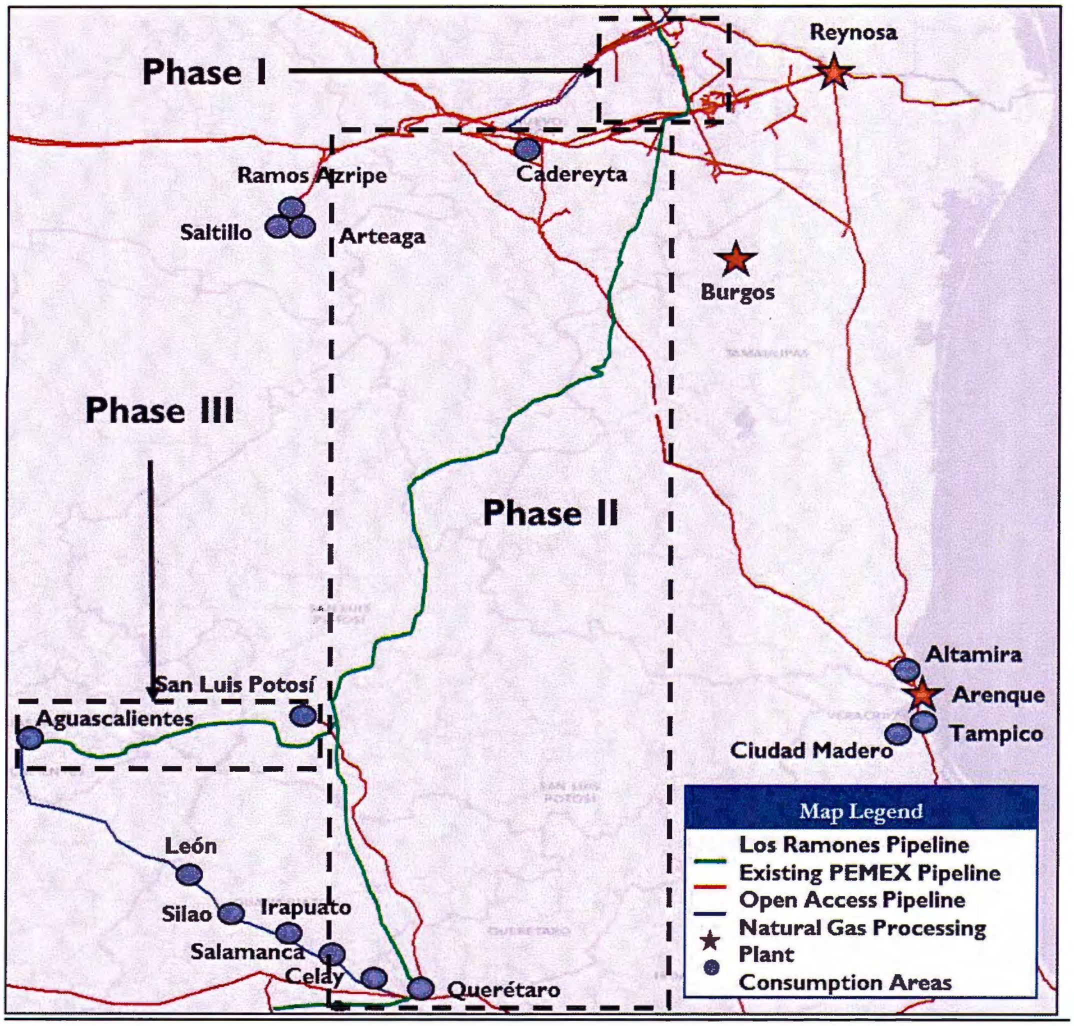
Phase I
Reynosa
Ramos Azripe
Cadereyta
Saltillo
Arteaga
Burgos
Phase III
Phase II
San Luis Potosí
Aguascalientes
Altamira
Arenque
Tampico
Ciudad Madero
León
Silao
Irapuato
Salamanca
Celay
Querétaro
Map Legend
Los Ramones Pipeline
Existing PEMEX Pipeline
Open Access Pipeline
Natural Gas Processing Plant
Consumption Areas

Figure 14: Geographic Evolution of Los Ramones Natural Gas Pipeline.

Data Description

The two underlying assets used in this work are the United States Dollar (USD)-Mexican Peso (MXN) exchange rate and the natural gas price (NGP hereafter). The choice of these variables derives from their relevance in the Mexican energetic industry.

The MXN-USD exchange rate is a key determinant in project evaluation in the country because, due to its geographic location and investment dynamic, most of the information used and presented are commonly expressed in USD rather than MXN. This work uses the indirect quotation (USD-MEX) of the monthly FIX average quote, from January 2001 to November 2012, in order to homogenize, in USD terms, the variables used in the valuation process.[42]

The value of the NGP considered in this work is the price of U.S. Natural Gas Pipeline Exports. It is being use as a proxy to the Mexican NGP because the gas industry in Mexico is not sufficiently developed in order to carry out the entire transformation process of it; as a consequence, most of the natural gas consumed in the country is being imported from Southern Texas. Therefore, Mexican NGP presents a high dependence to the movement of the U.S. NGP. In addition, as the Energy Ministry of Mexico does not keep a record of the evolution of the NGP prior to 2007, there is not a consistent time series for the Mexican case. The price used is a monthly publication by the US department of energy[43] and is expressed in USD per thousand cubic feet.

In order to capture the nature of the financial time series, the price of the underlying assets were computed as log-return rates, following the form:

$$r_{it} = \ln(P_{it}) - \ln(P_{it-1})$$

Where i represents the underlying asset and P_{it} its price in period t.

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[42] The FIX average exchange rate is the market reference exchange rate in Mexico and it is published by the Bank of Mexico (BANXICO).

[43] EIA (2013)

Proposed Methodology

The objective of this work is to exploit the advantages of the volatility treatment, through GARCH models, and the dependence structure determination, through copula modeling, and apply them in the ROA context. The general idea of this methodology can be summarized in three steps. First, the volatility and terminal value of the two underlyings assets, the USD-MXN exchange rate and the NGP, are determined by individual GARCH models. After that, copula modeling is used to determine a measure of association between them in order to define their joint volatility. In this step, three copulas are proposed the Clayton, Gumbel and Normal. Finally, the information obtained in the previous steps is being used as inputs in the ROA context for the valuation of an expansion real option.

In order to perform the first two steps, this work uses the maximum likelihood method to estimate copula models for the two log-return rate series. For an observation (x, y) the likelihood function under copula models follows the form:

$$f(x,y) = f_1(x)f_2(y)C_{12}[F_1(x), F_2(y)]$$

Where $f(x,y)$ represents the joint density function of $F(x,y)$, $fi(\cdot)$ is the density function of $F(\cdot)$, and C_{12} is defined as $\partial C(u,v)/\partial u \partial v$. Given the data, model parameters θ are estimated by maximizing the log likelihood function:

$$\ell(\theta) = \sum_{t=1}^{n} \log\big(f_1(x_t)\big) + \log\big(f_2(y_t)\big) + log\{C_{12}[F_1(x_t), F_2(y_t)]\}$$

A program was developed in order to simultaneously maximize the likelihood function parameters of the marginal distributions and the copula function.[44] In order to perform this maximization process, the proposed methodology considers an initial estimation of the individual log-return series made through a GARCH model. Instead of working with the basic GARCH model, this methodology considers the threshold GARCH (TGARCH hereafter) model. Acknowledged after the work of Zakoian (1994), but also developed by Glosten, Jagannathan, and Runkle (1993)[45], it is commonly used to handle the leverage effect presented in financial time series. A $TGARCH\ (m, s)$ model assumes a similar ARMA structure than the GARCH model, but the process for the volatility takes the form:

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[44] Using the software econometric views 6 (EViews) three copula programs were developed Clayton, Gumbel and Normal; however, the Gumbel copula is not further reported as it does not provide a good fit with the available information of the project.

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{s}(\alpha_i + \gamma_i I_{t-i})u_{t-i}^2 + \sum_{j=1}^{m}\beta_j \sigma_{t-j}^2$$

Where I_{t-i} is an indicator for negative u_{t-i}, that is,

$$I_{t-i}\begin{cases} 1 & if\ u_{t-i} < 0 \\ 0 & if\ u_{t-i} > 0 \end{cases}$$

α_i, γ_i and β_i are non-negative parameters satisfying conditions similar to those of GARCH models. From the model, it is seen that a positive u_{t-i} contributes $\alpha_i u_{t-i}^2$ to σ_t^2, while negative u_{t-i} impacts in $(\alpha_i + \gamma_i)u_{t-i}^2$ with $\gamma_i > 0$. This way the indicator will capture the leverage effect of the financial series; a missing consideration in GARCH models. For the empirical application, both series considered a $TGARCH\ (1, 1)$ model following the form:

$$\begin{gathered} r_{it} = \mu_i + u_{it} \\ u_{it} = \sigma_{it}\varepsilon_{it} \\ \sigma_{it}^2 = \alpha_{i0} + \alpha_{i1}u_{it-1}^2 + \beta_i \sigma_{it-1}^2 + \gamma_i I(u_{it-1} < 0) \end{gathered}$$

Where innovations ε_{it} are assumed to follow a standardized t-student distribution with v degrees of freedom. After it, the margins' estimated parameters are treated as known, and used to perform the evaluation of the likelihood function. At this point, the objective function of the estimation is reduced to:

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[45] Both works propose essentially the same model.

$$\ell(\theta) = \sum_{t=1}^{n} log\{C_{12}[\hat{F}_1(x_t), \hat{F}_2(y_t)]\}$$

Where the distribution functions $\hat{F}_1$ and $\hat{F}_2$ are obtained from the TGARCH models. As noted by Patton (2006a) and Chiou and Tsay (2008) this two-step perspective yields asymptotically efficient estimates.

The results obtained up to this part of the procedure are used as inputs to evaluate a project under the ROA perspective. In order to do so, a last element is needed: the joint volatility. The volatility treatment with multiple underlying assets is typically made through a spread perspective σ_S. It is assessed using a bivariate lognormal distribution with a constant correlation factor ρ. For the addition of two assets, it is expressed as:

$$\sigma_S = \sqrt{\sigma_{f_1}^2 + \sigma_{f_2}^2 + 2\rho\sigma_{f_1}\sigma_{f_2}}$$

Where σ_{f_1} and σ_{f_2}represent the volatility measure of the individual underlying assets. Due to the nature of the relationship between the exchange rate and the NGP, instead of working with the volatility of the spread of two variables, this work considers the volatility of the product of this two variables; being this, the main novel element in the Copula-GARCH real option literature. Its treatment then is made through: [46]

$$Var(xy) = Cov(x^2, y^2) - 2\rho\ \sigma_x\sigma_y\ \mu_x\ \mu_y - \rho^2\sigma_x^2\sigma_y^2 + \sigma_x^2\sigma_y^2 + \sigma_y^2\ \mu_x^2 + \sigma_x^2\ \mu_y^2$$

Where μ_x and μ_y represent the expected value of the log-return series, σ_x and σ_y the volatility measure of the individual underlying asset, obtained from the TGARH models, and ρ is the measure of association obtained through copula modeling (Spearman's rho).

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[46] Refer to the Annex for the explanation.

Results

Los Ramones natural gas pipeline investment project can be seen as an option to alter operating scale; particularly an expansion real option. This work is considering the product of two underlying assets, therefore, the intrinsic value of this option can be seen as:

$$Max\big((Expansion * e_{USD-MXN} * P_{NG}) - Investment,\ \ Continue\big)$$

The expansion factor used in the valuation is 11.63 as it represents what PEMEX has published the production enhancement will be, with an investment cost of $3.1 billion USD. The valuation period considered in this analysis is from 2012 to 2020 with a market return rate of 12%[47].

Value	Clayton	Normal
Spearman's rho	0.069847	0.063280
Kendall's tau	0.033745	0.075251
Joint Volatility	0.003104	0.003106
ROA Value USD	$1,241,774.68	$ 1,237,861.79
ROA Value MXN[48]	$ 15,845,044.86	$ 15,795,116.42

Figure 15: Summary of Estimated Results.

Figure 15 shows the results obtained by the proposed methodology for the Clayton and the Normal copulas. It is interesting to observe that, even though Normal copulas do not capture tail dependence, the result is very similar to the one from the Clayton copula; used when variables exhibit lower tail dependence. Both cases yield a positive ROA valuation, indicating that the project should be accepted.

The original objective of this work was to compare the NPV resulting from PEMEX's procedure with the valuation result from this methodology. The expected result for this comparison was to illustrate that, by combining the advantages of the Copula-TGARCH modeling into the ROA context, a higher value for the investment project will be obtained. Unfortunately, despite my efforts to obtain such information, at the time this thesis was finished the information was not available, as PEMEX argued that as it is an ongoing project, no official information could be published. Is in my best interest to perform this missing comparison but, as in real options, I have to defer that plan until information is available.

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[47] Used as a generally accepted value for investment projects in Mexico.

[48] Considering a 12.76 MXN per USD exchange rate, value reported by PEMEX (2012) by the time of the publication of the project description.

Conclusions and Final Remarks

The proposed methodology is an alternative process to value investment opportunities that seeks to harness the benefits of the ROA, T-GARCH and Copula models. The main argument is that the three components are best fitted to capture and describe the nature of the financial series. It has been established that the ROA perspective outperforms the traditional valuation techniques as it incorporates flexibility, uncertainty, irreversibility, discipline and strategic perspective into the valuation process. Special attention has been paid to the volatility treatment as it is a fundamental variable in the investment project valuation.

For doing so, a T-GARCH model was used as it outperforms the traditional volatility models by incorporating the clustering and leverage effect into its value. On the other hand, copula modeling enables the establishment of a correlation structure for variables that are not normally distributed. It provides a flexible tool to analyze nonlinear and asymmetric dependence structure between markets and risk factors, preserving the specification of the individual marginal distributions and eliminating their influence in the joint structure. The bivariate Gumbel and Clayton copula are useful to work with variables that present tail dependence; a main focus of risk management.

The objective of this work is to exploit the advantages of the volatility treatment, through GARCH models, and the dependence structure determination, through copula modeling, and apply them in the ROA context. By implementing a Copula-TGARCH model, the treatment for the volatility and terminal value of the margins is made, in a first step, through TGARCH individual models. Afterwards, copula modeling is used to determine a measure of association between them in order to define their joint volatility. The proposed methodology suggests a third step that uses the previous information as inputs in the ROA context for the valuation of an expansion real option.

Even though notable contributions are found in order to develop a "rainbow real option" approach, to the best of my knowledge, there has been no application of a Copula-TGARCH methodology into ROA pricing context considering the effect of two underlying assets. Instead, most of the existing work uses the combined effect of two (or more) real options. Moreover, when working with the volatility, the common treatment is to perform a spread analysis. The novel of this work is to consider the volatility of the product of two variables, in order incorporate their combined effect over the joint volatility in a linear and non-linear sense; consistent with the Copula-TGARCH model.

In order for this methodology to be enhanced, some final recommendations must be established. The use of high frequency data is consistent with the intention of this procedure. Working, for example, with daily information will enhance the properties of the Copula-TGARCH model. For this work, this type of information was not available for the Mexican Natural Gas Price due to the limited data infrastructure on the matter. I highly encourage future research to focus on this. After reviewing and comparing the energetic investment opportunities in the world, the Mexican energetic sector presents lags in terms of the development of projects with a real option perspective; expansion or contraction projects are predominant in the country. New types of real options should be considered in the sector to reinforce its strategic perspective.

Finally, two main expansions are suggested for this methodology. The methodology used for a bivariate case can be directed to develop multivariate Copula-GARCH models for real option analysis. By doing so, the number of relevant variables considered in the analysis increases. If done correctly, this clearly enlarges the possibility of capturing their effect in the value of the project. Also, the Copula-GARCH model (in any form) can be enriched by the addition of a discount rate model that adequately estimates and captures the nature of the energetic industry; particularly the Mexican. This will eliminate the arbitrary selection of a discount rate by the manager, increasing the possibilities of estimating a value for the project that completely, or at least in the maximum possible way, captures and reflects the characteristics of its financial environment.

Annex 1: Volatility of a product

By definition, the variance of a random variable x is $Var(x) = E(x) - E(x)^2$ and the covariance of two random variables x and y is $Cov(x,y) = E(xy) - E(x)E(y)$. Consequently the variance of the product of two random variables x and y is:

$$Var(xy) = E(x^2y^2) - E(xy)^2 \tag{1}$$

And the covariance of the square of two random variables x and y

$$Cov(x^2,y^2) = E(x^2y^2) - E(x)^2E(y)^2 \tag{2}$$

Solving (2) for $E(x^2y^2)$ and substituting in (1)

$$Var(xy) = Cov(x^2,y^2) + E(x)^2E(y)^2 - E(xy)^2 \tag{3}$$

From the covariance definition, (3) can be rewritten as

$$Var(xy) = Cov(x^2,y^2) + E(x^2)E(y^2) - [E(x)E(y) + Cov(x,y)]^2 \tag{4}$$

By performing the square and taking $E(x^2) = Var(x) + E(x)^2$ and $E(y^2) = Var(y) + E(y)^2$ from the variance definition, (4) can be written as

$$\begin{aligned} Var(xy) = {} & Cov(x^2,y^2) - 2Cov(x,y)E(x)E(y) - Cov(x,y)^2 + Var(x)Var(y) \\ & + Var(y)E(x)^2 + Var(x)E(y)^2 \end{aligned}$$

By definition, the Pearson's correlation coefficient of two random variables x and y, with expected values of μ_x and μ_y and standard deviations σ_x and σ_y is

$$\rho_{x,y} = \frac{Cov(x,y)}{\sigma_x\sigma_y}$$

Therefore, the variance of a product can also be expressed as

$$Var(xy) = Cov(x^2,y^2) - 2\rho\,\sigma_x\sigma_y\,\mu_x\,\mu_y - \rho^2\sigma_x^2\sigma_y^2 + \sigma_x^2\sigma_y^2 + \sigma_y^2\,\mu_x^2 + \sigma_x^2\,\mu_y^2$$

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